# HOW TO SLICE A POLYTOPE

joint work with Chiara Meroni and Jesús A. De Loera.
arXiv: 2304.14239

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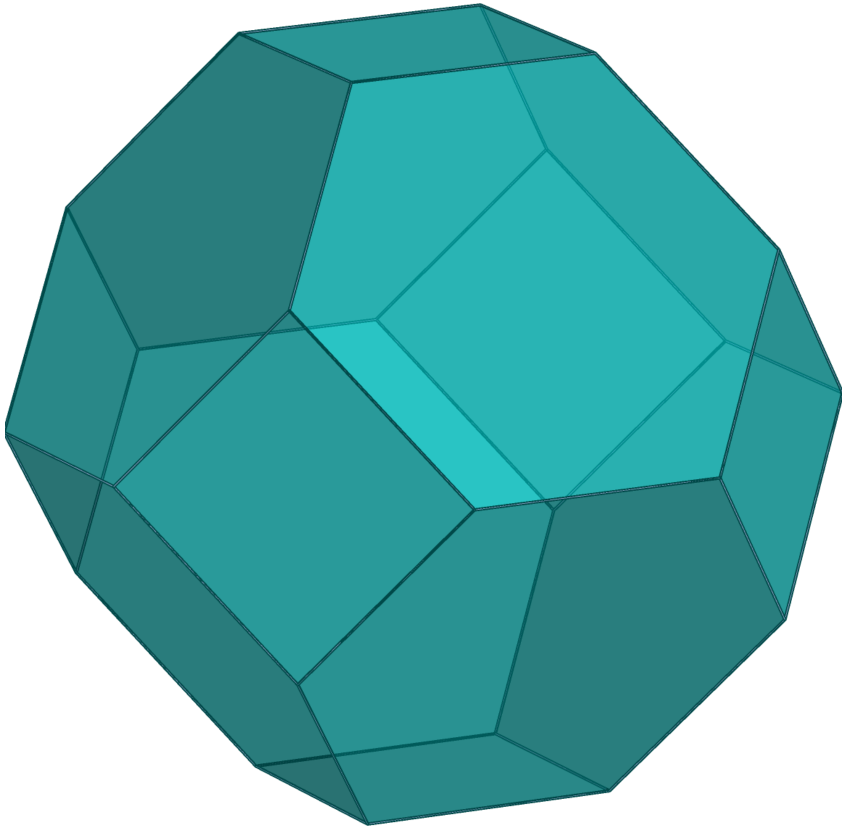
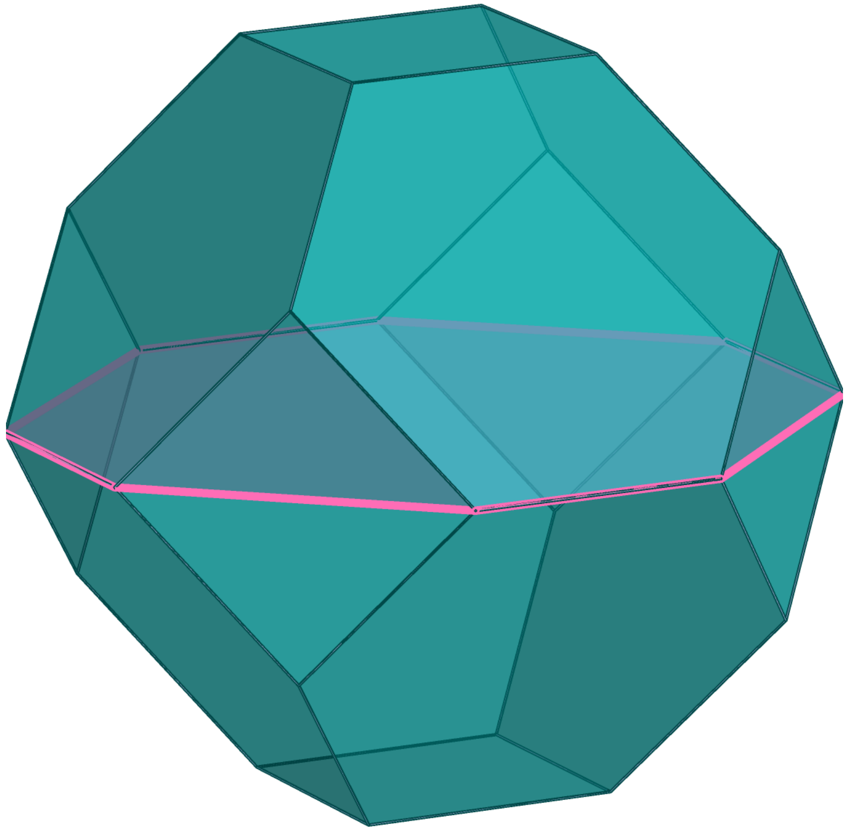
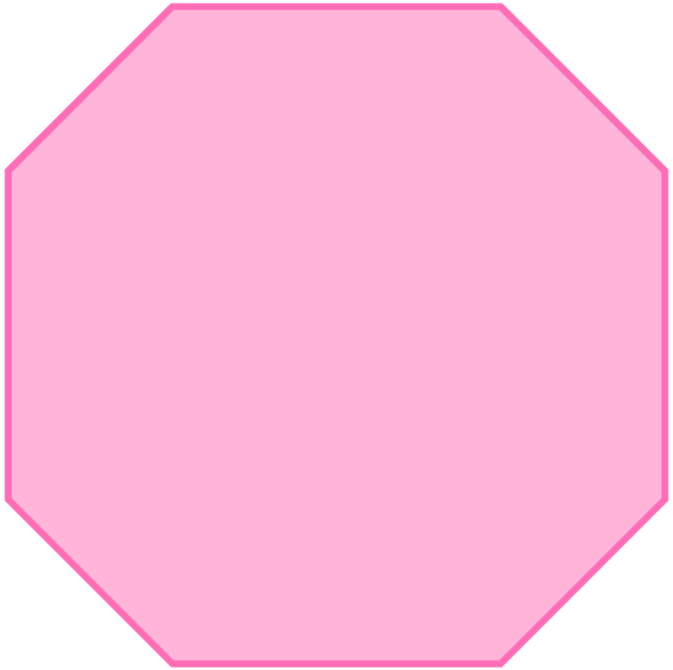
# Slices of the Permutohedron



$$P = \text{conv}(\ (\sigma(1), \sigma(2), \sigma(3), \sigma(4))\ \mid \sigma \in S_4) - \tfrac{3}{2}(1,1,1,1)$$
$$= \text{conv}(\ (1,2,3,4),\ (1,2,4,3),\ \ldots,\ (4,3,2,1)\ ) - \tfrac{3}{2}(1,1,1,1)$$

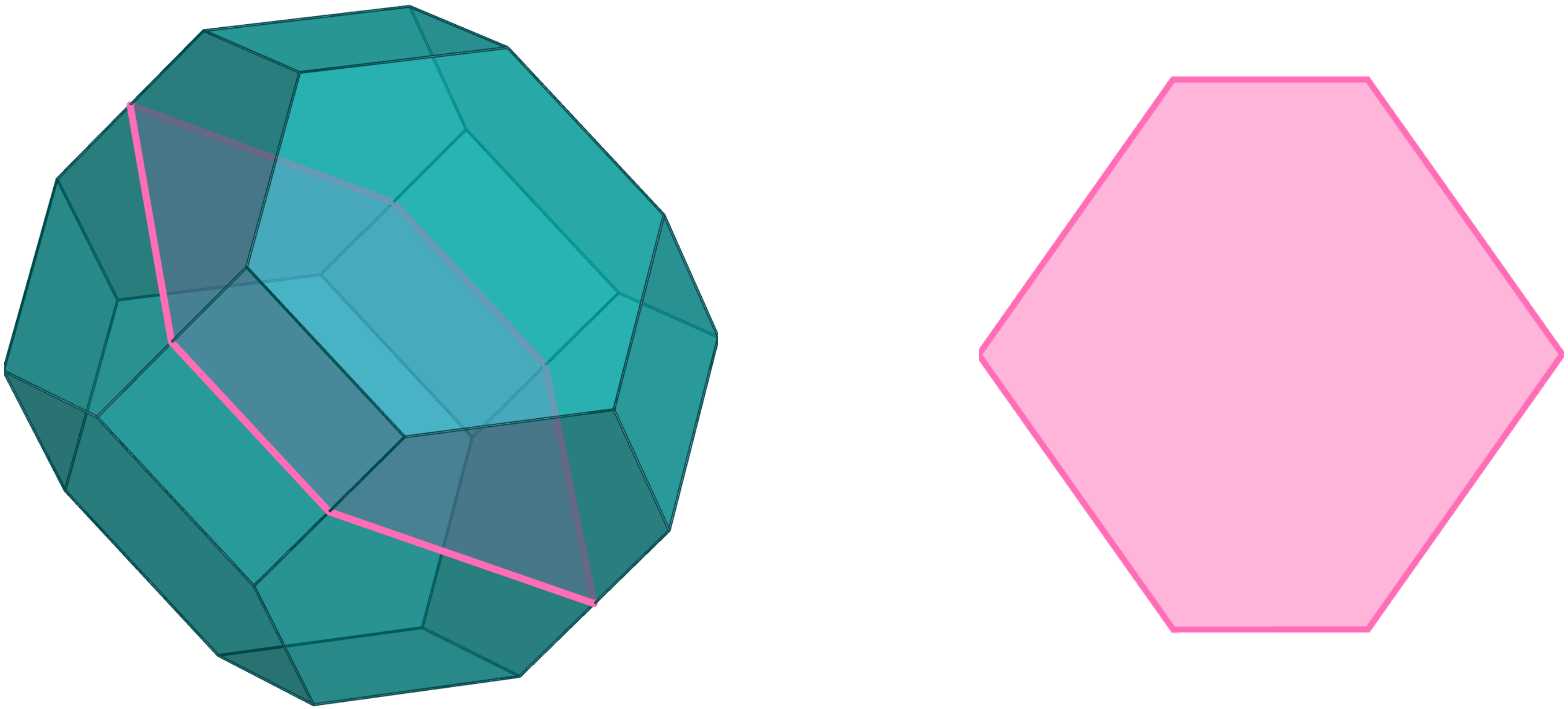
# Slices of the Permutohedron





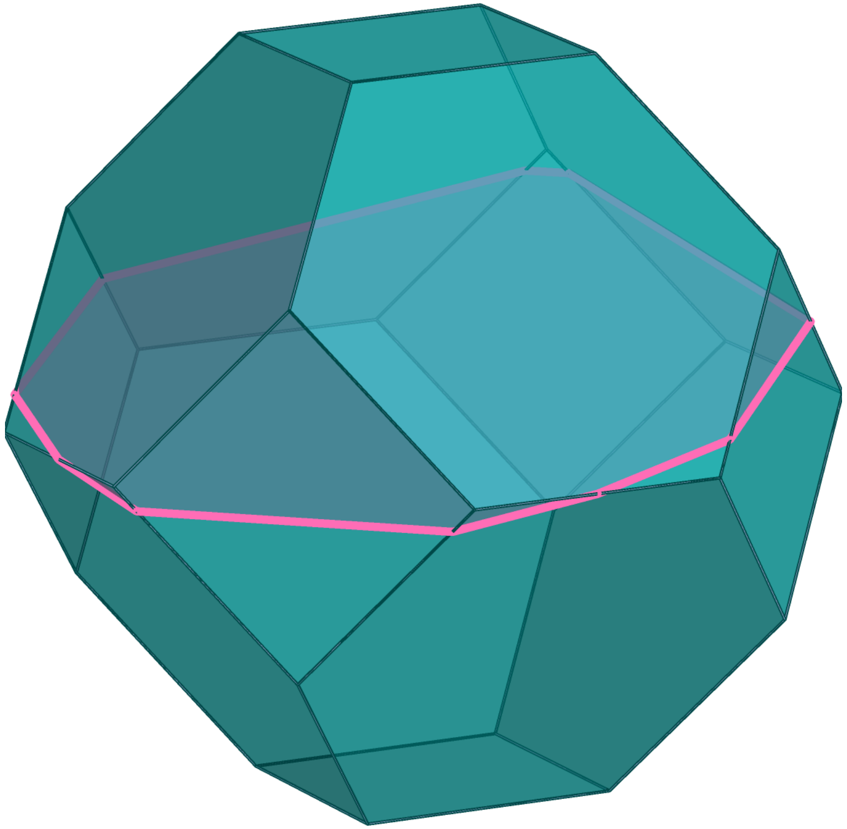
Affine slice of maximum volume

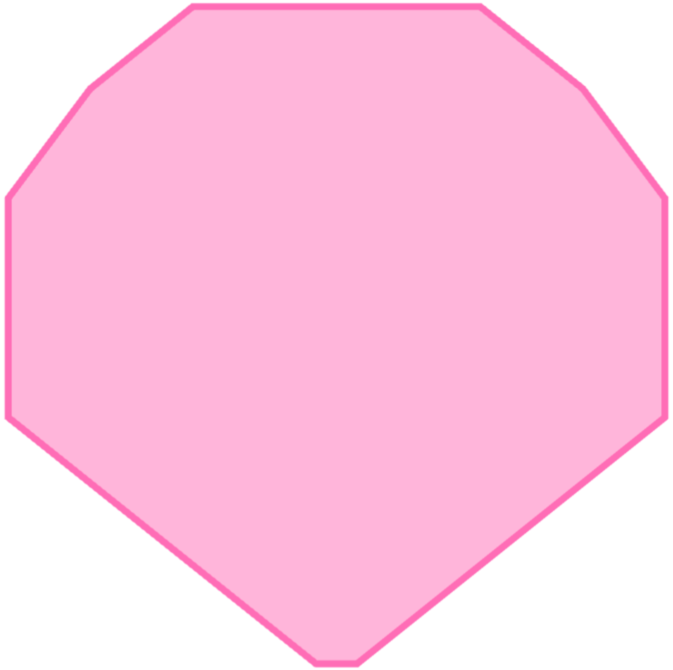
# Slices of the Permutohedron



Central slice of minimum volume

# Slices of the Permutohedron





Affine slice with maximum number of vertices

# Joint work with


**Chiara Meroni**

Harvard

**Jesús A. De Loera**

UC Davis

MB, Chiara Meroni, and Jesús A. De Loera. *The Best Ways to Slice a Polytope*. 2023. arXiv: 2304.14239

# WHO WANTS TO COMPUTE (EXTREMAL) SLICES OF POLYTOPES?

# Motivation

- **Maximal volume slice**: What is the slice of $P$ with maximal volume?
  [Ball '89, Meyer-Pajor '88, Webb '96, Pournin '22, …]

- **Bourgain's slicing problem**: Does there exist $c > 0$ such that for any convex body $K \subset \mathbb{R}^d$ with vol$(K) = 1$ there exist a hyperplane $H$ such that vol$(K \cap H) > c$?
  [Bourgain '84, Klartag-Lehec '22, Klartag '23,…]

- How does the **$h$-vector** of $P$ compare with the $h$-vector of a generic hyperplane section of $P$?
  [Khovanskii '06]

- **Densest hemisphere problem**: Given points on the sphere, how can we find the hemisphere with the most points?
  [Johnson-Preparata '78,…]

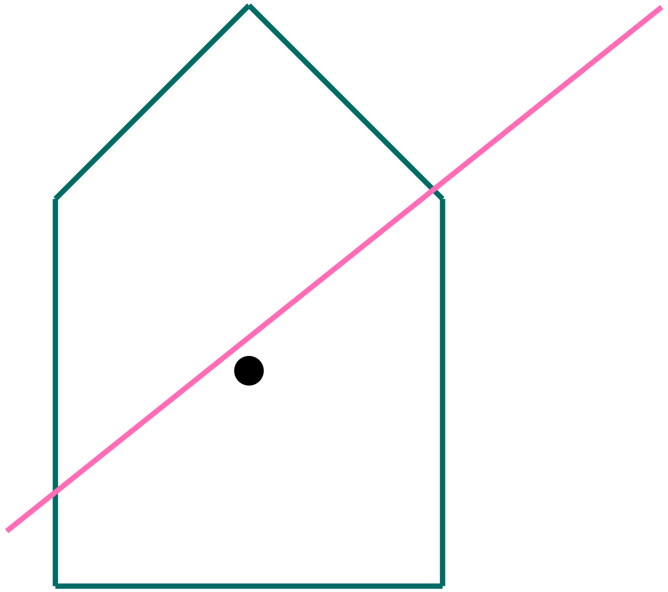
- Volumes of **slices of the permutahedron** fixed by action of a permutation
  [Ardila-Schindler-(Vindas-Meléndez) '21,…]

## HOW CAN WE COMPUTE THESE “EXTREMAL” SLICES?

# 2 Approaches

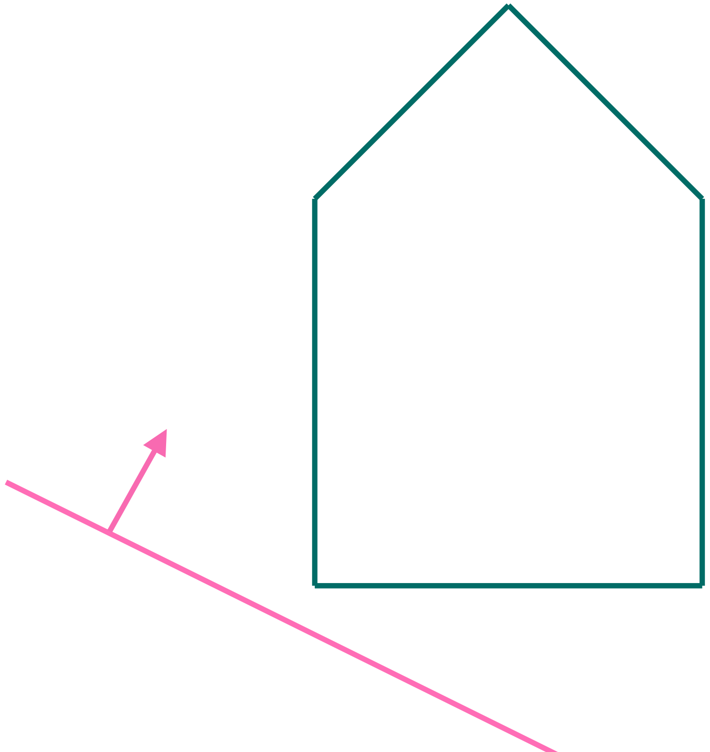
## Rotational Approach

1. Choose a position of the origin
2. Consider all hyperplanes through the origin



## Translational Approach

1. Choose a normal direction
2. Consider all affine translates of the orthogonal hyperplane



# 2 Approaches

## Key Observations:

- $H$ generic hyperplane $\Longrightarrow$ vertices of $P \cap H$ = intersections of $H$ with edges of $P$
- $H, H'$ intersect the same set of edges of $P$
  $\Longrightarrow P \cap H, P \cap H'$ have the same combinatorial type

## Main Idea for both approaches:

Collect all hyperplanes which intersect $P$ in the same set of edges

$\longrightarrow$ regions of hyperplane arrangements

| | **Hyperplane Arrangement** | **Notation** | **Reference Object** |
|---|---|---|---|
| $\circlearrowleft$ | central arrangement | $\mathcal{C}_{\circlearrowleft}$ | intersection body |
| | cocircuit arrangement | $\mathcal{R}_{\circlearrowleft}$ | oriented matroid |
| $\uparrow$ | parallel arrangement | $\mathcal{C}^{\mathbf{u}}_{\uparrow}$ | fiber polytope |
| | sweep arrangement | $\mathcal{R}_{\uparrow}$ | sweep polytope |

# ROTATIONAL APPROACH

# Rotational Approach

Fix the position of the origin.

$u^\perp$ = central hyperplane orthogonal to $u$

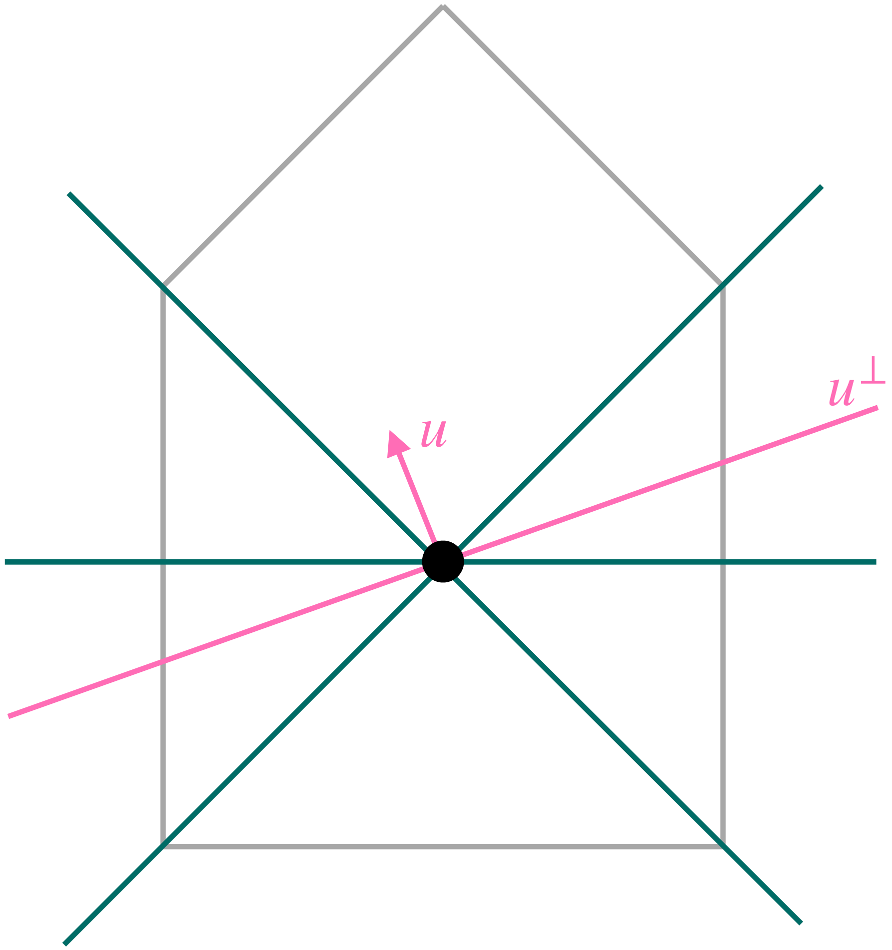
Consider the central hyperplane arrangement

$$\mathscr{C}_{\circlearrowleft}(P) = \{v^\perp \mid v \text{ is a vertex of } P\}.$$

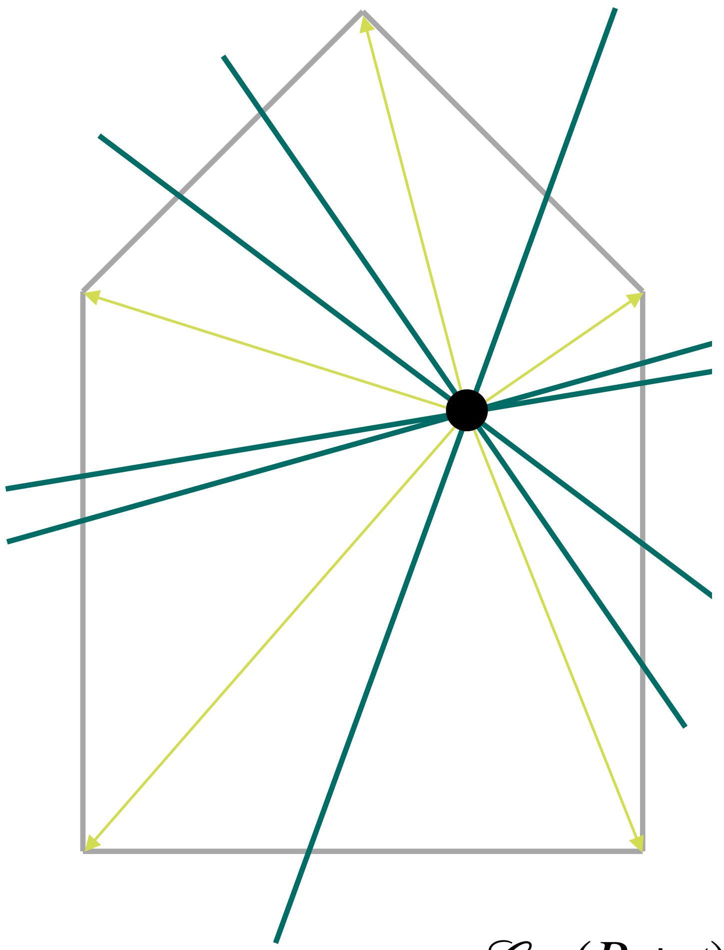
$\longrightarrow$ The combinatorial type of $P \cap u^\perp$ is constant in each cell of $\mathscr{C}_{\circlearrowleft}(P)$.

We refer to the maximal cells of $\mathscr{C}_{\circlearrowleft}(P)$ as chambers.

*What happens if we translate $P$, i.e. vary the position of the origin?*



# Rotational Approach



$$\mathscr{C}_{\circlearrowleft}(P + t) = \{(v + t)^{\perp} \mid v \text{ is a vertex of } P\}$$

# Rotational Approach

Translation $P + t \longleftrightarrow$ rotation of hyperplanes $(v + t)^{\perp}$ in central arrangement $\mathscr{C}_{\circlearrowleft}(P + t)$

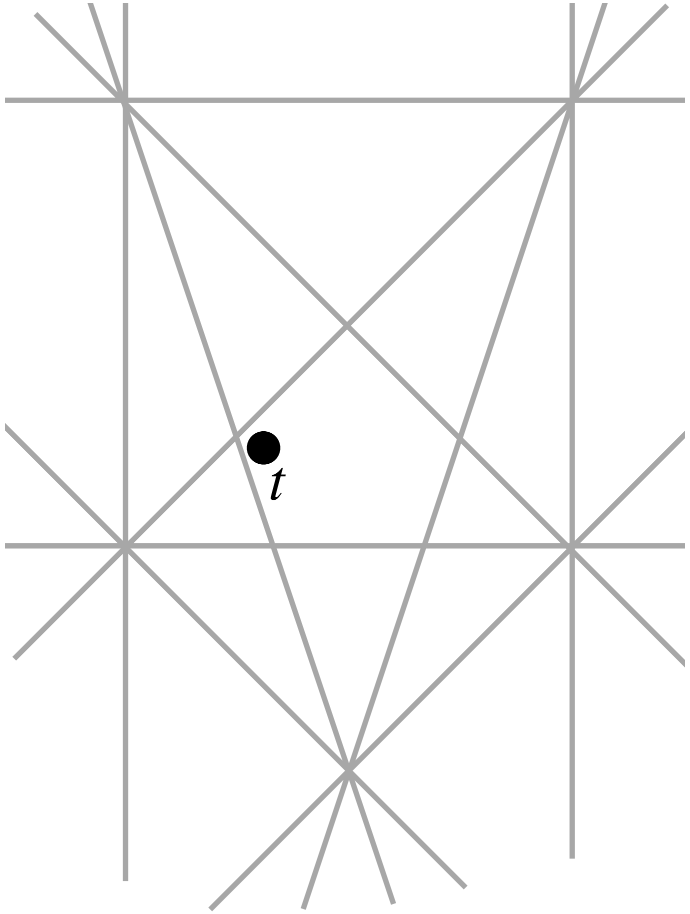
*For which $t \in \mathbb{R}^d$ does $\mathscr{C}_{\circlearrowleft}(P + t)$ have the same combinatorics?*
*(i.e. the same oriented matroid)*

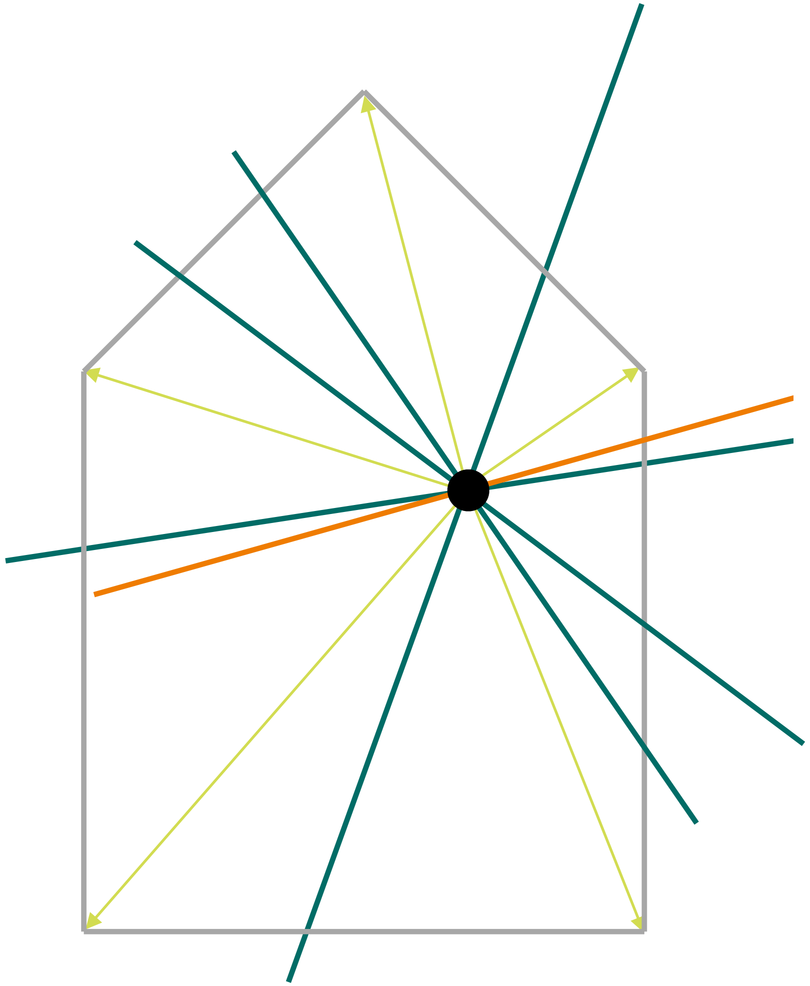
Consider the affine hyperplane arrangement (called cocircuit arrangement)

$$\mathscr{R}_{\circlearrowleft}(P) = \{\text{aff}(-v_1, \ldots, -v_d) \mid v_k \text{ are vertices of } P\}$$

$\longrightarrow$ within each region of $\mathscr{R}_{\circlearrowleft}(P)$ the combinatorics of $\mathscr{C}_{\circlearrowleft}(P + t)$ are fixed

# Rotational Approach



$\mathscr{R}_{\circlearrowleft}(P) = \{\text{aff}(-v_1, \ldots, -v_d) \mid v_k \text{ are vertices of } P\}$ $\mathscr{C}_{\circlearrowleft}(P+t) = \{(v+t)^{\perp} \mid v \text{ is a vertex of } P\}$

# Rotational Approach

## Theorem (B.-Meroni-De Loera '23):

Let $P \subseteq \mathbb{R}^d$ be a polytope, and $f(x) = \sum_\alpha c_\alpha x^\alpha$ be a polynomial in variables $x_1, \ldots, x_d$.
Fix a region $R \in \mathscr{R}_{\circlearrowleft}(P)$ of the cocircuit arrangement, a translation $t \in R$ and a chamber $C(t) \in \mathscr{C}_{\circlearrowleft}(P + t)$ of the central arrangement.

Restricted to $t \in R$ and $u \in C(t) \cap S^{d-1}$, the integral

$$\int_{(P+t)\cap u^\perp} f(x)\ \mathrm{d}x$$

is a rational function in variables $t_1, \ldots, t_d, u_1, \ldots, u_d$.
(and we have an algorithm to compute it)

## Note:

If $f(x) = 1$ then $\int_{(P+t)\cap u^\perp} f(x)\ \mathrm{d}x = \mathrm{vol}((P + t) \cap u^\perp)$.

# Proof Idea

1. Fix region $R \in \mathscr{R}_{\circlearrowleft}(P)$, chamber $C(t) \in \mathscr{C}_{\circlearrowleft}(P+t)$

   Let $Q(t,u) = (P+t) \cap u^{\perp}$ for $t \in R$, $u \in C(t)$

2. We can choose a fixed triangulation of $Q(t,u)$ for all $t \in R, u \in C(t)$

3. Coordinates of vertices of $Q(t,u)$ are rational functions in $u_1, \ldots, u_d, t_1, \ldots, t_d$

4. The volume of a simplex $\Delta$ in the triangulation is a determinant (in terms of vertices of $\Delta$)

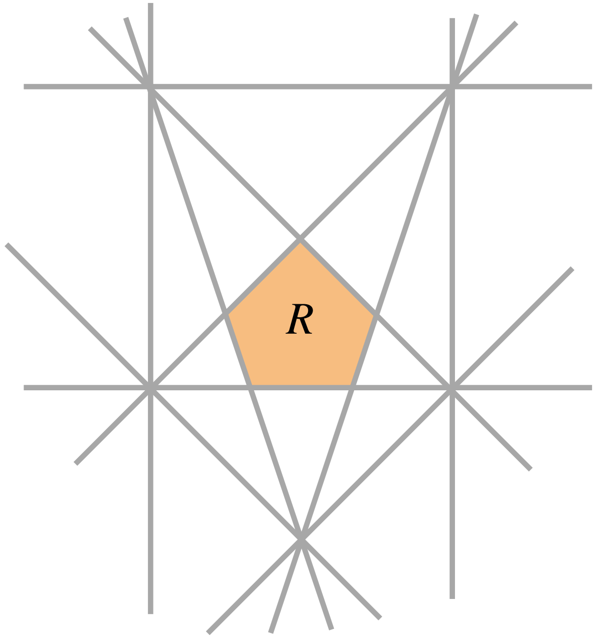
   $\Longrightarrow$ vol$(\Delta)$ is a rational function in $u_1, \ldots, u_d, t_1, \ldots, t_d$

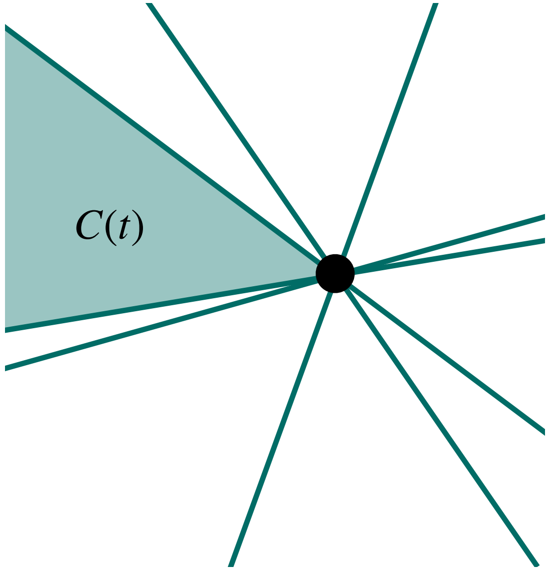
5. There is a formula for $\int_{\Delta}$ (linear form) in terms of vol$(\Delta)$ [Lasserre-Avrachenkov ’01, Baldoni-Berline-DeLoera-Köppe-Vergne ’11]

6. A Waring decomposition of $f(x)$ is a decomposition into sums of powers of linear forms

$\Longrightarrow$ formula for computing $\int_{Q(u,t)} f(x)$ in terms of $u_1, \ldots, u_d, t_1, \ldots, t_d$

# Rotational Approach




Let the computer find the biggest slice:

| | |
|---|---|
| maximize | $\dfrac{-(t_1u_1+t_2u_2+u_1-u_2)}{u_1(u_1-u_2)}$ |
| s.t | $(t_1,t_2)\in R$ |
| | $(u_1,u_2)\in C(t)\cap S^{d-1}$ |

$(t_1,t_2)\in R \iff$

$$-t_1-t_2\geq 0,\qquad t_1-t_2\geq 0$$
$$-3t_1+t_2\geq -2,\ 3t_1+t_2\geq -2$$
$$t_2\geq -1$$

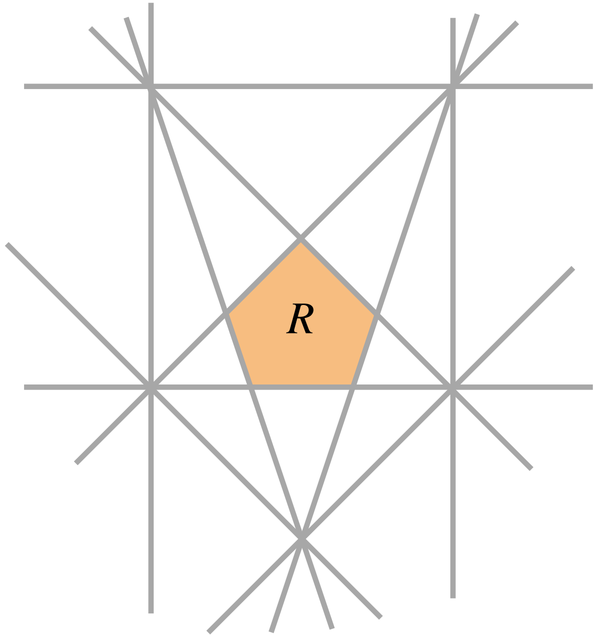
If $(t_1,t_2)\in R$ then $(u_1,u_2)\in C(t) \iff$

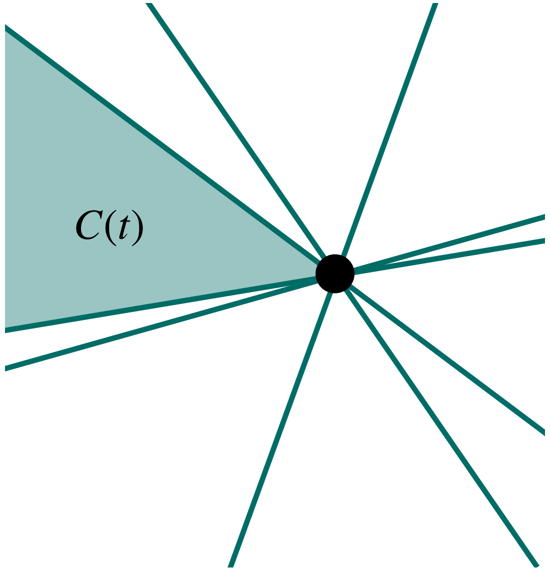
$$2u_2+t_1u_1+t_2u_2\geq 0$$
$$-u_1-u_2+t_1u_1+t_2u_2\geq 0$$

If $t\in R$ and $u\in C(t)\cap S^{d-1}$ then

$$\mathrm{vol}((P+t)\cap u^{\perp}) = \int_{(P+t)\cap u^{\perp}} 1\mathrm{d}x = \frac{-(t_1u_1+t_2u_2+3u_1-u_2)}{u_1(u_1-u_2)}$$

# Rotational Approach




$(t_1, t_2) \in R \iff$
$-t_1 - t_2 \geq 0, \quad t_1 - t_2 \geq 0$
$-3t_1 + t_2 \geq -2, \; 3t_1 + t_2 \geq -2$
$t_2 \geq -1$

*If* $(t_1, t_2) \in R$ *then* $(u_1, u_2) \in C(t) \iff$
$2u_2 + t_1u_1 + t_2u_2 \geq 0$
$-u_1 - u_2 + t_1u_1 + t_2u_2 \geq 0$

If $t \in R$ and $u \in C(t) \cap S^{d-1}$ then

$$\mathrm{vol}((P+t) \cap u^{\perp}) = \int_{(P+t)\cap u^{\perp}} 1 \mathrm{d}x = \frac{-(t_1u_1 + t_2u_2 + 3u_1 - u_2)}{u_1(u_1 - u_2)}$$

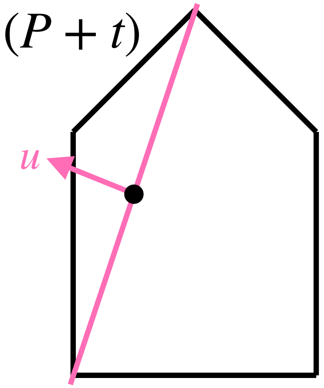
Let the computer find the biggest slice:

maximize $\dfrac{-(t_1u_1 + t_2u_2 + u_1 - u_2)}{u_1(u_1 - u_2)}$

s.t
$-t_1 - t_2 \geq 0,$
$t_1 - t_2 \geq 0$
$-3t_1 + t_2 \geq -2,$
$3t_1 + t_2 \geq -2$
$t_2 \geq -1$

$2u_2 + t_1u_1 + t_2u_2 \geq 0$
$-u_1 - u_2 + t_1u_1 + t_2u_2 \geq 0$
$u_1^2 + u_2^2 + u_3^2 = 1$

Compute this for all regions $R \in \mathscr{R}_{\circlearrowleft}(P)$ and chambers $C(t) \in \mathscr{C}_{\circlearrowleft}(P+t)$
$\longrightarrow$ largest slice!



# TRANSLATIONAL APPROACH

# Translational Approach

Fix a normal direction $u \in S^{d-1}$

$H(\beta) = \{x \in \mathbb{R}^d \mid \langle x, u \rangle = \beta\}$

hyperplane parallel to $u^\perp$

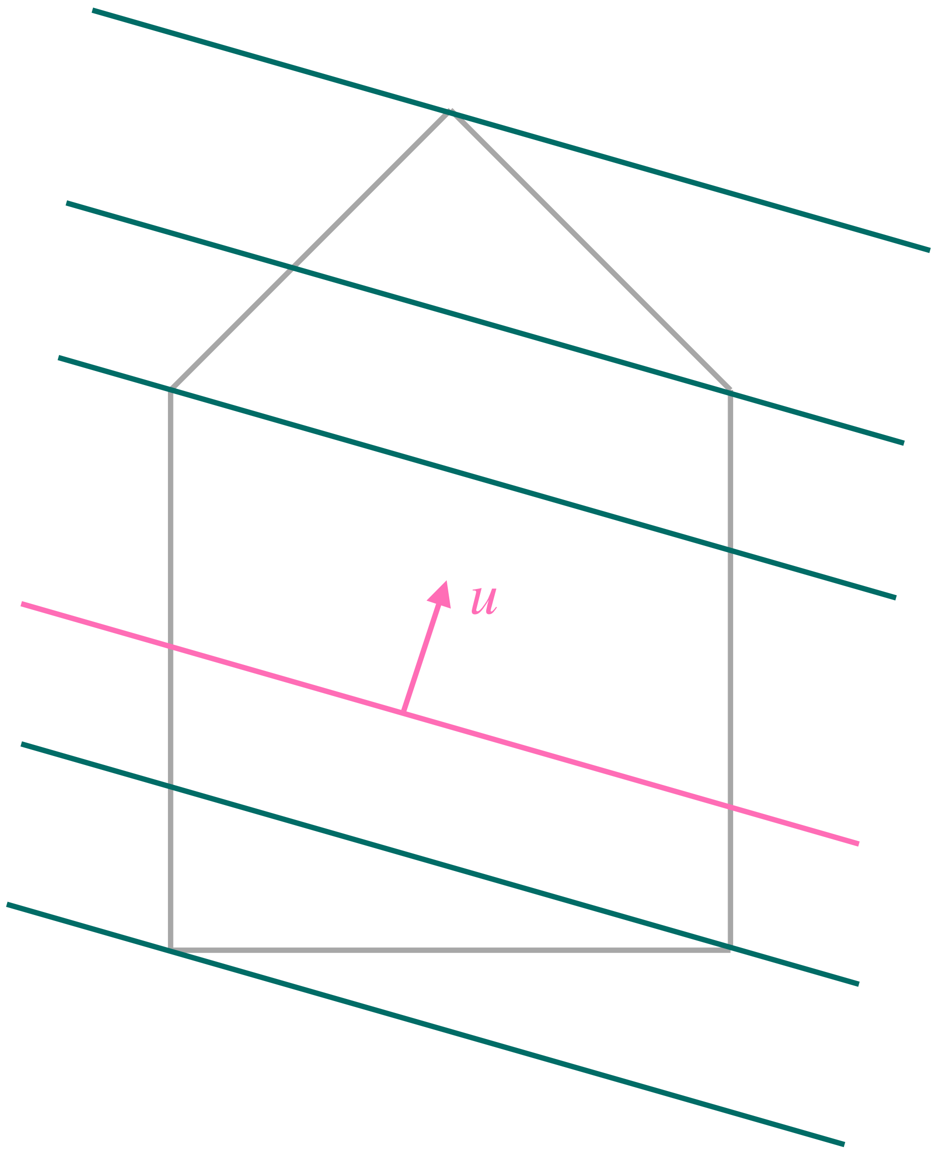
Consider the parallel hyperplane arrangement

$$\mathscr{C}^u_\uparrow(P) = \{H(\langle v, u \rangle \mid v \text{ is a vertex of } P\}.$$

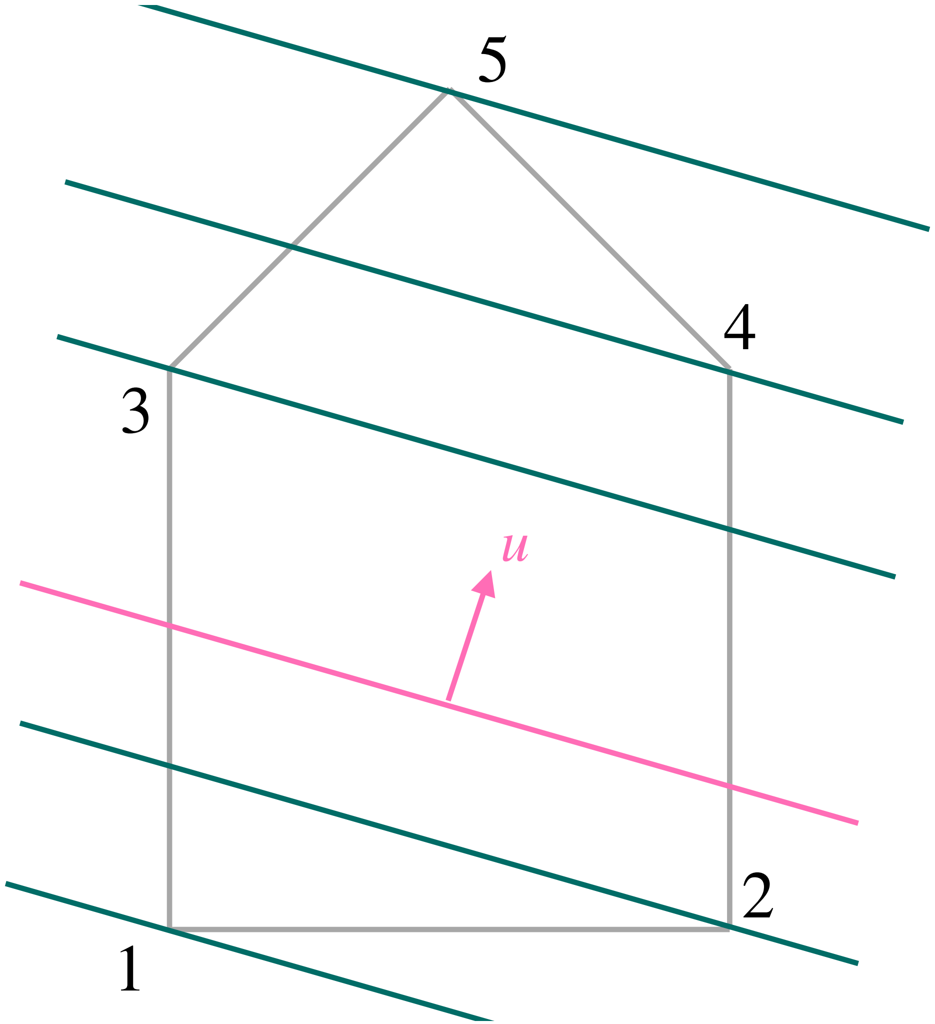
$\longrightarrow$ The combinatorial type of $P \cap H(\beta)$ is constant in each cell of $\mathscr{C}^u_\uparrow(P)$.

We refer to the maximal cells of $\mathscr{C}^u_\uparrow$ as chambers.

*What happens if we vary the direction $u \in S^{d-1}$?*



# Translational Approach




$$\mathscr{C}_{\uparrow}^{u}(P) = \{H(\langle v, u\rangle \mid v \text{ is a vertex of } P\}$$

# Translational Approach

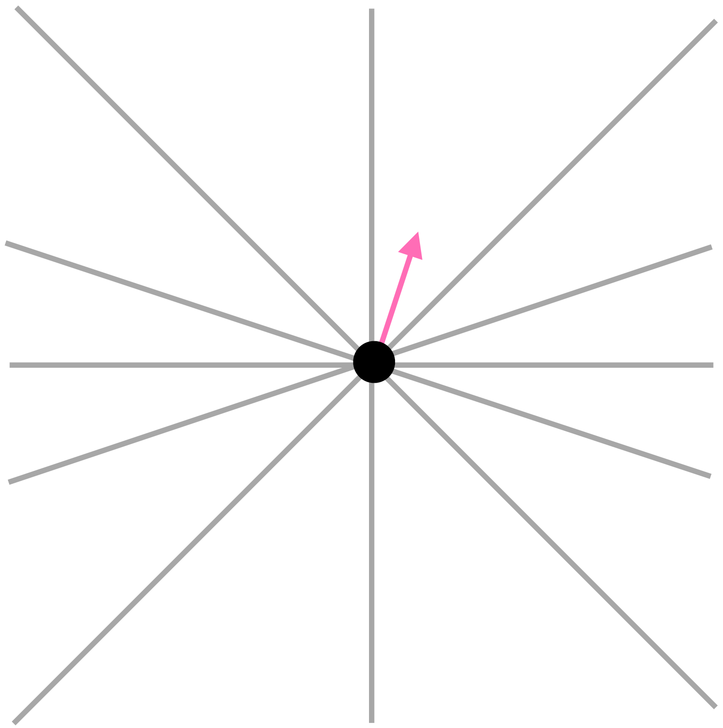
*For which $u \in S^{d-1}$ does $\mathscr{C}^u_\uparrow(P)$ induce the same ordering of the vertices?*

Consider the sweep arrangement

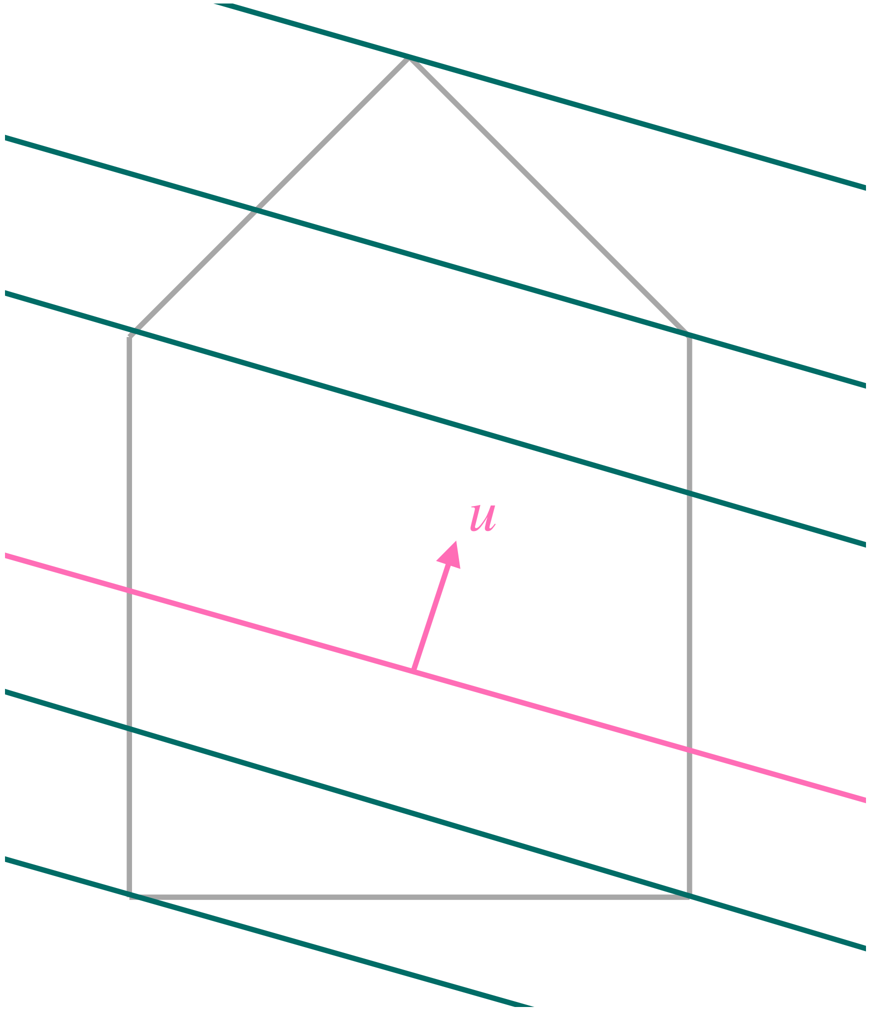
$$\mathscr{R}_\uparrow(P) = \{(v_i - v_j)^\perp \mid v_i, v_j \text{ are vertices of } P\}$$

$\longrightarrow$ with each region of $\mathscr{R}_\uparrow(P)$ the induced ordering given by $\mathscr{C}^u_\uparrow(P)$ is fixed

# Translational Approach



$\mathscr{R}_{\uparrow}(P) = \{(v_i - v_j)^{\perp} \mid v_i, v_j \text{ are vertices of } P\}$




$\mathscr{C}^{u}_{\uparrow}(P) = \{H(\langle v, u \rangle \mid v \text{ is a vertex of } P\}$

# Translational Approach

**Theorem (B.-Meroni-De Loera ’23):**

Let $P \subseteq \mathbb{R}^d$ be a polytope and $f(x) = \sum_\alpha c_\alpha x^\alpha$ be a polynomial in variables $x_1, \ldots, x_d$. Fix a region $R \in \mathscr{R}_\uparrow(P)$ of the sweep arrangement, a unit direction $u \in R \cap S^{d-1}$ and a chamber $C(u) \in \mathscr{C}^u_\uparrow(P)$ of the parallel arrangement.

Restricted to $u \in R \cap S^{d-1}$ and $H(\beta) \in C(u)$, the integral

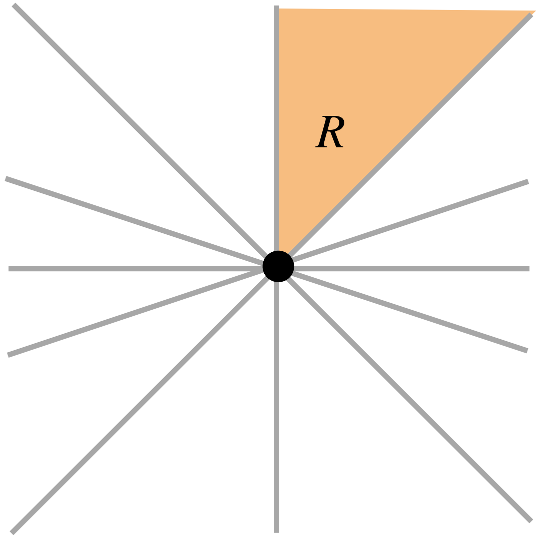
$$\int_{P \cap H(\beta)} f(x) \, \mathrm{d}x$$

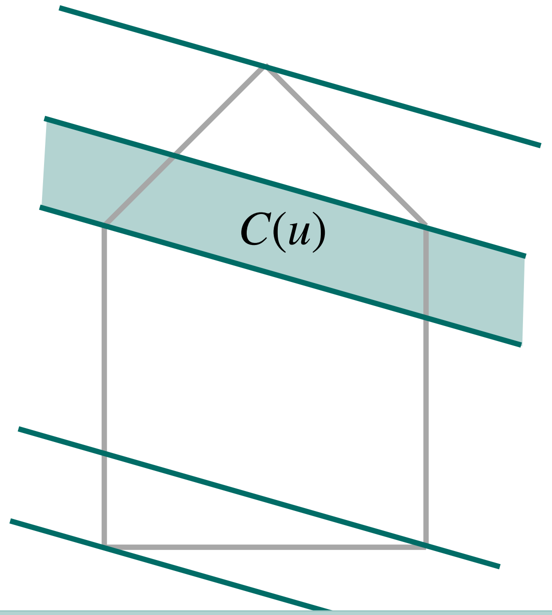
is a rational function in variables $u_1, \ldots, u_d, \beta$.

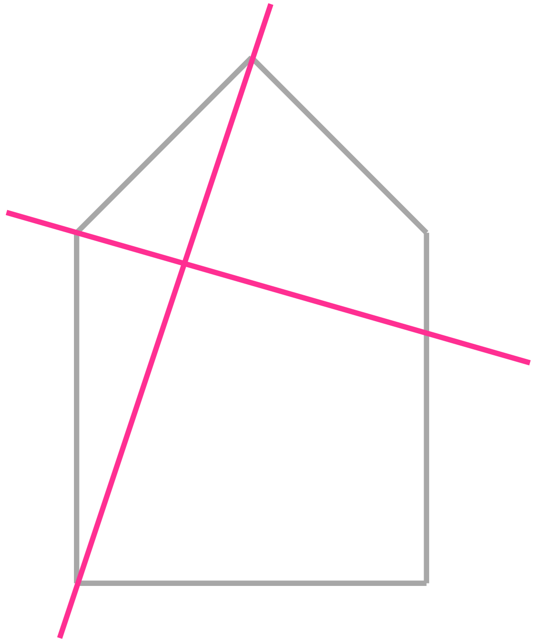
# Translational Approach

Let the computer find the biggest slice:

| | |
|---|---|
| maximize | $\dfrac{-(\beta - u_1 - 3u_2)}{u_2(u_1 + u_2)}$ |
| s.t | $(u_1, u_2) \in R \cap S^{d-1}$ |
| | $H(\beta) \in C(u)$ |



$(u_1, u_2) \in R \iff u_1 \geq 0,\ u_1 - u_2 \leq 0$

If $(u_1, u_2) \in R \cap S^{d-1}$ then $\beta \in C(u) \iff u_1 - u_2 \leq \beta \leq -u_1 + u_2$

If $u \in R \cap S^{d-1}$ and $H(\beta) \in C(u)$ then

$$\text{vol}((P + t) \cap u^{\perp}) = \frac{-(\beta - u_1 - 3u_2)}{u_2(u_1 + u_2)}$$

# ROTATION VS TRANSLATION

# COMPARISON OF THE APPROACHES

# Comparison

Running time of the algorithm $\longleftrightarrow$ number of chambers in the arrangements

$n =$ # vertices of $P$

**Rotational Approach**

| Arrangement | #Chambers (incl. lower-dim. cells) |
| --- | --- |
| $\mathscr{C}_{\circlearrowleft}(P)$ | $O(n^d 2^d)$ |
| $\mathscr{R}_{\circlearrowleft}(P)$ | $O(n^{d^2} 2^d)$ |
| Total | $O(n^{d^2+d} 2^d)$ |

**Translational Approach**

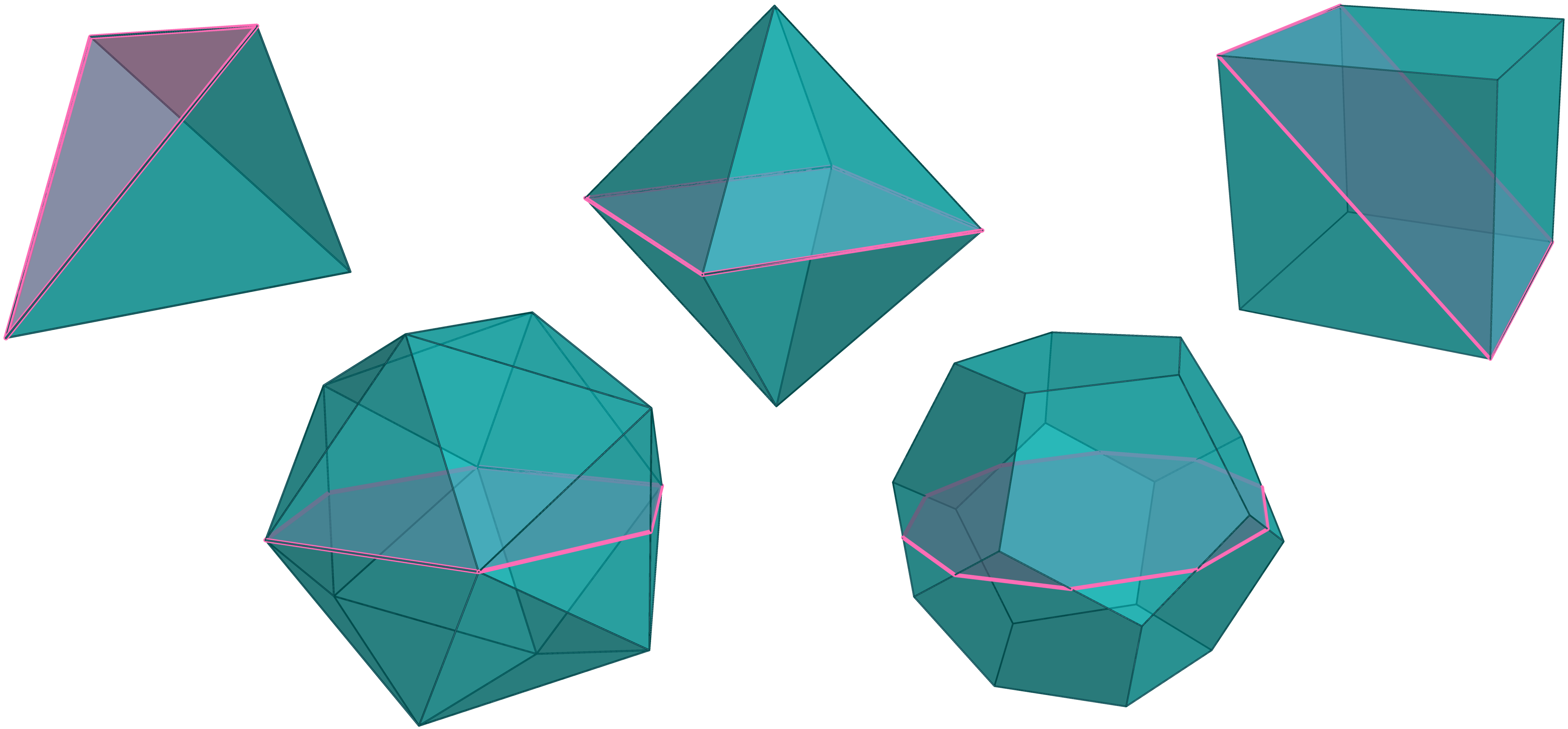
| Arrangement | #Chambers (incl. lower-dim.) |
| --- | --- |
| $\mathscr{C}_{\uparrow}(P)$ | $O(n)$ |
| $\mathscr{R}_{\uparrow}(P)$ | $O(n^{2d} 2^d)$ |
| Total | $O(n^{2d+1} 2^d)$ |

If $d \in \mathbb{N}$ is fixed then all of these are polynomials in $n$

$\longrightarrow$ both approaches yield algorithms in polynomial running time

$\longrightarrow$ Translational approach runs much faster

# Maximum Volume Slices of Platonic solids



# VARIATIONS

# What Else?

## With the same methods we can understand...

- Intersections with half-spaces
- Projections onto hyperplanes
- Combinatorial types

## We can optimize for...

- volume
- Integral of a polynomial
- (Weighted) Number of $k$-dimensional faces

## Theorem (B.-De Loera-Meroni)

- We can compute all of these in polynomial time in fixed dimension

non-fixed dimension:

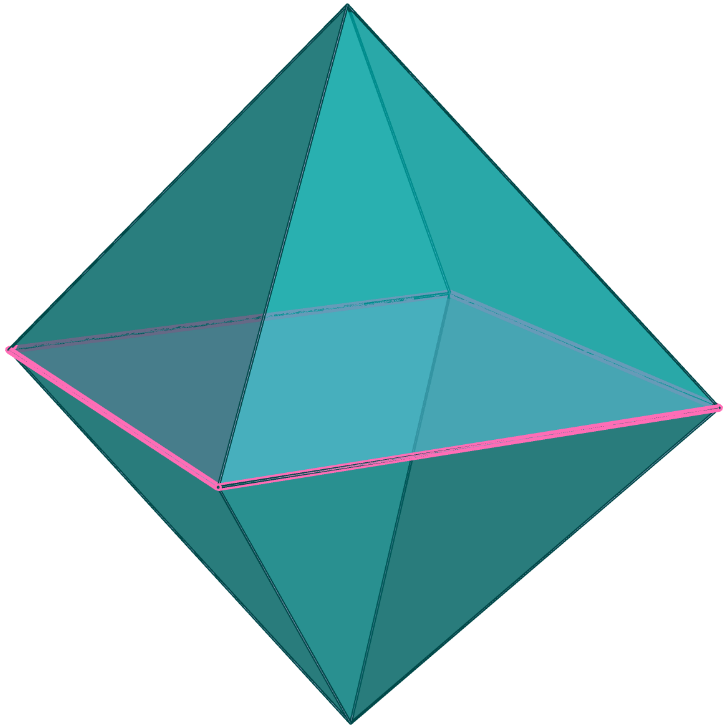
- It is #P-hard to compute the volume of $P \cap H$ with largest volume
- It is NP-hard to compute $P \cap H$ that maximizes the sum of weights of edges it intersects
- It is NP-hard to find the central halfspace $H_0^+$ that contains the most vertices of $P$, if $P$ is inscribed

## Conjecture

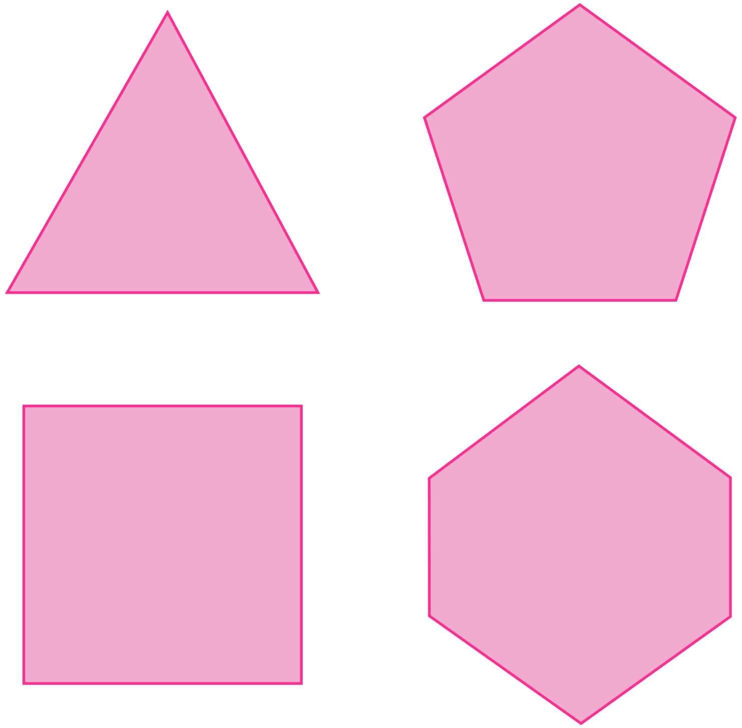
- It is NP-hard to find $H$ which maximizes $f_k(P \cap H)$
- It is #P-hard to find $H$ which maximizes $\mathrm{vol}(\pi_H(P))$
- It is NP-hard to find a central halfspace $H_0^+$ which maximizes $\mathrm{vol}(P \cap H_0^+)$

# Combinatorial Types of Sections of the Cross-Polytope

$P = \text{conv}(\pm e_i \mid i \in [d])$



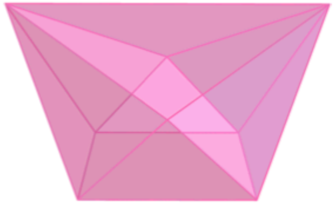
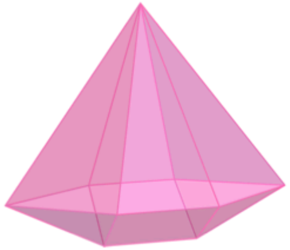
$d = 3$



KTH VETENSKAP OCH KONST

# Combinatorial Types of Sections of the Cross-Polytope

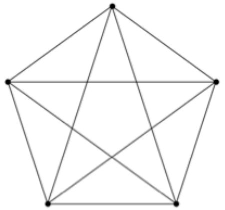
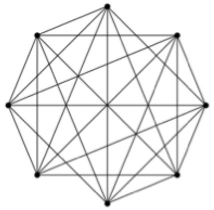
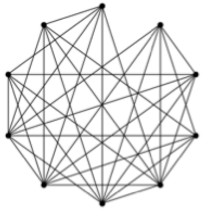
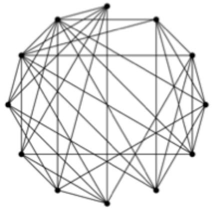
$P = \text{conv}(\pm e_i \mid i \in [d])$ $\qquad$ $d = 4$

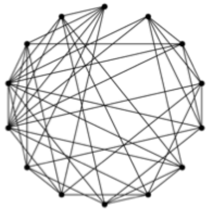
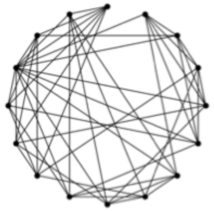
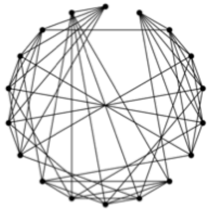
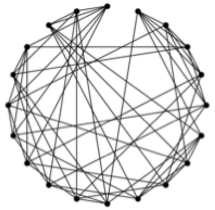
| | | | | | |
|---|---|---|---|---|---|
| $P \cap H$ |  |  |  |  |  |
| $f$-vector | $(4, 6, 4)$ | $(6, 12, 8)$ | $(8, 18, 12)$ | $(8, 17, 11)$ | $(9, 19, 12)$ |
| $H$ | $x_1 + x_2 + x_3 + x_4 = 1$ | $2x_1 = 1$ | $x_1 + x_2 + x_3 = 0$ | $2x_1 + 2x_2 + x_3 + x_4 = 1$ | $2x_1 + 2x_2 + x_3 = 1$ |

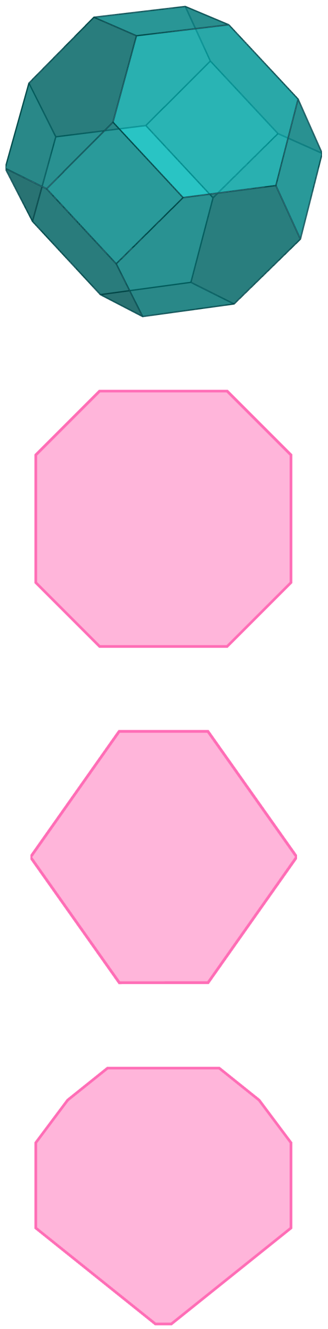
| | | | | |
|---|---|---|---|---|
| $P \cap H$ |  |  |  |  |
| $f$-vector | $(8, 18, 12)$ | $(10, 21, 13)$ | $(12, 24, 14)$ | $(12, 24, 14)$ |
| $H$ | $x_1 + x_2 + x_3 = 0$ | $2x_1 + 2x_2 + 2x_3 + x_4 = 1$ | $x_1 + x_2 + x_3 + x_4 = 0$ | $2x_1 + 2x_2 + 2x_3 = 1$ |

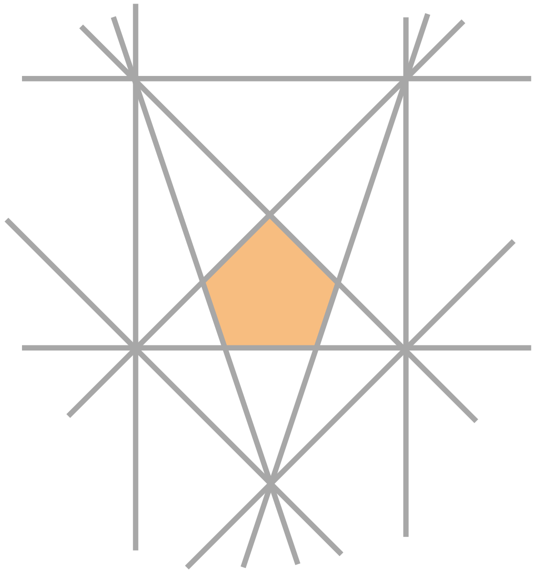
# Combinatorial Types of Sections of the Cross-Polytope

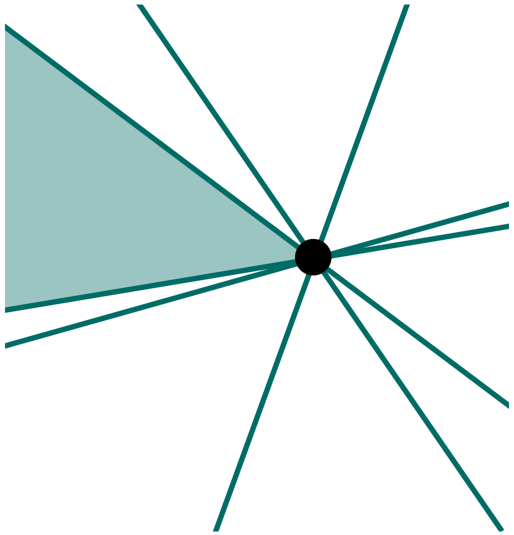
$P = \text{conv}(\pm e_i \mid i \in [d])$ $\qquad d = 5$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $P \cap H$ |  |  |  |  |  |  |  |  |
| $f$-vector | $(5, 10, 10, 5)$ | $(8, 24, 32, 16)$ | $(10, 34, 48, 24)$ | $(11, 36, 48, 23)$ | $(12, 39, 51, 24)$ | $(13, 41, 52, 24)$ | $(14, 42, 52, 24)$ | $(14, 48, 62, 28)$ |
| $H$ | $x_1 + x_2 + x_3 + x_4 + x_5 = 1$ | $2x_1 = 1$ | $x_1 + x_2 + x_3 = 0$ | $2x_1 + 2x_2 + x_3 + x_4 + x_5 = 1$ | $2x_1 + 2x_2 + x_3 + x_4 = 1$ | $2x_1 + 2x_2 + x_3 = 1$ | $2x_1 + 2x_2 = 1$ | $x_1 + x_2 + x_3 + x_4 = 0$ |

| | | | | | | |
|---|---|---|---|---|---|---|
| $P \cap H$ |  |  |  |  |  |  |
| $f$-vector | $(14, 46, 59, 27)$ | $(16, 51, 63, 28)$ | $(17, 54, 66, 29)$ | $(18, 54, 64, 28)$ | $(20, 60, 70, 30)$ | $(20, 60, 70, 30)$ |
| $H$ | $2x_1 + 2x_2 + 2x_3 + x_4 + x_5 = 1$ | $2x_1 + 2x_2 + 2x_3 + x_4 = 1$ | $2x_1 + 2x_2 + 2x_3 + 2x_4 + x_5 = 1$ | $2x_1 + 2x_2 + 2x_3 = 1$ | $2x_1 + 2x_2 + 2x_3 + 2x_4 = 1$ | $x_1 + x_2 + x_3 + x_4 + x_5 = 0$ |







# THANK YOU!