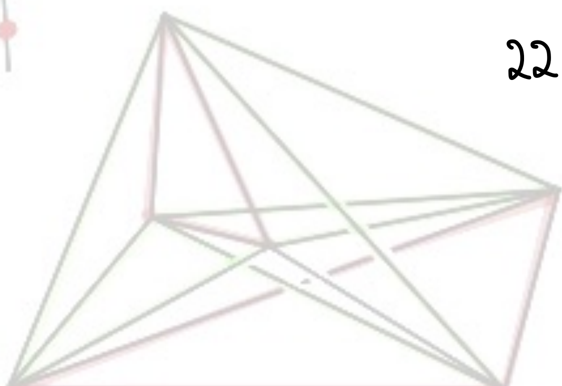
The positive tropicalization of low rank matrices

Marie Brandenburg

joint work with Georg Loho and Rainer Sinn

ECCO

22 June 2022



Recap : Tropicalization

Complex Puiseux series $\mathcal{C} = \mathbb{C}\{\{t\}\}$:

$x(t) \in \mathcal{C} \iff \qquad x(t) = \sum_{k=k_0}^{\infty} c_k t^{k} \quad , \; c_k \in \mathbb{C}$

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$$\mathcal{C}^{2\times 2} \ni \tilde{A} = \begin{bmatrix} 1-t & 2t^{-3} \\ t^{1/2}+t^{2} & 3 \end{bmatrix} \qquad \operatorname{trop}(\tilde{A}) = \begin{bmatrix} \; & \; \\ \; & \; \end{bmatrix}$$

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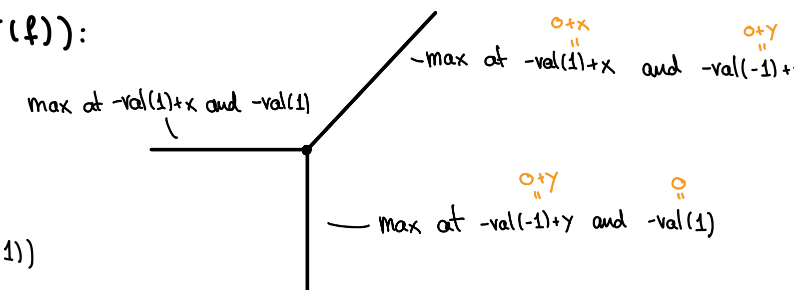

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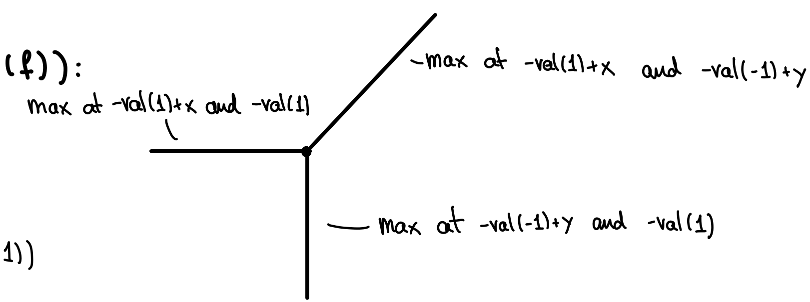

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Speyer-Williams '05 $= \{ \omega \mid \mathrm{in}_\omega(I) \not\ni \text{monomial and } \mathrm{in}_\omega(I) \cap \mathbb{R}_+[x_1,\dots,x_n] = \emptyset \}$

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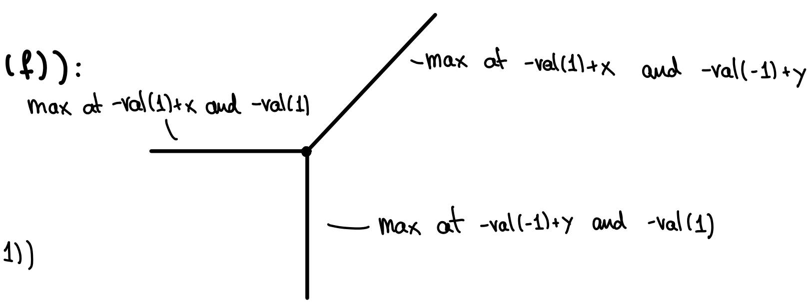

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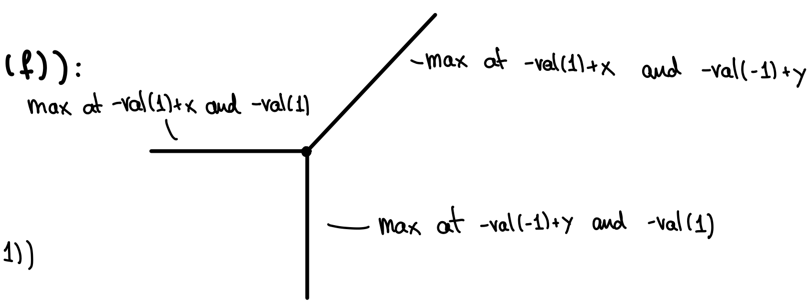

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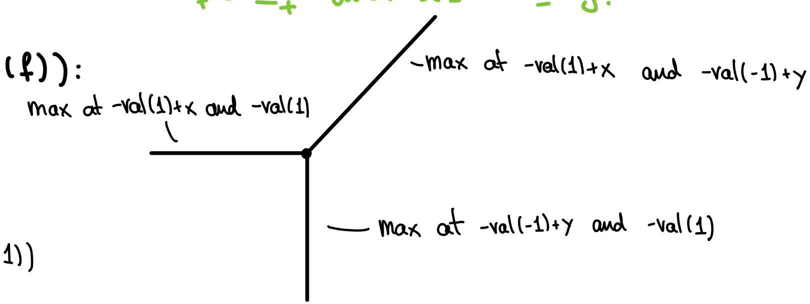

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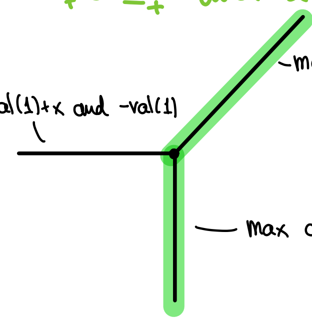

Determinantal Varieties

$$V(I_r) = \{ \tilde{A} \in \mathbb{C}^{d \times n} \mid \operatorname{rk} \tilde{A} \leq r \}, \quad I_r = \langle (r+1) \times (r+1) \text{-minors} \rangle$$

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Birkhoff polytope

special case of transportation polytope!

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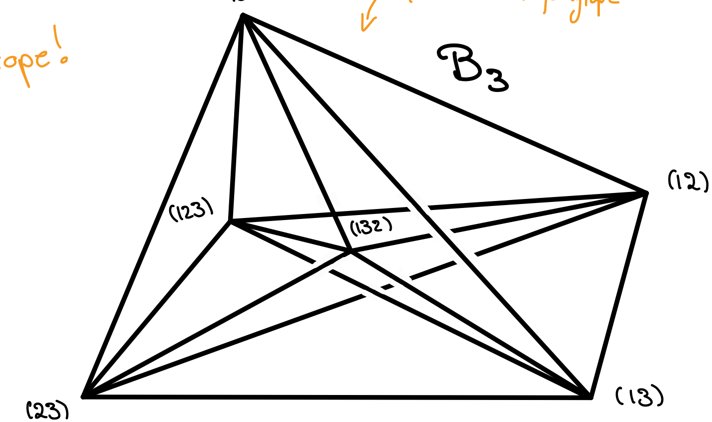

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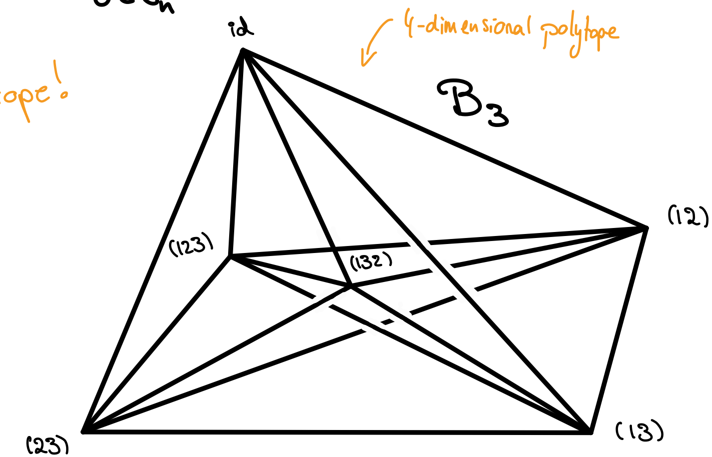

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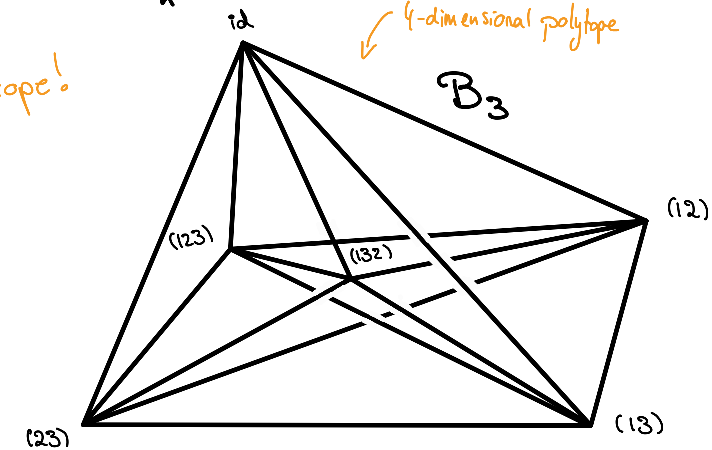

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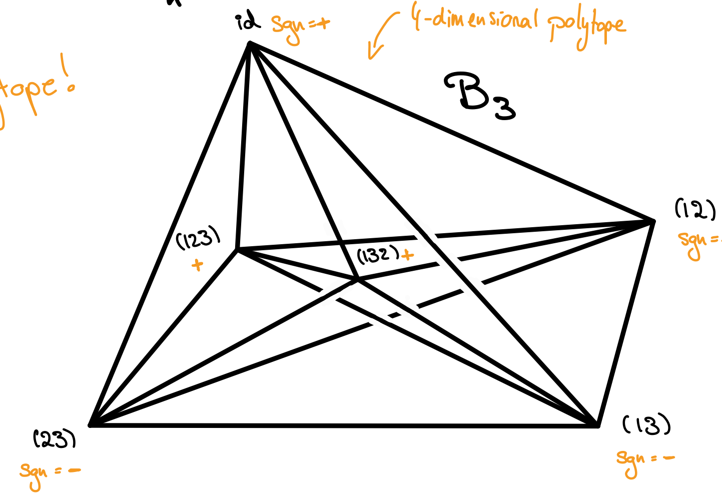

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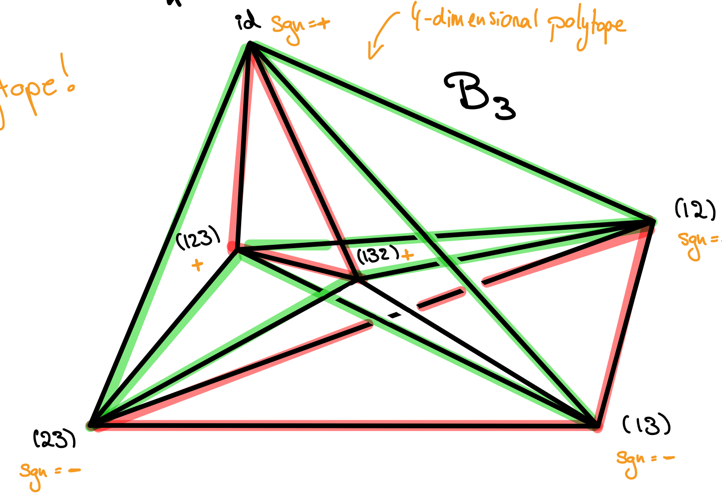

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Example: $d = n$, $r = n-1$

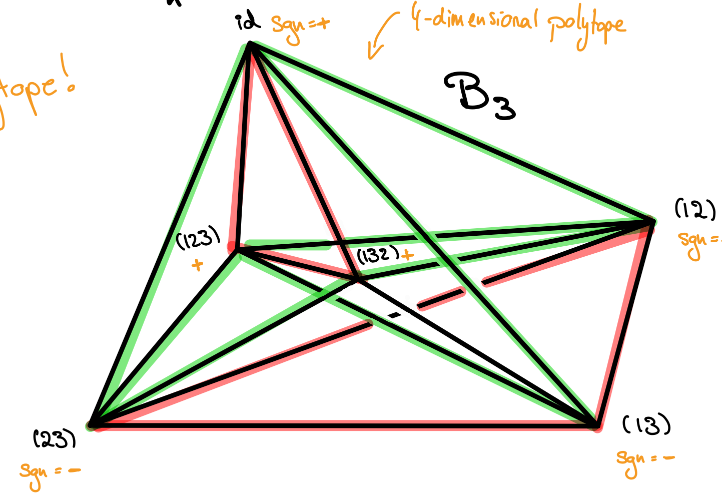
$V(I_{n-1}) = V(\langle \det \rangle) = \{ \tilde{A} \in \mathbb{C}^{n \times n} \mid \det(\tilde{A}) = 0 \}$ $\qquad \det \tilde{A} = \sum_{\sigma \in S_n} \operatorname{sgn} \sigma \prod_{i=1}^{n} \tilde{A}_{i\sigma(i)}$

Newton polytope of det:

Birkhoff polytope

special case of transportation polytope!

$B_n = \operatorname{conv}\big((n \times n)\text{-permutation matrices} \big) \subseteq \mathbb{R}^{n \times n}$



C maximal cone of $\operatorname{trop}^{+}(V(\det))$

$\Leftrightarrow$ C is normal cone of edge $\operatorname{conv}(\sigma, \pi)$ of B_n

and $\operatorname{sgn} \sigma \neq \operatorname{sgn} \pi$

non-positive edges: $\{ (\sigma, \pi) \mid \operatorname{sgn}(\sigma \pi) = 1 \} \simeq A_n$ alternating group

Point configurations

$\tilde{A} \in \mathbb{C}^{d \times n}$, $\operatorname{rk} \tilde{A} \le r \rightsquigarrow$ columns of $\tilde{A}$

$\simeq$ n points on r-dimensional linear space in $\mathbb{C}^{d-1}$

Point configurations

$\tilde{A} \in \mathbb{C}^{d \times n}$, $\operatorname{rk} \tilde{A} \leq r$ $\rightsquigarrow$ columns of $\tilde{A}$

$\simeq$ n points on $(r-1)$-dimensional linear space in $\mathbb{C}\mathbb{P}^{d-1}$

Point configurations

$\tilde{A} \in \mathbb{C}^{d \times n}$, $\operatorname{rk} \tilde{A} \le r$ $\rightsquigarrow$ columns of $\tilde{A}$

$\simeq$ n points on $(r-1)$-dimensional linear space in $\mathbb{CP}^{d-1}$

$A = \operatorname{val}(\tilde{A})$ $\rightsquigarrow$ columns of A

$\simeq$ n points on $(r-1)$-dimensional linear space in $\mathbb{TP}^{d-1}$

$(\mathbb{R} \cup \{\infty\})^n / \mathbb{R} \cdot (1, \dots, 1)$

Point configurations

$\tilde{A} \in \mathbb{C}^{d \times n}$, $\operatorname{rk} \tilde{A} \leq r \rightsquigarrow$ columns of $\tilde{A}$

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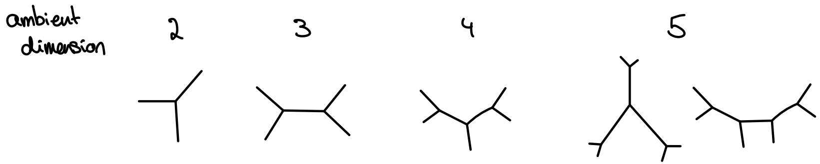
$A = \operatorname{val}(\tilde{A}) \rightsquigarrow$ columns of A

$\simeq n$ points on $(r-1)$-dimensional linear space in $\mathbb{TP}^{d-1}$

$(\mathbb{R} \cup \{\infty\})^n / \mathbb{R} + (1, \ldots, 1)$

Rank 2 & tropical lines



Point configurations

$\tilde{A} \in \mathbb{C}^{d\times n}$, $\mathrm{rk}\,\tilde{A} \le r \rightsquigarrow$ columns of $\tilde{A}$

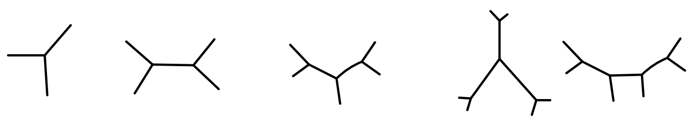
$\simeq n$ points on $(r-1)$-dimensional linear space in $\mathbb{CP}^{d-1}$

$A = \mathrm{val}(\tilde{A}) \rightsquigarrow$ columns of A

$\simeq n$ points on $(r-1)$-dimensional linear space in $\mathbb{TP}^{d-1}$ $\leftarrow (\mathbb{R}\cup\{\infty\})^n / \mathbb{R}+(1,\dots,1)$

Rank 2 & tropical lines

ambient dimension 2 3 4 5



Th [Ardila '04]: Let $A \in \mathrm{trop}(V(I_2))$. Then $A \in \mathrm{trop}^+(V(I_2))$

$\Leftrightarrow$ the points form a "consecutive chain" on the tropical line

positive

not positive

Point configucations

Rank 3

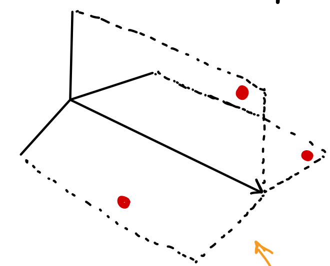
Recall: A tropical plane is a 2-dimensional polyhedral complex

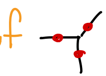
Point configurations

Rank 3

Recall: A tropical plane is a 2-dimensional polyhedral complex

Definition: A point configuration of 3 points forms a starship on a tropical plane if they lie on 3 distinct 2-dimensional faces, which intersect in an unbounded 1-dimensional face



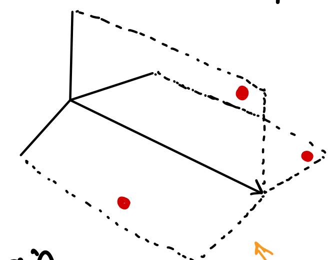
2d-analogue of 

Point configurations

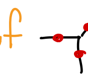
Rank 3

Recall: A tropical plane is a 2-dimensional polyhedral complex

Definition: A point configuration of 3 points forms a **starship** on a tropical plane if they lie on 3 distinct 2-dimensional faces, which intersect in an unbounded 1-dimensional face



2d-analogue of



Th [B-Loho-Sinn] "Starship criterion"

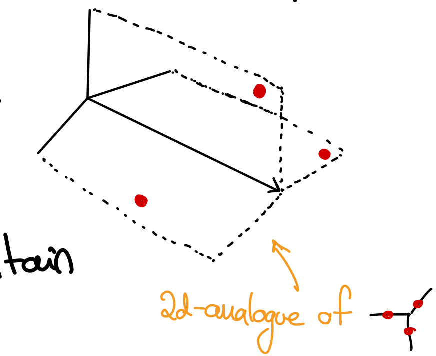
$A \in \operatorname{trop}^+(V(I_3)) \Rightarrow$ the point configuration does not contain a starship

Point configurations

Rank 3

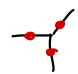
Recall: A tropical plane is a 2-dimensional polyhedral complex

Definition: A point configuration of 3 points forms a starship on a tropical plane if they lie on 3 distinct 2-dimensional faces, which intersect in an unbounded 1-dimensional face



Th [B-Loho-Sinn] "Starship criterion"

$A \in \operatorname{trop}^+(V(I_3)) \Rightarrow$ the point configuration does not contain a starship

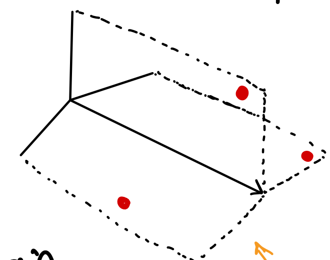
Rank ≥ 4: Higher dimensional analogues of  can occur.

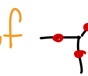
Point configurations

Rank 3

Recall: A tropical plane is a 2-dimensional polyhedral complex

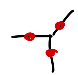
Definition: A point configuration of 3 points forms a starship on a tropical plane if they lie on 3 distinct 2-dimensional faces, which intersect in an unbounded 1-dimensional face

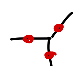


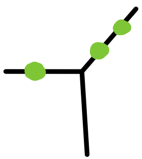
2d-analogue of 

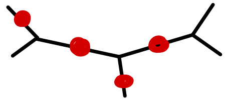
Th [B-Loho-Sinn] "Starship criterion"

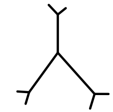
$A \in \operatorname{trop}^+(V(I_3)) \implies$ the point configuration does not contain a starship

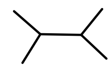
Rank ≥ 4: Higher dimensional analogues of  can occur.

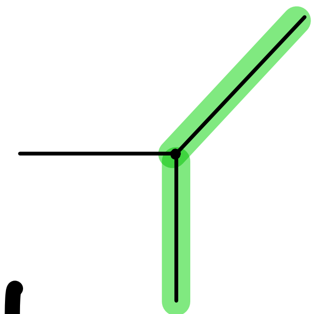
Summary:
- Rank 2: positive $\Rightarrow$ no 
- Rank 3: positive $\Rightarrow$ no starship
- Rank ≥ 4: Everything can happen.











Thank you!