Tropical Positivity and Determinantal Varieties

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joint work with Georg Loho and Rainer Sinn

Copenhagen-Jerusalem Combinatorics Seminar

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MAX PLANCK INSTITUTE
FOR MATHEMATICS IN THE SCIENCES

Overview

1. Tropicalization
2. Positive Tropicalization
3. Determinantal Varieties

tropical semiring $\mathbb{T} = (\mathbb{R} \cup \{\infty\}, \oplus, \odot) = (\mathbb{R} \cup \{\infty\}, \min, +)$

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Example.

$$1 \odot \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \oplus \begin{pmatrix} 2 \\ \infty \\ -1 \end{pmatrix} = \begin{pmatrix} \min(1+0, 2) \\ \min(1+1, \infty) \\ \min(1+2, -1) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \in \mathbb{T}^3$$

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tropicalization: transform algebraic varieties into polyhedral fans

complex Puisseux series $\mathcal{C} := \mathbb{C}\{\{t\}\}$

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$w \in \mathbb{R}^n$ weight vector, $f \in \mathbb{R}[x_1, \dots, x_n]$ polynomial, i.e.

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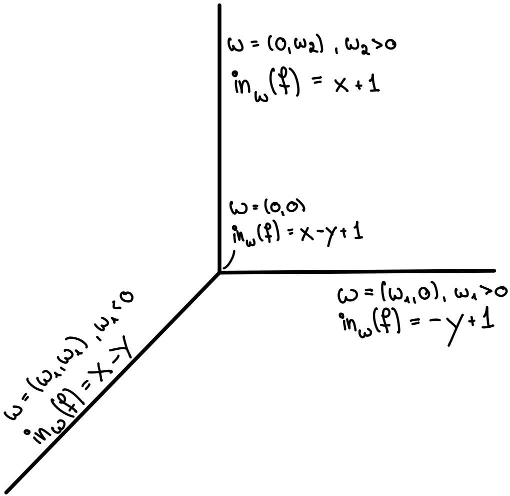

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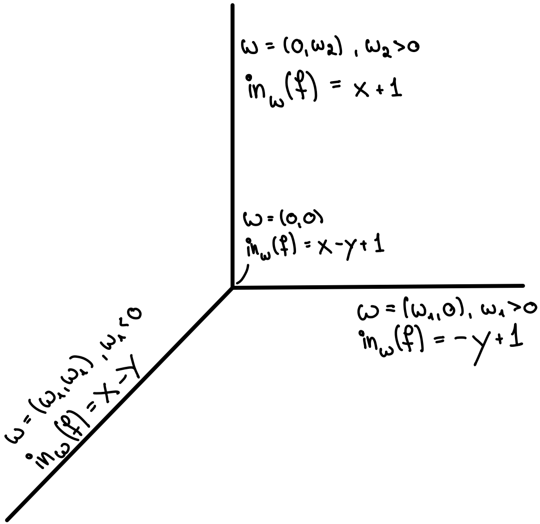
Theorem (Speyer-Williams '05)

Let $w \in \mathrm{trop}(V(I))$. Then $w \in \mathrm{trop}^+(V(I))$

$\iff$ *all polynomials in $\mathrm{in}_w(I)$ have coefficients of both signs.*

$f = x - y + 1$


ω = (0,ω2) , ω2>0
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inω(f) = x-y

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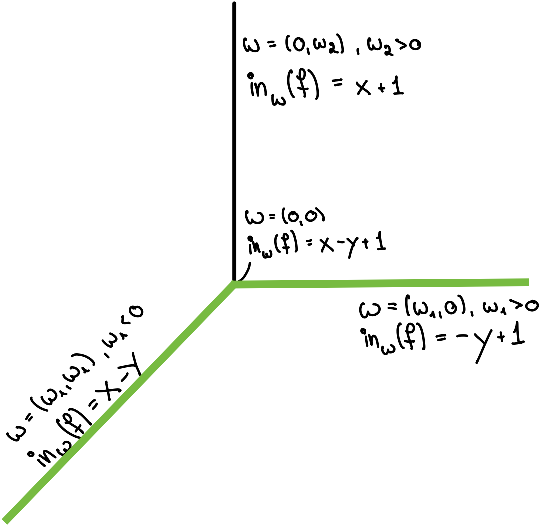
ω = (0,ω₂) , ω₂>0
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- [Ruiz-Santos '22]
  positive part of tropical Pfaffian prevariety

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Theorem (DSS05, CJR11, Shi13)

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Theorem (B.-Loho-Sinn)

If $d = n = r+1$ *or* $r = 2$*, then the* $(r+1) \times (r+1)$*-minors form a set of positive-tropical generators of* $\mathrm{trop}(V(I_r))$*.*

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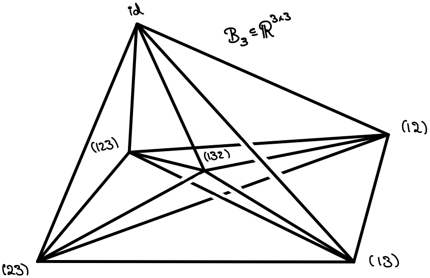

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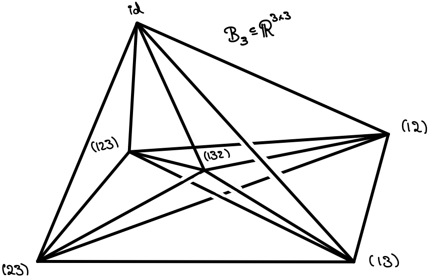

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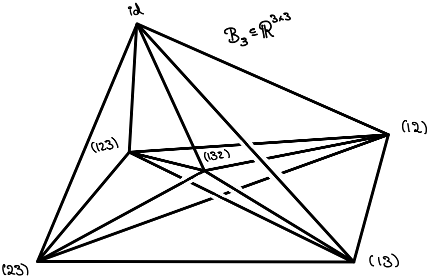

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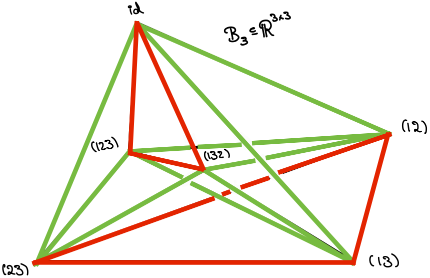

Case $d = n = r + 1$.

$V(\det) = \{\tilde{A} \mid \det(\tilde{A}) = 0\}$, $\det(\tilde{A}) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} \tilde{A}_{i\sigma(i)}$

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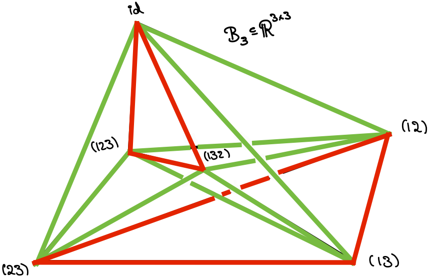

non-positive edges: $\{(\sigma, \pi) \mid \operatorname{sgn}(\pi\sigma) = 1\} \cong A_n$ alternating group

Point Configurations

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$\rightarrow$ columns of $\tilde{A} \cong n$ points on r-dim'l linear space in $\mathcal{C}^d$

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Example

$$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cong \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \cong \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \in \mathbb{TP}^2$$

Tropical Lines

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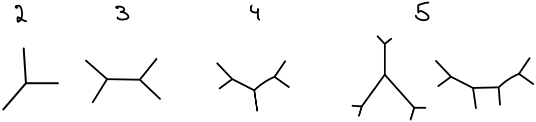

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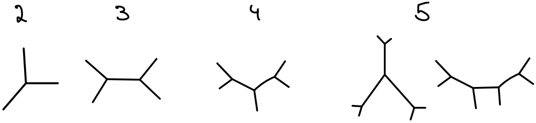

Theorem (follows from [Ardila '04])

Let $A \in \operatorname{trop}(V(I_2))$. *Then* $A \in \operatorname{trop}^+(V(I_2))$
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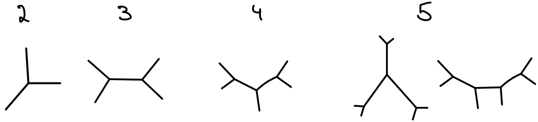
Determinantal Varieties ⟩ Rank 2

Tropical Lines



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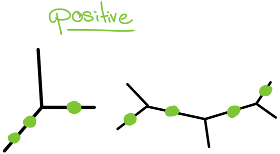

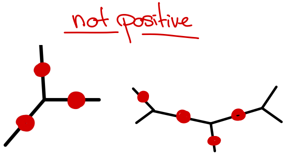

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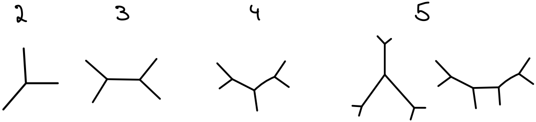

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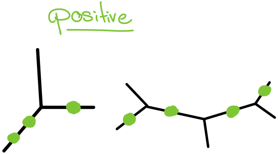
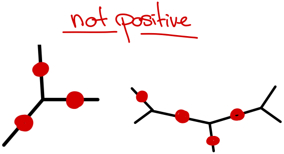
($\iff$ A *has Barvinok rank* 2)
($\iff$ *the ass. bicolored phylogenetic tree [Markwig-Yu'09] is a caterpillar tree*)





A tropical plane is a 2-dimensional polyhedral complex.

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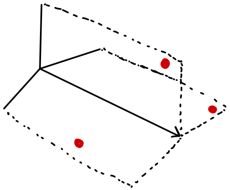
Definition.

A point configuration of 3 points form a starship on a tropical plane if they lie on 3 distinct 2-dimensional faces, which intersect in an unbounded 1-dimensional face.



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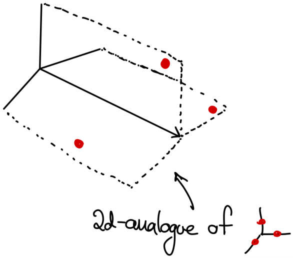

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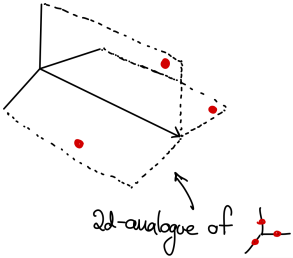
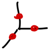
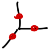


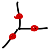
Theorem (B.-Loho-Sinn, "Starship Criterion")

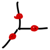
$A \in \operatorname{trop}^+(V(I_3))$

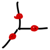
$\implies$ *the point configuration does not contain a starship*

For ranks $r \geq 4$, higher-dimensional analogues of  may occur:

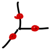
For ranks $r \geq 4$, higher-dimensional analogues of  may occur:

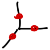
Counterexamples for rank $r \geq 4$ of a positive point configuration $A \in \mathrm{trop}^+(V(I_r))$ containing an analogue of .

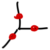
For ranks $r \geq 4$, higher-dimensional analogues of  may occur:

Counterexamples for rank $r \geq 4$ of a positive point configuration $A \in \mathrm{trop}^+(V(I_r))$ containing an analogue of .

Recap

- Rank 2: positive $\Longrightarrow$ no 

For ranks $r \geq 4$, higher-dimensional analogues of  may occur:

Counterexamples for rank $r \geq 4$ of a positive point configuration $A \in \mathrm{trop}^+(V(I_r))$ containing an analogue of .

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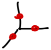
- Rank 2: positive $\implies$ no 
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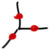
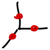
For ranks $r \geq 4$, higher-dimensional analogues of  may occur:

Counterexamples for rank $r \geq 4$ of a positive point configuration $A \in \mathrm{trop}^{+}(V(I_r))$ containing an analogue of .

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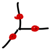
- Rank 2: positive $\implies$ no 
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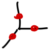
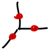
For ranks $r \geq 4$, higher-dimensional analogues of  may occur:

> Counterexamples for rank $r \geq 4$ of a positive point configuration $A \in \mathrm{trop}^+(V(I_r))$ containing an analogue of .

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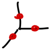
- Rank 2: positive $\implies$ no 
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Thank you!