# Tropical Positivity and Determinantal Varieties

Marie Brandenburg
joint work with Georg Loho and Rainer Sinn

## MOSAiC

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MAX PLANCK INSTITUTE
FOR MATHEMATICS IN THE SCIENCES

# Overview

1. Tropicalization

2. Positive Tropicalization

3. Determinantal Varieties

tropical semiring $\mathbb{T} = (\mathbb{R} \cup \{\infty\}, \oplus, \odot) = (\mathbb{R} \cup \{\infty\}, \min, +)$

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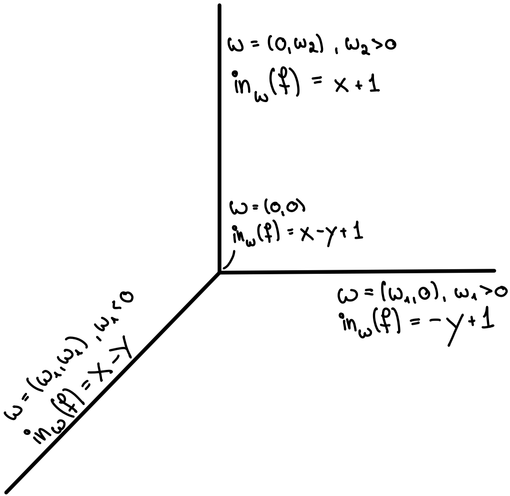

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**Theorem (Speyer-Williams '05)**

*Let* $w \in \mathrm{trop}(V(I))$. *Then* $w \in \mathrm{trop}^+(V(I))$

$\iff$ *all polynomials in* $\mathrm{in}_w(I)$ *have coefficients of both signs.*

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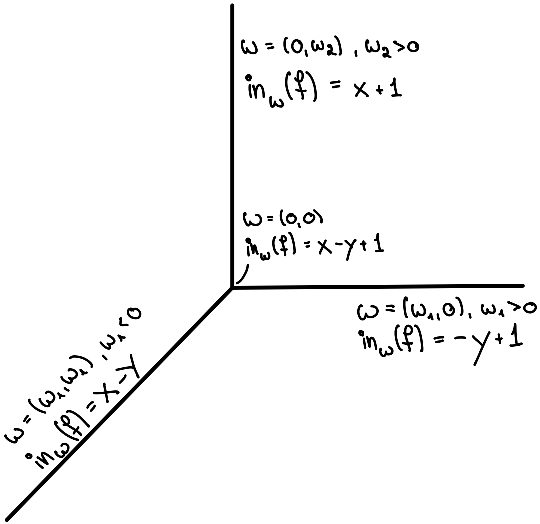
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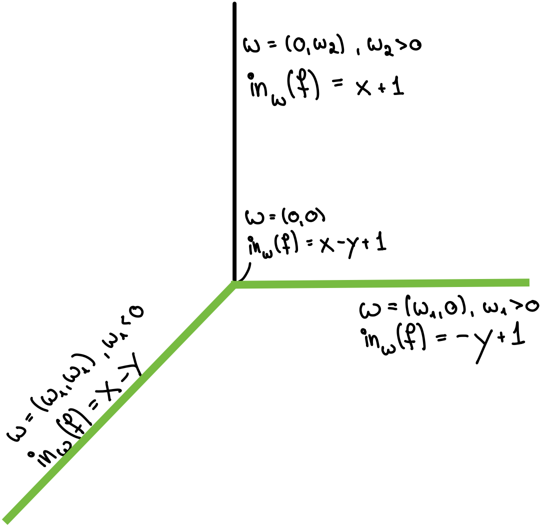
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A finite set $P$ of polynomials is a set of positive-tropical generators if $\mathrm{trop}^+(V(I)) = \bigcap_{f \in P} \mathrm{trop}^+(V(f))$

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*The* $(r+1)\times(r+1)$*-minors form a tropical basis of* $\mathrm{trop}(V(I_r))$ *if and only if* $r \leq 2$*, or* $r+1 = \min\{d,n\}$*, or if* $r = 3$ *and* $\min\{d,n\} \leq 4$*.*

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## Theorem (B.-Loho-Sinn)

*If $d = n = r+1$ or $r = 2$, then the $(r+1) \times (r+1)$-minors form a set of positive-tropical generators of $\mathrm{trop}(V(I_r))$.*

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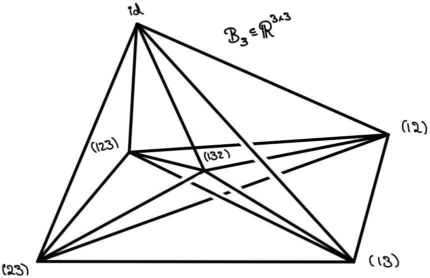

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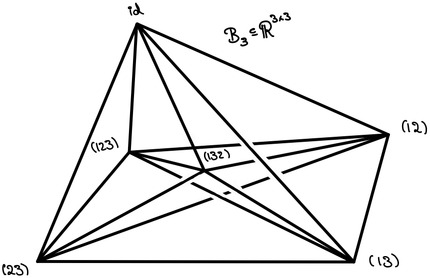

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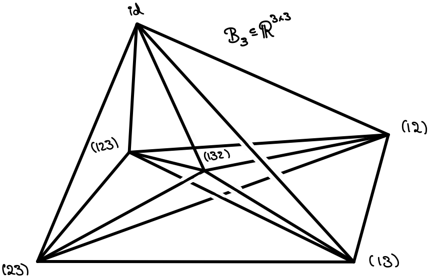

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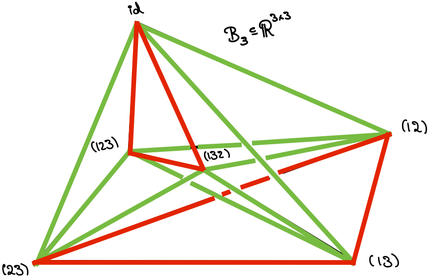

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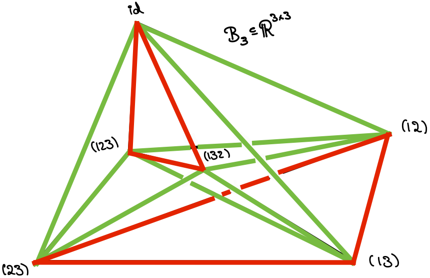
Newton polytope of $\det$: Birkhoff polytope

$B_n = \operatorname{conv}((n \times n)$-permutation matrices$) \subseteq \mathbb{R}^{n \times n}$

$C$ is a maximal cone of $\operatorname{trop}^{+}(V(\det)) \iff$
$C$ is normal cone of edge $\operatorname{conv}(\sigma, \pi)$ of $B_n$ and $\operatorname{sgn}(\sigma) \neq \operatorname{sgn}(\pi)$




non-positive edges: $\{(\sigma, \pi) \mid \operatorname{sgn}(\pi\sigma) = 1\} \cong A_n$ alternating group

# Point Configurations

$\tilde{A} \in \mathcal{C}^{d \times n}, \mathrm{rk}(\tilde{A}) \leq r$

$\rightarrow$ columns of $\tilde{A} \cong n$ points on $r$-dim'l linear space in $\mathcal{C}^d$

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## Example

$$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cong \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \cong \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \in \mathbb{TP}^2$$

## Tropical Lines

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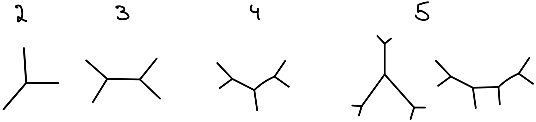

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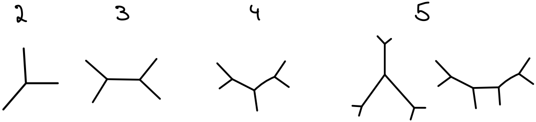


## Theorem (follows from [Ardila '04])

*Let* $A \in \mathrm{trop}(V(I_2))$. *Then* $A \in \mathrm{trop}^+(V(I_2))$
$\iff$ *points form "consecutive chain" on tropical line*

# Determinantal Varieties ⟩ Rank $2$

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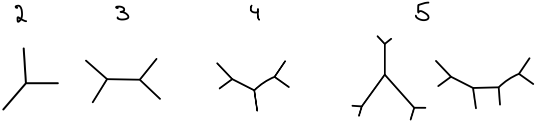


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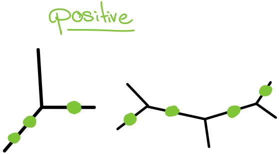
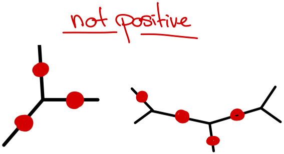
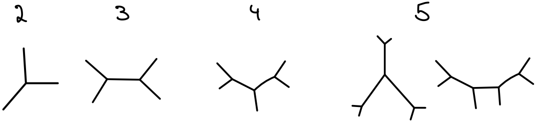


# Determinantal Varieties 〉 Rank $2$

## Tropical Lines



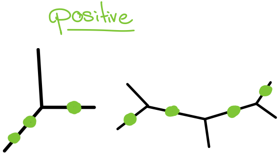

## Theorem (follows from [Ardila '04])

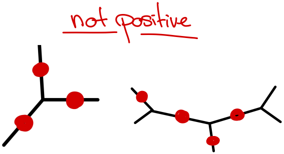
*Let* $A \in \mathrm{trop}(V(I_2))$. *Then* $A \in \mathrm{trop}^+(V(I_2))$

$\iff$ *points form "consecutive chain" on tropical line*

( $\iff$ $A$ *has Barvinok rank* $2$)

( $\iff$ *the ass. bicolored phylogenetic tree [Markwig-Yu'09] is a caterpillar tree*)



A tropical plane is a $2$-dimensional polyhedral complex.

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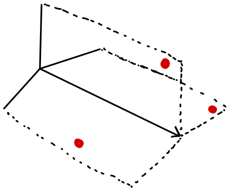
## Definition.

A point configuration of $3$ points form a starship on a tropical plane if they lie on $3$ distinct $2$-dimensional faces, which intersect in an unbounded $1$-dimensional face.



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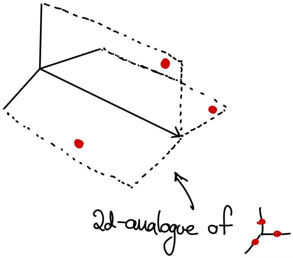

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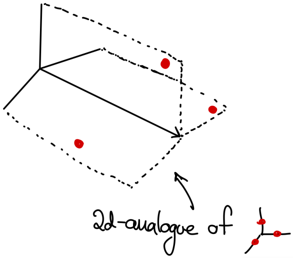
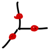
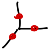
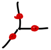


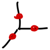

**Theorem (B.-Loho-Sinn, "Starship Criterion")**

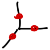
$A \in \operatorname{trop}^+(V(I_3))$
$\implies$ *the point configuration does not contain a starship*

For ranks $r \geq 4$, higher-dimensional analogues of  may occur:

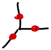
For ranks $r \geq 4$, higher-dimensional analogues of  may occur:

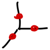
Counterexamples for rank $r \geq 4$ of a positive point configuration $A \in \mathrm{trop}^+(V(I_r))$ containing an analogue of .

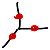
For ranks $r \geq 4$, higher-dimensional analogues of  may occur:

Counterexamples for rank $r \geq 4$ of a positive point configuration $A \in \mathrm{trop}^+(V(I_r))$ containing an analogue of .

## Recap

- Rank 2: positive $\implies$ no 

For ranks $r \geq 4$, higher-dimensional analogues of  may occur:

Counterexamples for rank $r \geq 4$ of a positive point configuration $A \in \operatorname{trop}^{+}(V(I_r))$ containing an analogue of .

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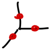
- Rank 2: positive $\Longrightarrow$ no 
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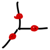
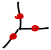
For ranks $r \geq 4$, higher-dimensional analogues of  may occur:

Counterexamples for rank $r \geq 4$ of a positive point configuration $A \in \operatorname{trop}^+(V(I_r))$ containing an analogue of .

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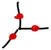
- Rank 2: positive $\implies$ no 
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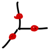
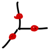
For ranks $r \geq 4$, higher-dimensional analogues of  may occur:

> Counterexamples for rank $r \geq 4$ of a positive point configuration $A \in \mathrm{trop}^+(V(I_r))$ containing an analogue of .

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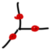
- Rank 2: positive $\implies$ no 
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# Thank you!