MAX PLANCK INSTITUTE
FOR MATHEMATICS IN THE SCIENCES

How to slice a polytope

joint work with Chiara Meroni and Jesús A. De Loera.
arXiv: 2304.14239

Marie-Charlotte Brandenburg

Discrete Geometry and Topological Combinatorics Seminar
Freie Universität Berlin
23 November 2023

Joint work with

Chiara Meroni

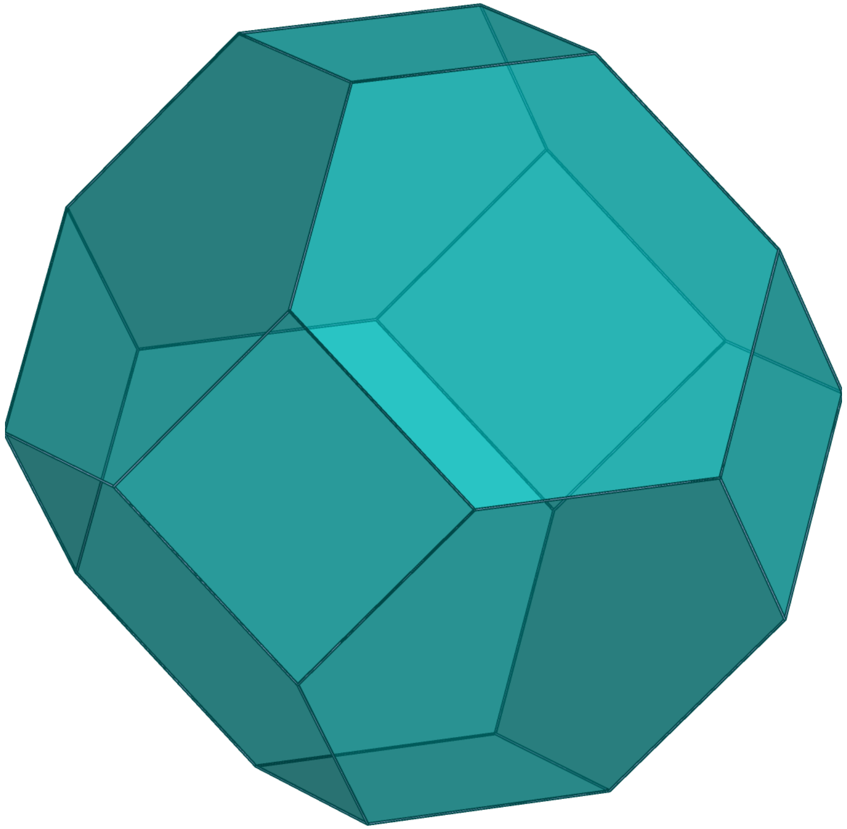
Harvard

Jesús A. De Loera

UC Davis

MB, Chiara Meroni, and Jesús A. De Loera. *The Best Ways to Slice a Polytope*. 2023. arXiv: 2304.14239

Slices of the Permutohedron



$$P = \text{conv}(\ (\sigma(1), \sigma(2), \sigma(3), \sigma(4))\ \mid \sigma \in S_4) - \tfrac{3}{2}(1,1,1,1)$$
$$= \text{conv}(\ (1,2,3,4),\ (1,2,4,3),\ \ldots,\ (4,3,2,1)\ ) - \tfrac{3}{2}(1,1,1,1)$$

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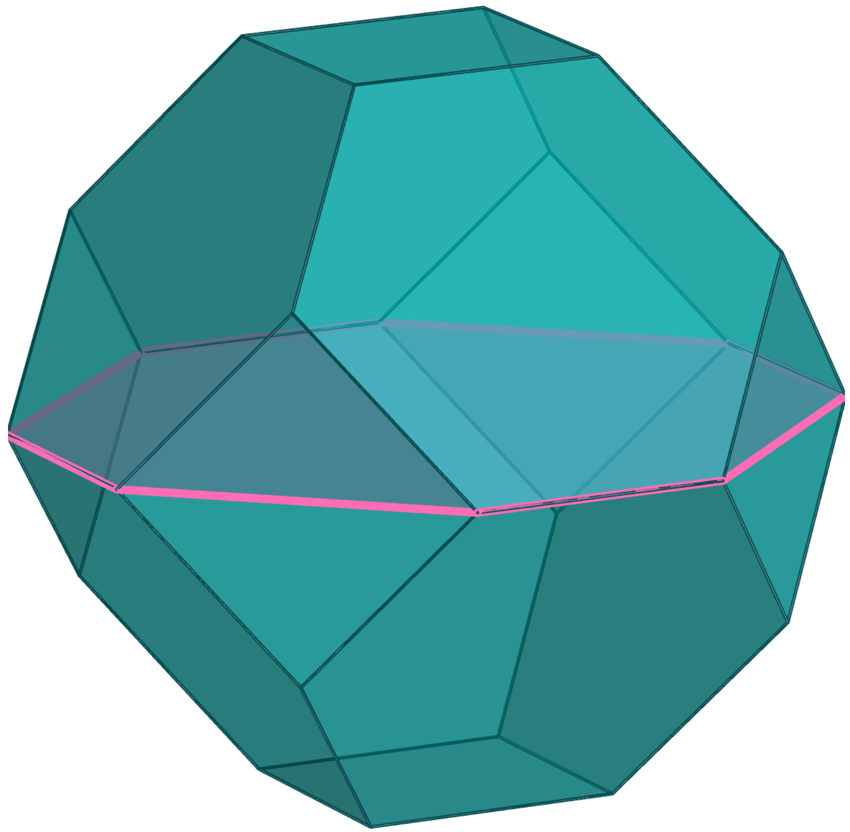
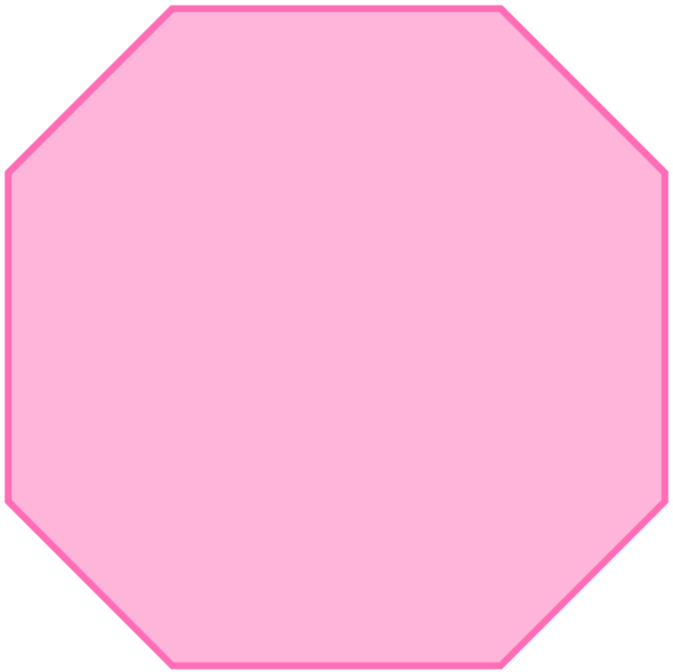
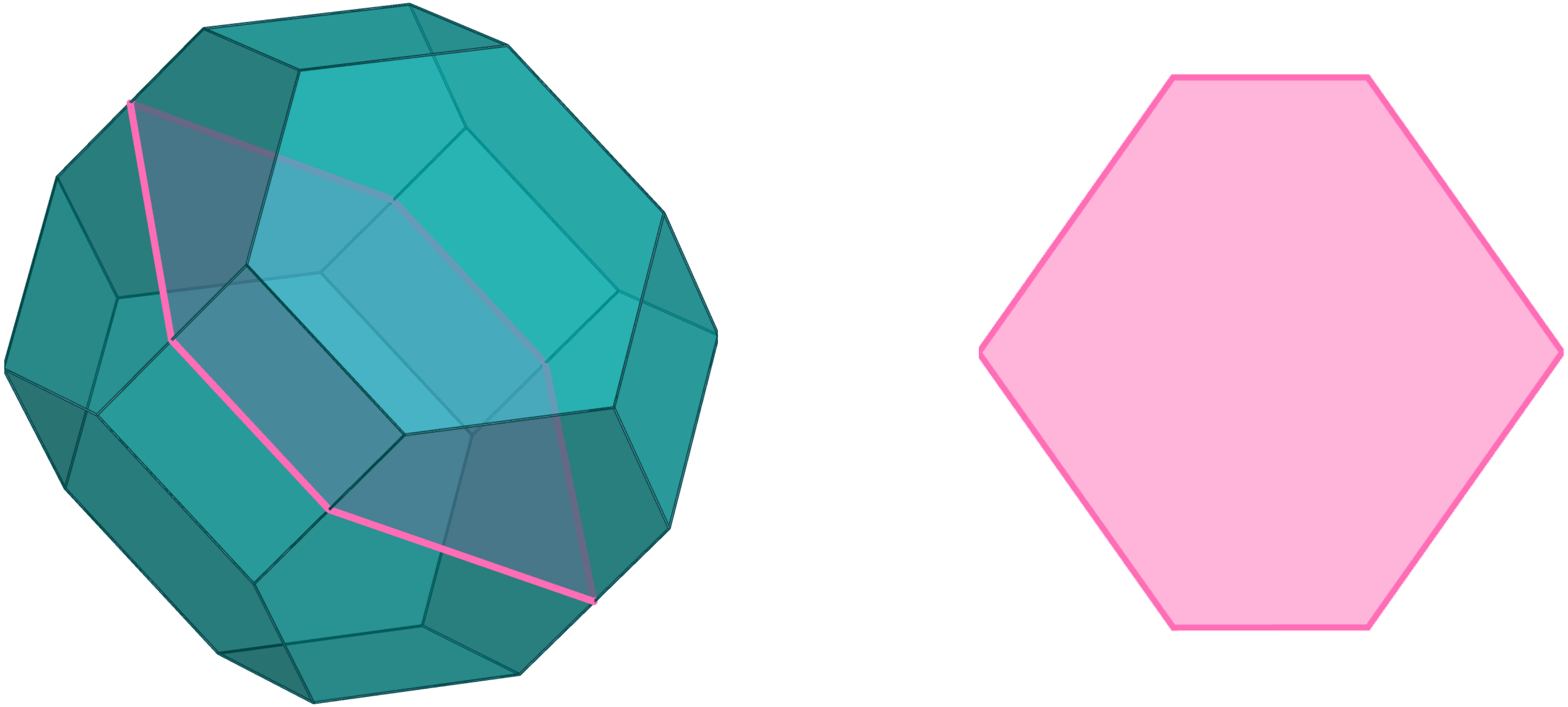




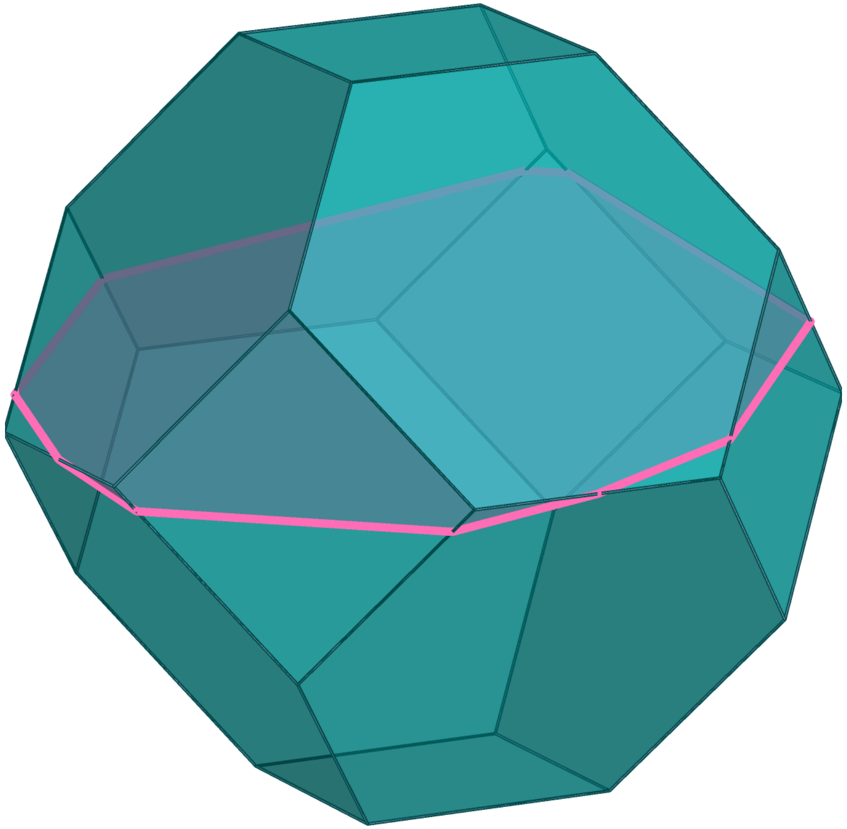
Affine slice of maximum volume

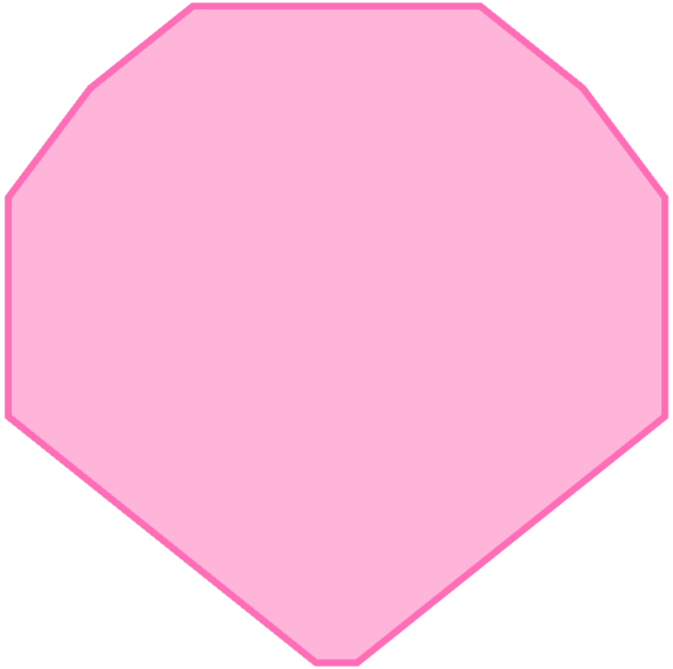
Slices of the Permutohedron



Central slice of minimum volume

Slices of the Permutohedron





Affine slice with maximum number of vertices

WHO WANTS TO COMPUTE (EXTREMAL) SLICES OF POLYTOPES?

Motivation

- **Maximal volume slice**: What is the slice of P with maximal volume?

  [Ball '89, Meyer-Pajor '88, Webb '96, Pournin '22, …]

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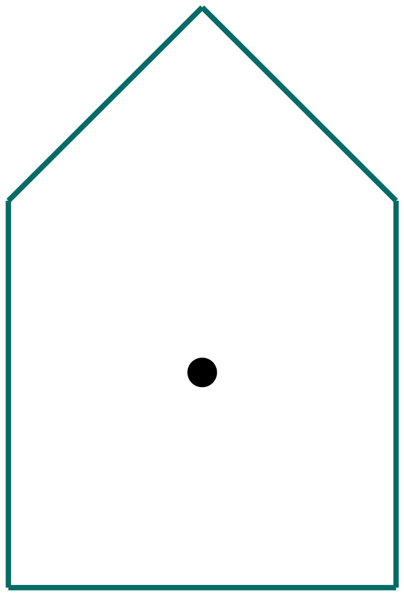
- Volumes of **slices of the permutahedron** fixed by action of a permutation
  [Ardila-Schindler-(Vindas-Meléndez) ‘21,…]

HOW CAN WE COMPUTE THESE “EXTREMAL” SLICES?

2 Approaches

Rotational Approach

1. Choose a position of the origin



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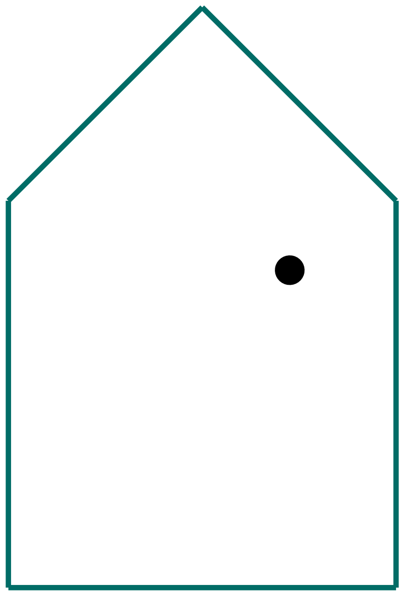

2 Approaches

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1. Choose a position of the origin
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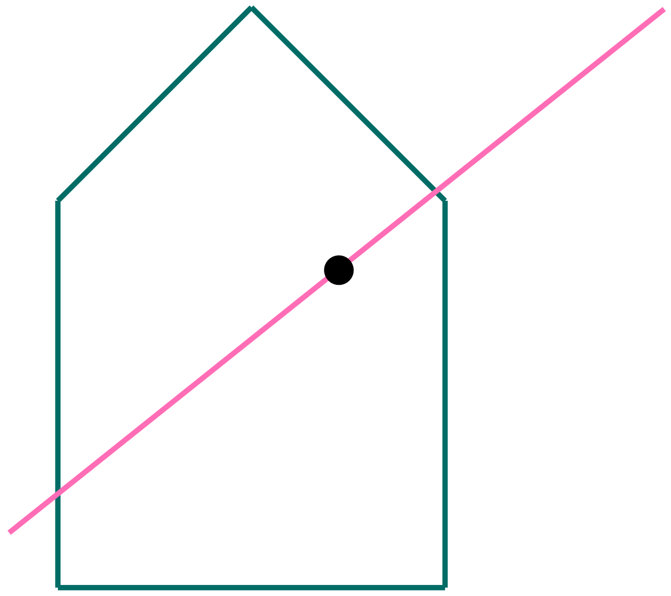

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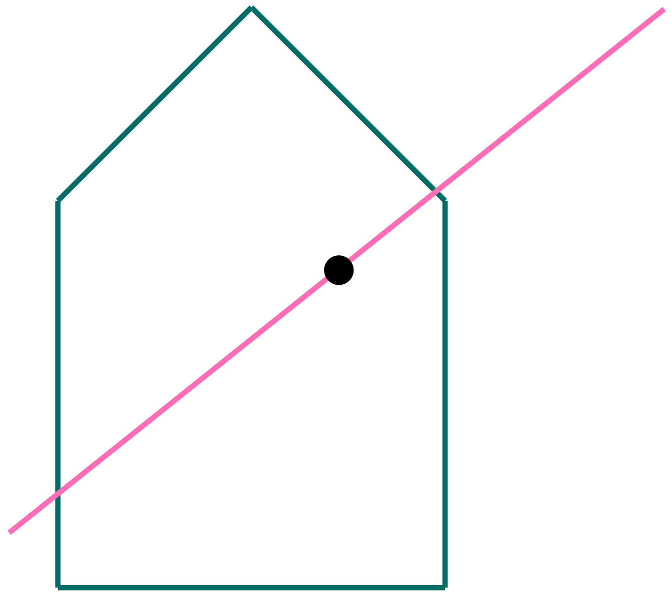

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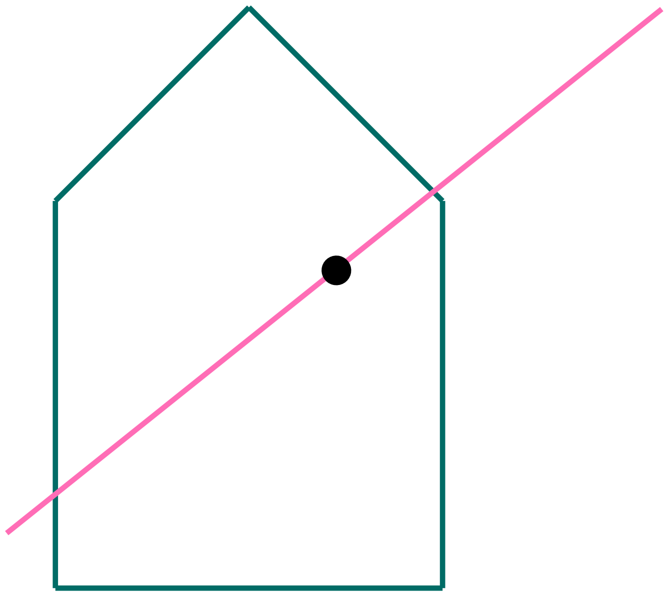

Translational Approach

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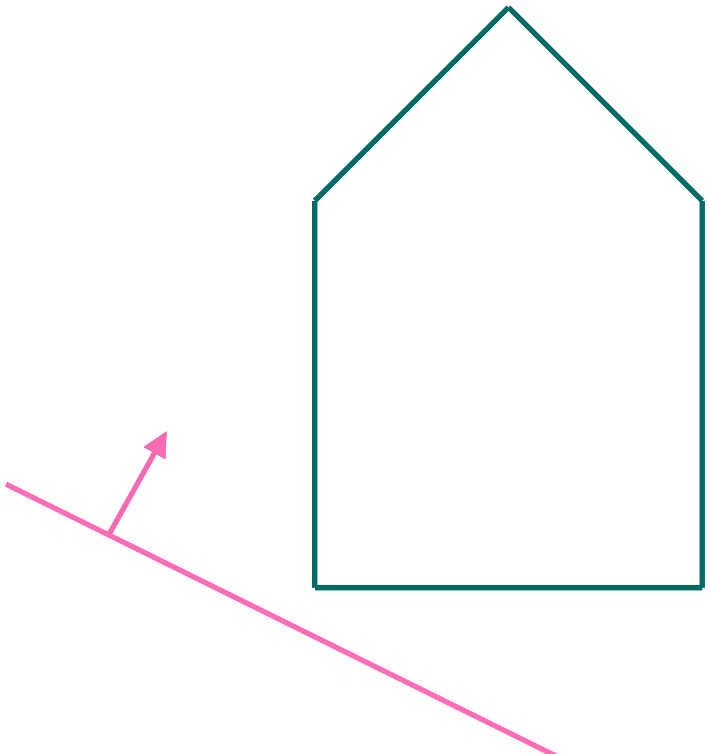

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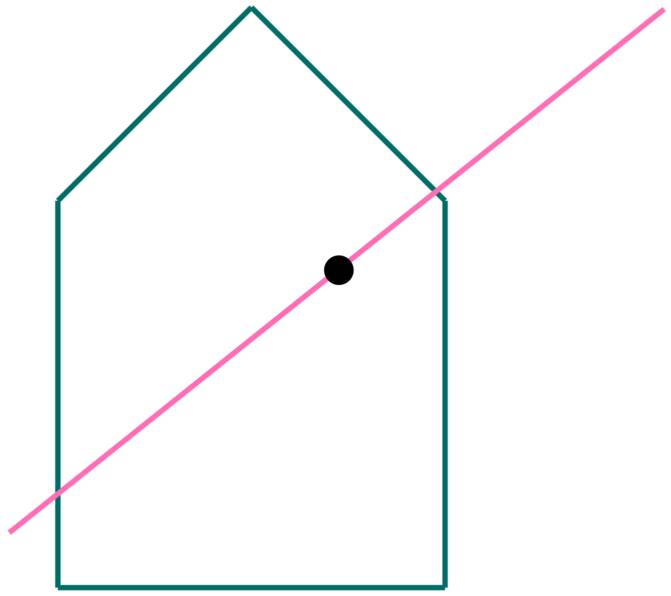

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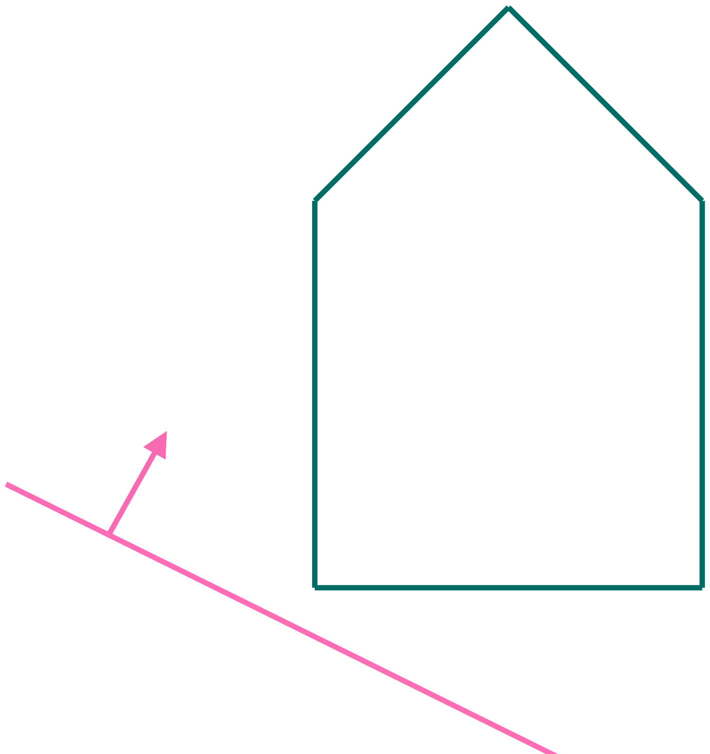

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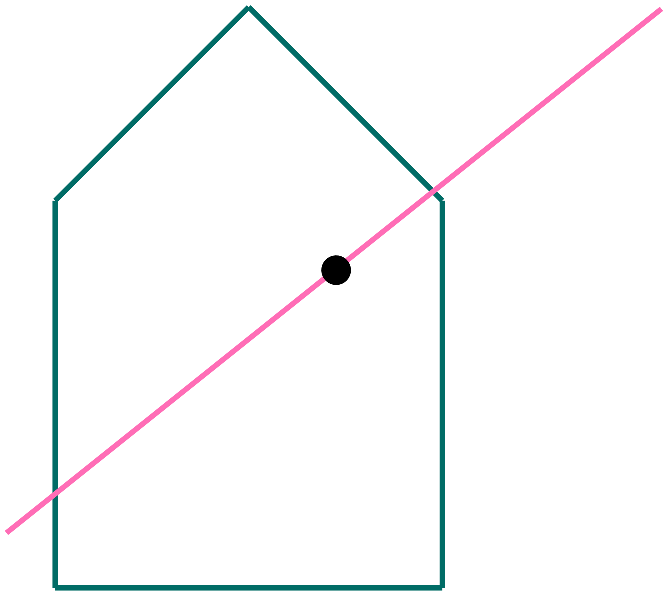

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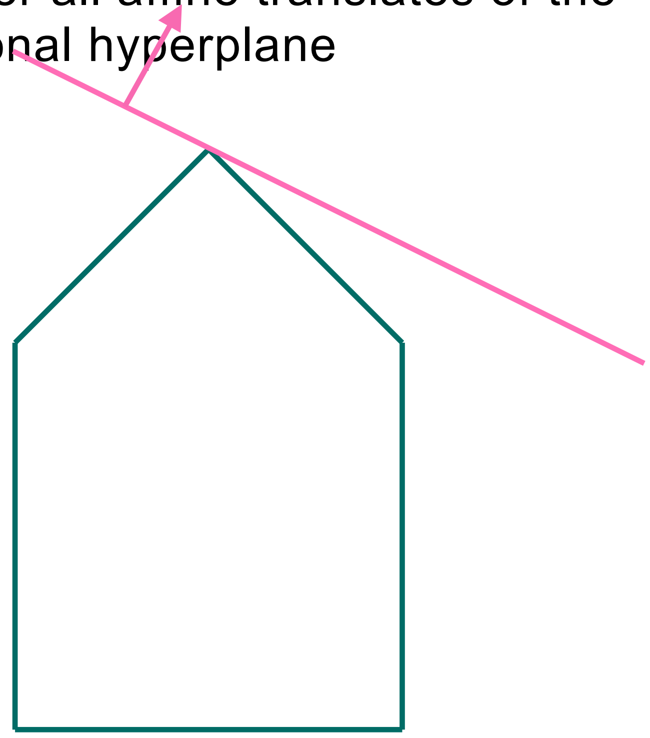

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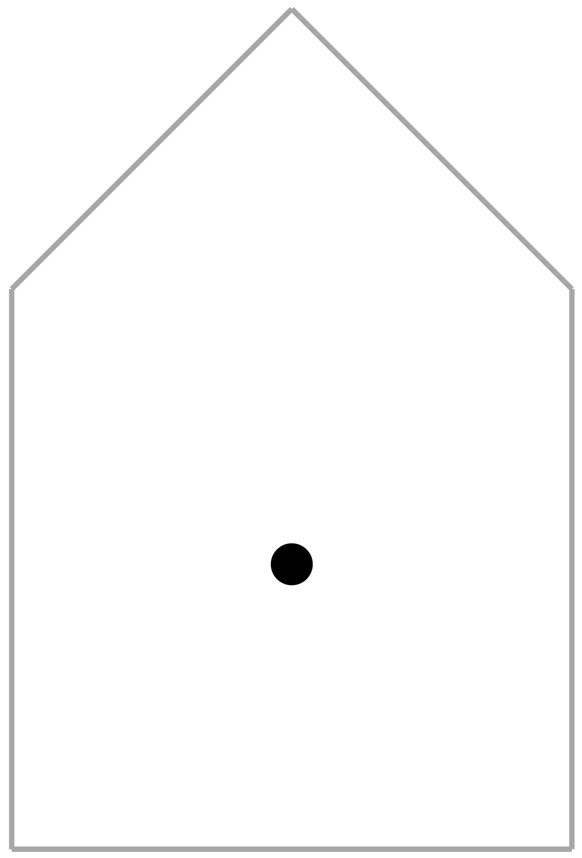
	Hyperplane Arrangement	**Notation**	**Reference Object**
$\circlearrowleft$	central arrangement	$\mathcal{C}_{\circlearrowleft}$	intersection body
	cocircuit arrangement	$\mathcal{R}_{\circlearrowleft}$	oriented matroid
$\uparrow$	parallel arrangement	$\mathcal{C}^{\mathbf{u}}_{\uparrow}$	fiber polytope
	sweep arrangement	$\mathcal{R}_{\uparrow}$	sweep polytope

ROTATIONAL APPROACH

Rotational Approach

Fix the position of the origin.

$u^{\perp}$ = central hyperplane orthogonal to u



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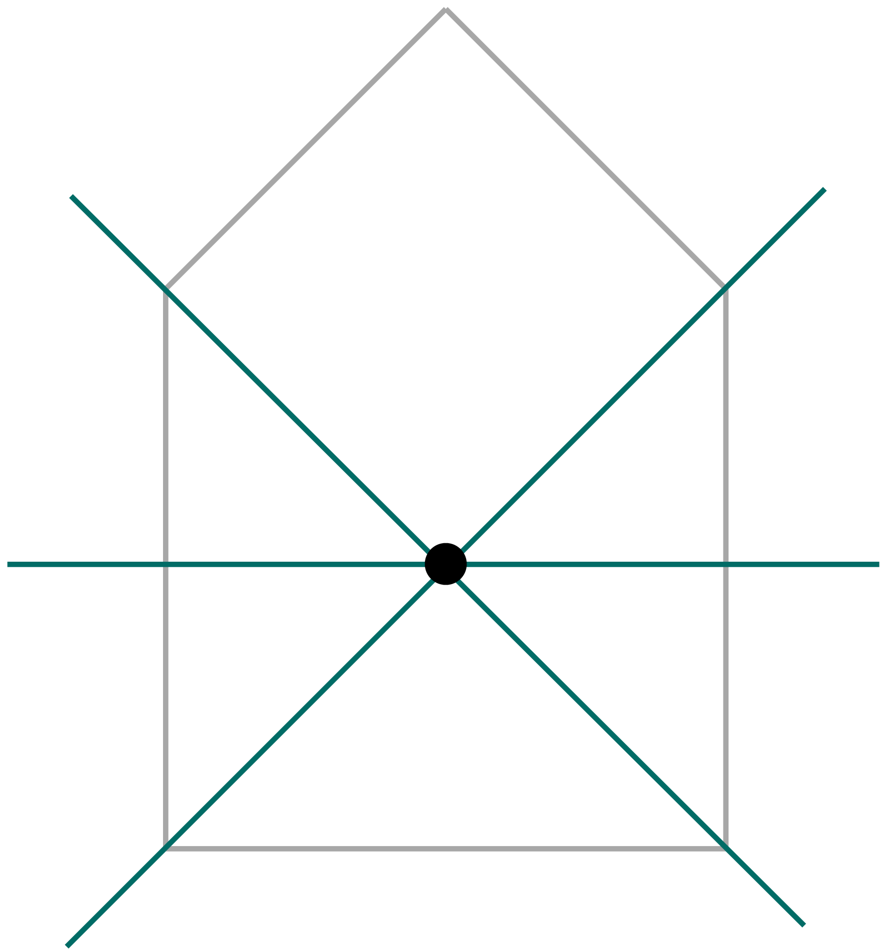
Consider the central hyperplane arrangement

$$\mathscr{C}_{\circlearrowleft}(P) = \{v^\perp \mid v \text{ is a vertex of } P\}.$$



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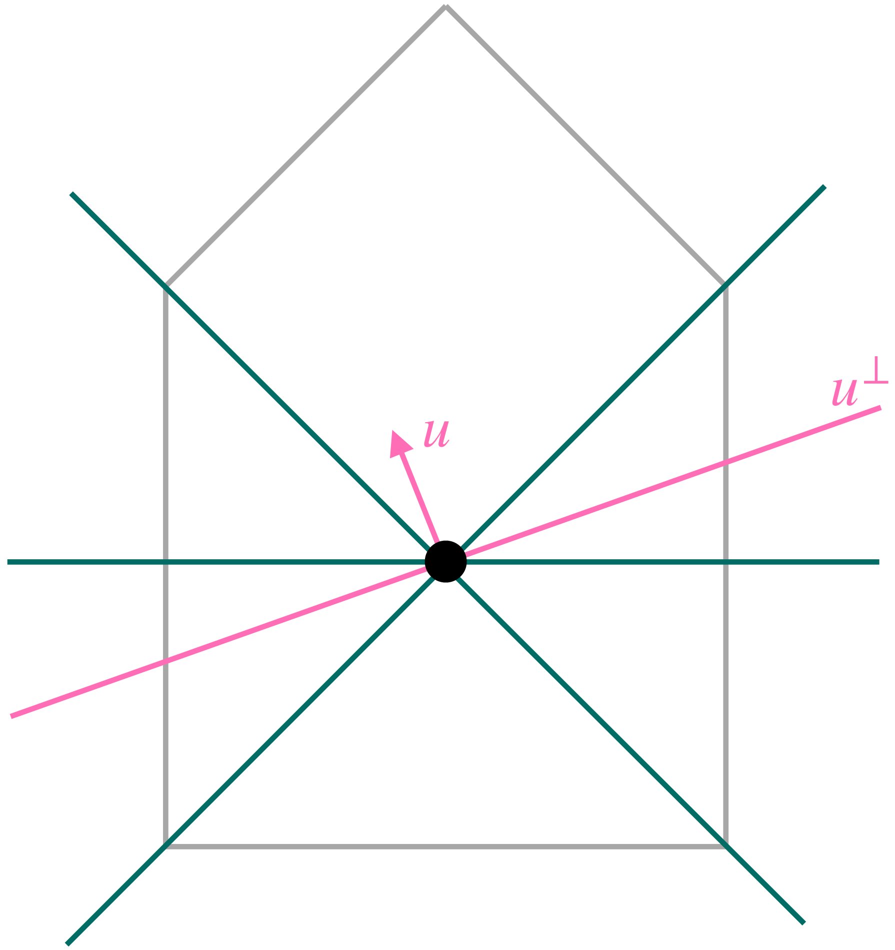
$\longrightarrow$ The combinatorial type of $P \cap u^\perp$ is constant in each cell of $\mathscr{C}_{\circlearrowleft}(P)$.

We refer to the maximal cells of $\mathscr{C}_{\circlearrowleft}(P)$ as chambers.



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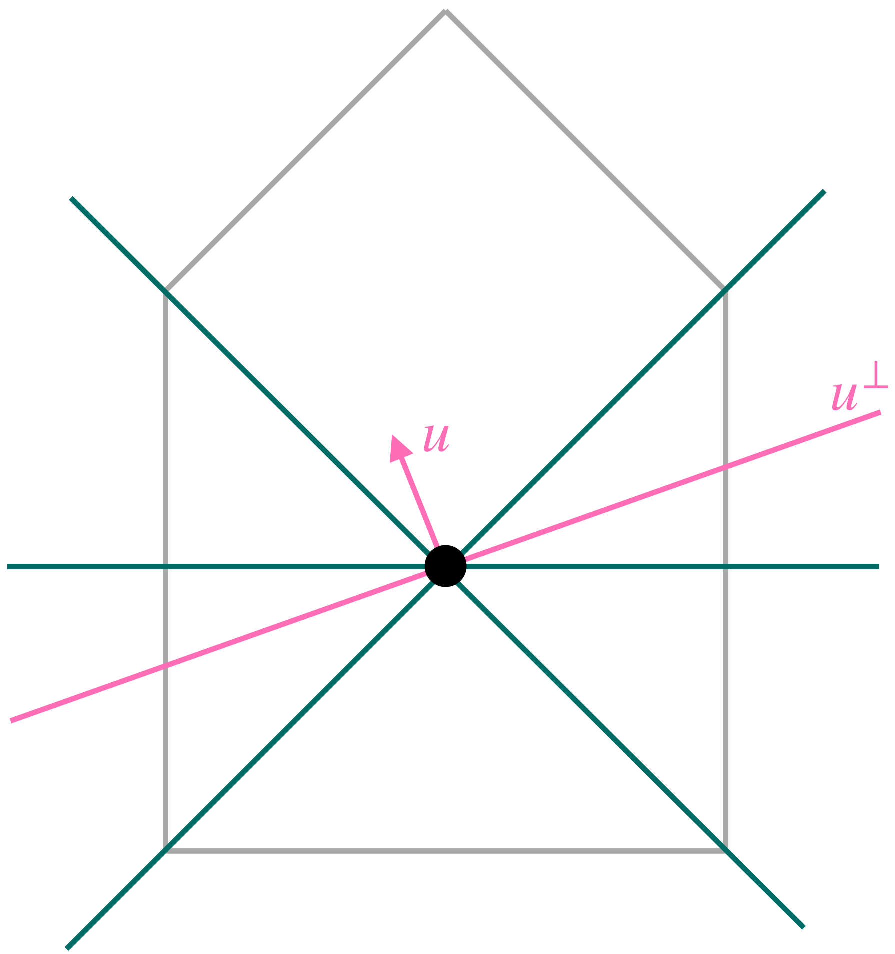
What happens if we translate P, i.e. vary the position of the origin?



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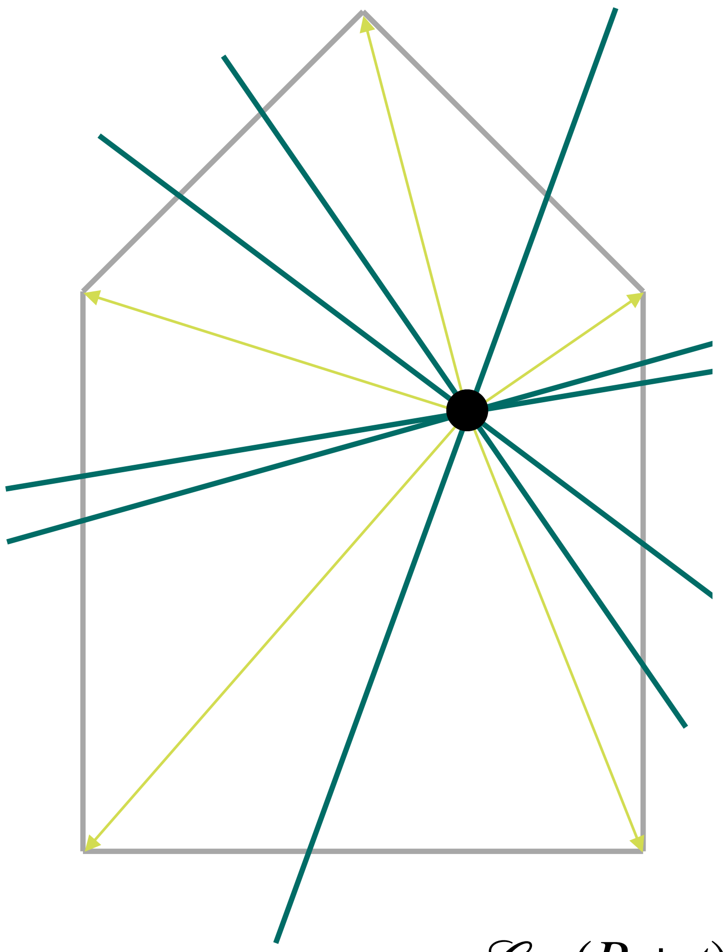

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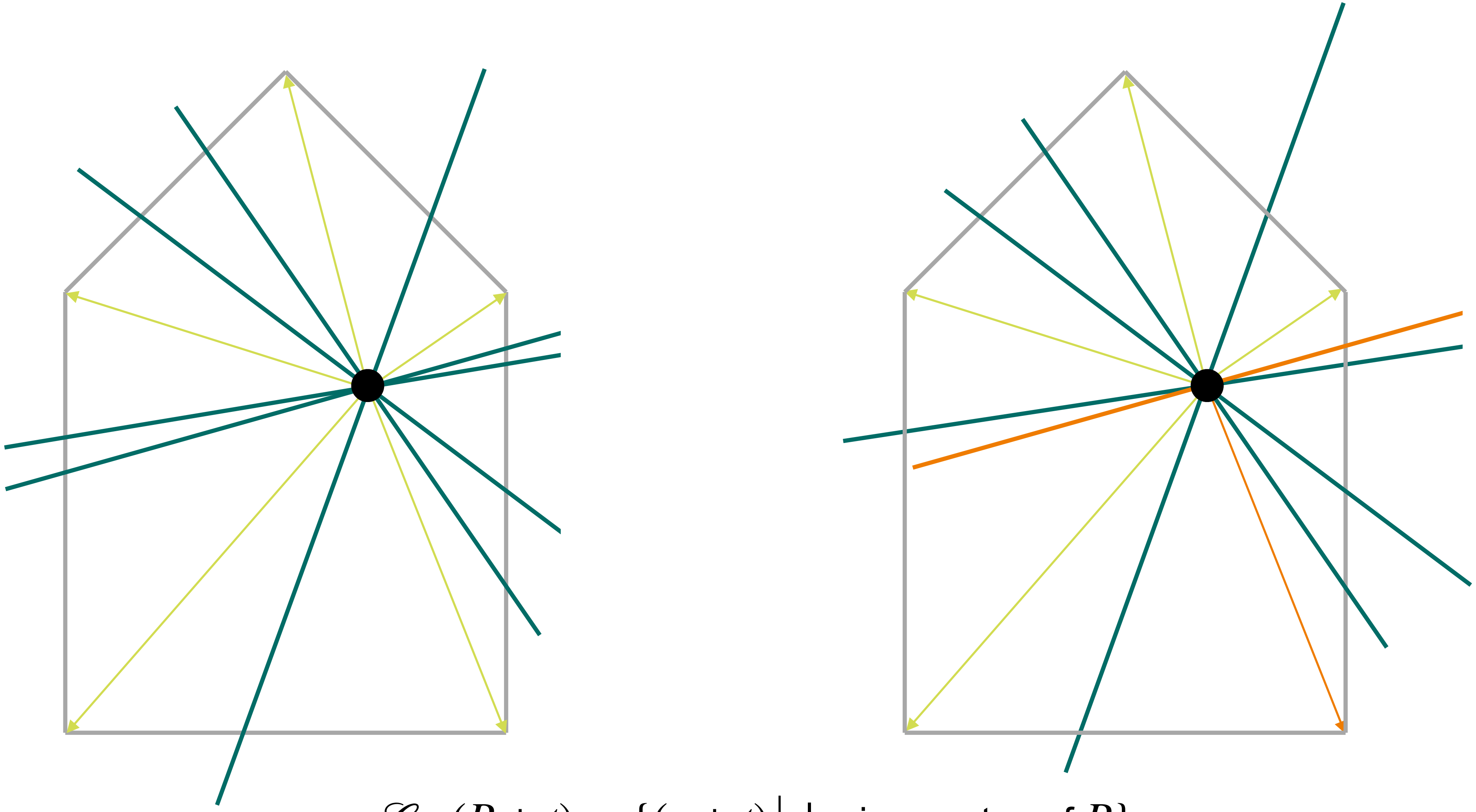

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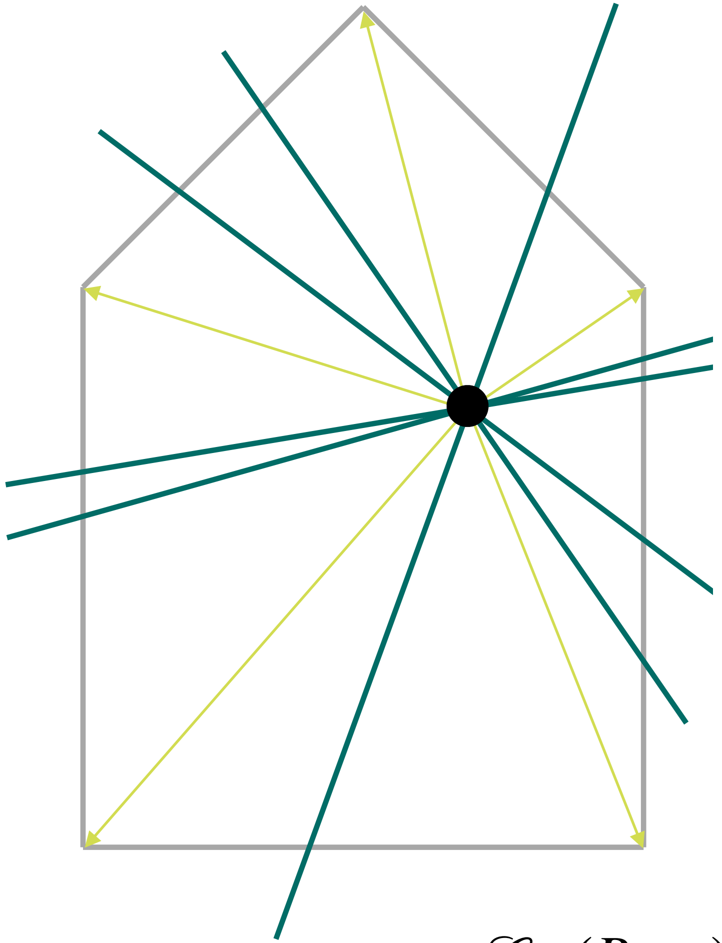

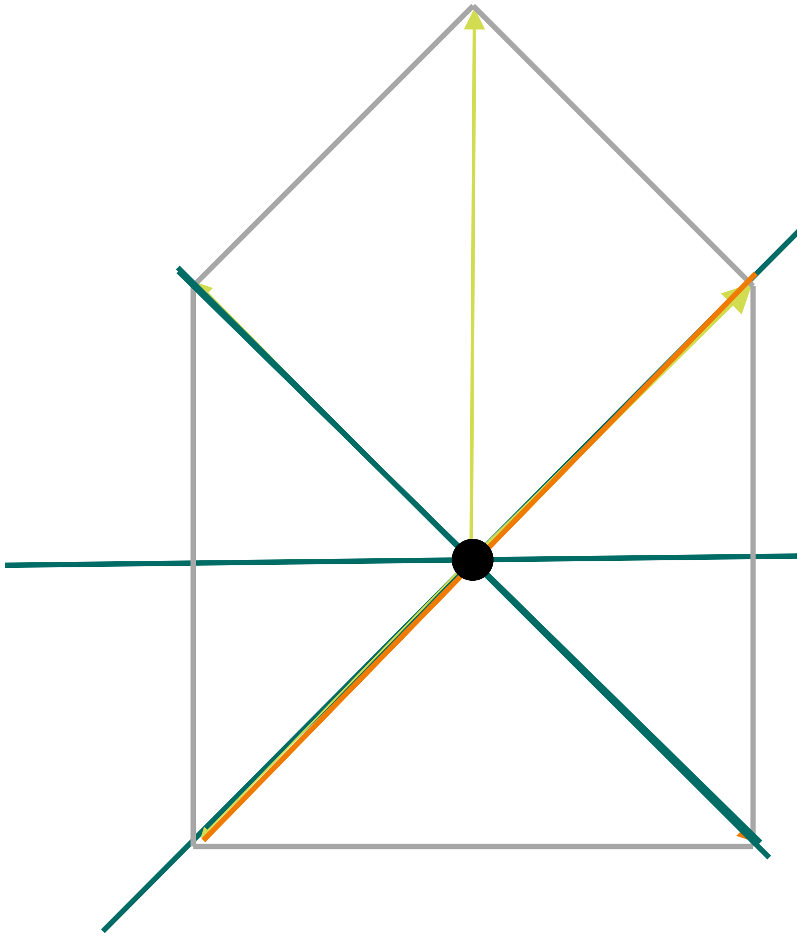

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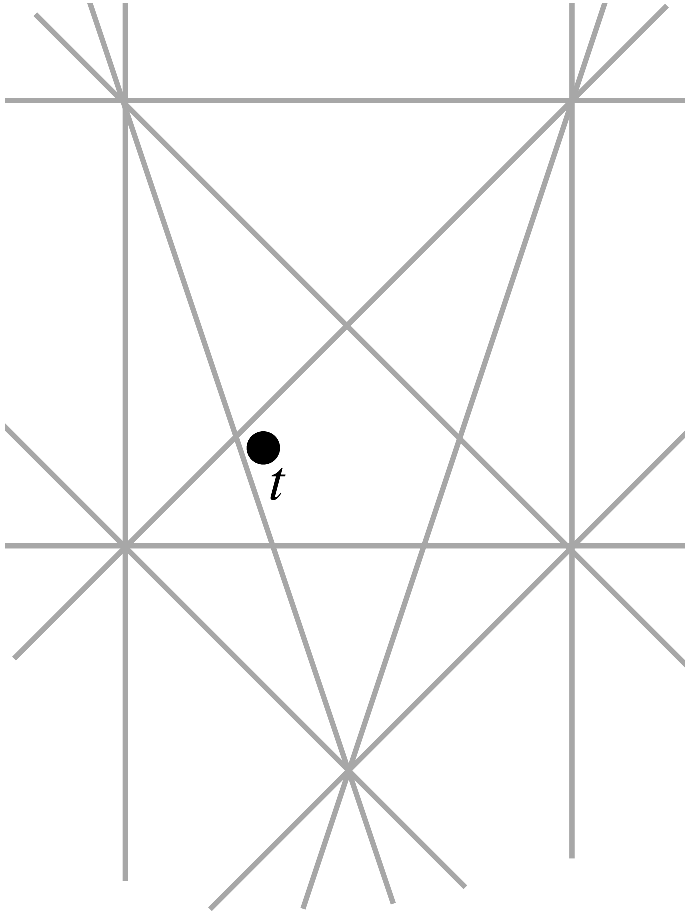
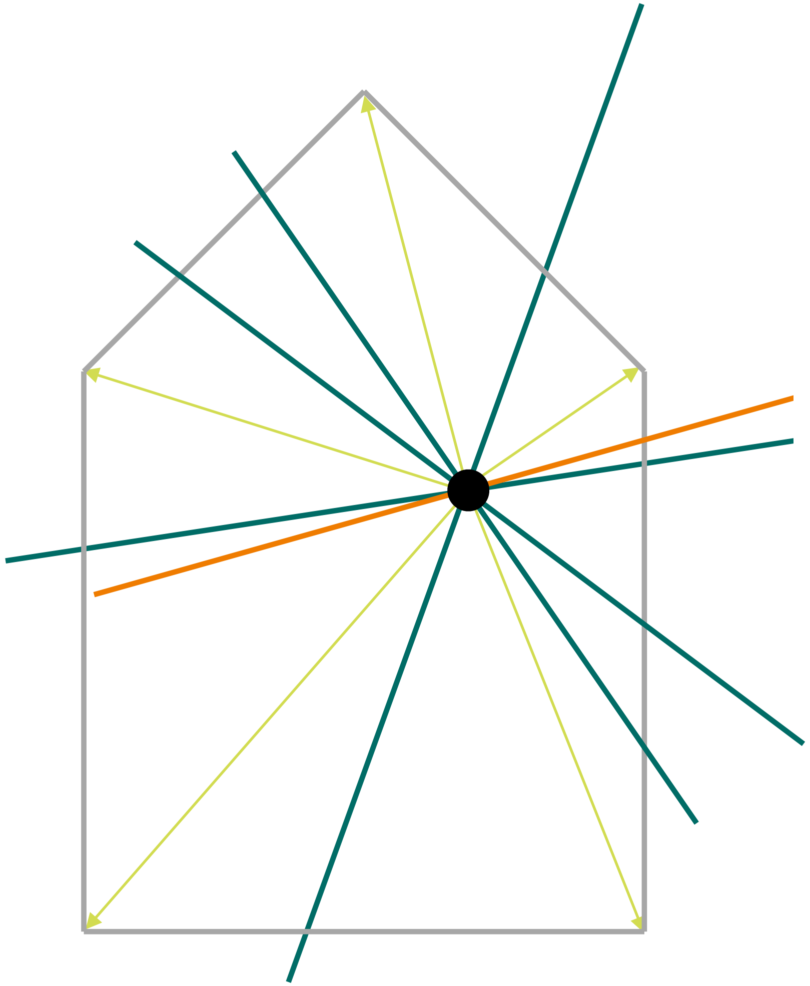
Consider the affine hyperplane arrangement (called cocircuit arrangement)

$$\mathscr{R}_{\circlearrowleft}(P) = \{\text{aff}(-v_1, \ldots, -v_d) \mid v_k \text{ are vertices of } P\}$$

$\longrightarrow$ within each region of $\mathscr{R}_{\circlearrowleft}(P)$ the combinatorics of $\mathscr{C}_{\circlearrowleft}(P+t)$ are fixed

Rotational Approach





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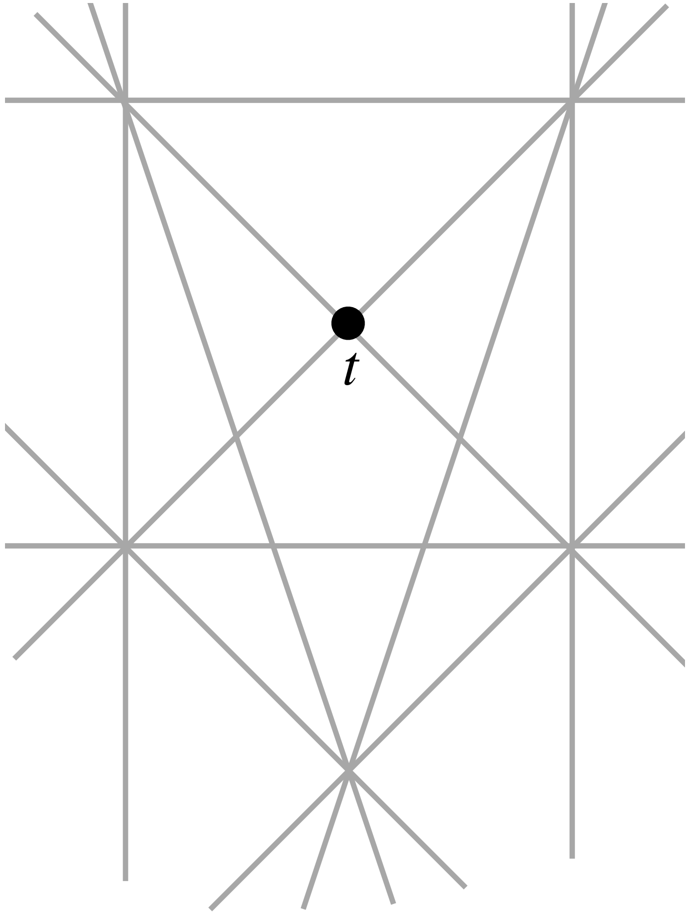

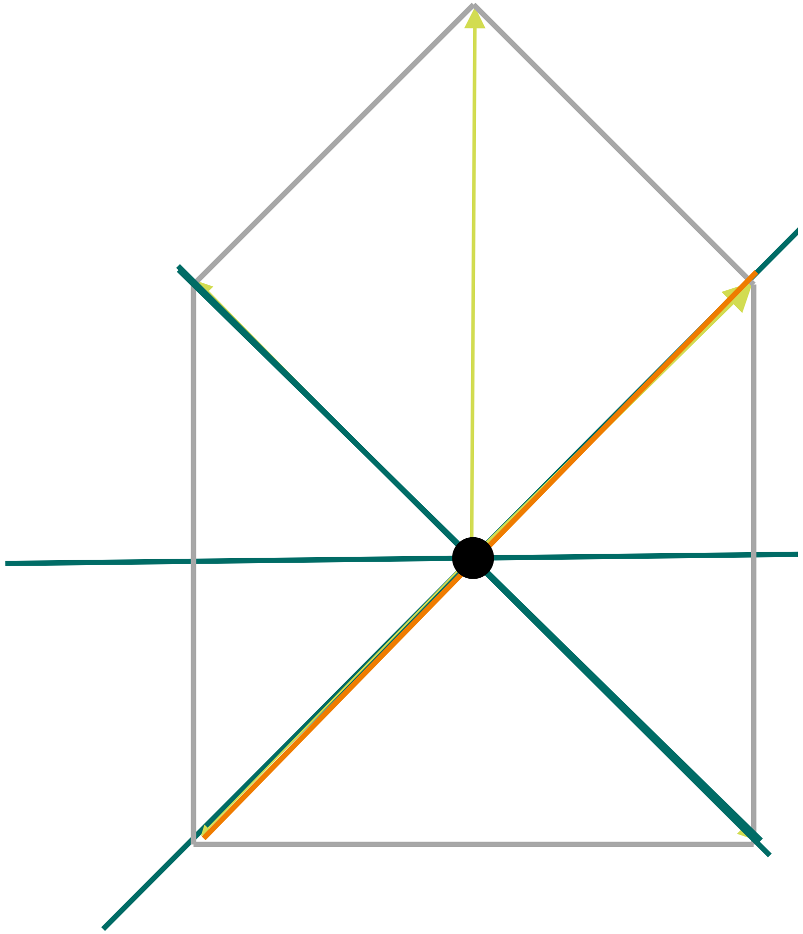

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Theorem (B.-Meroni-De Loera '23):

Let $P \subseteq \mathbb{R}^d$ be a polytope, and $f(x) = \sum_\alpha c_\alpha x^\alpha$ be a polynomial in variables $x_1, \ldots, x_d$.

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Fix a region $R \in \mathscr{R}_{\circlearrowleft}(P)$ of the cocircuit arrangement, a translation $t \in R$ and a chamber $C(t) \in \mathscr{C}_{\circlearrowleft}(P + t)$ of the central arrangement.

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Restricted to $t \in R$ and $u \in C(t) \cap S^{d-1}$, the integral

$$\int_{(P+t)\cap u^\perp} f(x)\ \mathrm{d}x$$

is a rational function in variables $t_1, \ldots, t_d, u_1, \ldots, u_d$.

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Note:

If $f(x) = 1$ then $\int_{(P+t)\cap u^\perp} f(x)\ \mathrm{d}x = \mathrm{vol}((P+t) \cap u^\perp)$.

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   $\Longrightarrow$ vol(Δ) is a rational function in $u_1, \ldots, u_d, t_1, \ldots, t_d$

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6. There is a formula for $\int_{\Delta}$ (linear form) in terms of vol(Δ) [Lasserre-Avrachenkov '01, Baldoni-Berline-DeLoera-Köppe-Vergne '11]

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4. The volume of a simplex Δ in the triangulation is a determinant (in terms of vertices of Δ)

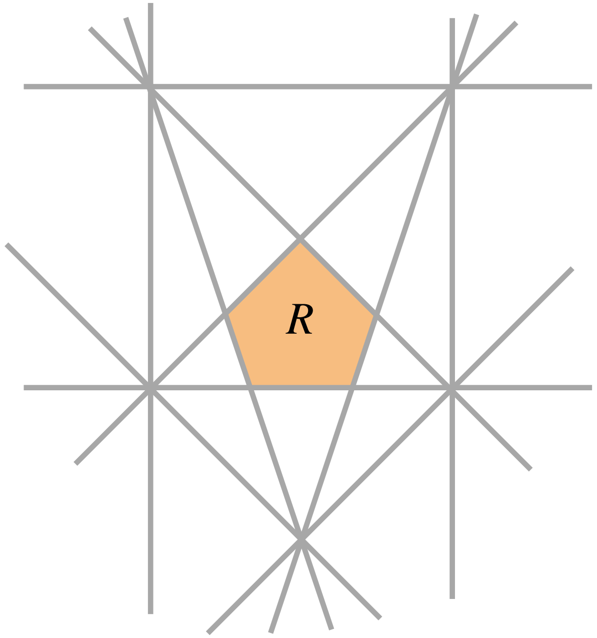
   $\Longrightarrow$ vol(Δ) is a rational function in $u_1, \ldots, u_d, t_1, \ldots, t_d$

5. A Warring decomposition of $f(x)$ is a decomposition into sums of powers of linear forms

6. There is a formula for $\int_{\Delta}$ (linear form) in terms of vol(Δ) [Lasserre-Avrachenkov '01, Baldoni-Berline-DeLoera-Köppe-Vergne '11]

$\Longrightarrow$ formula for computing $\int_{Q(u,t)} f(x)$ in terms of $u_1, \ldots, u_d, t_1, \ldots, t_d$

Rotational Approach



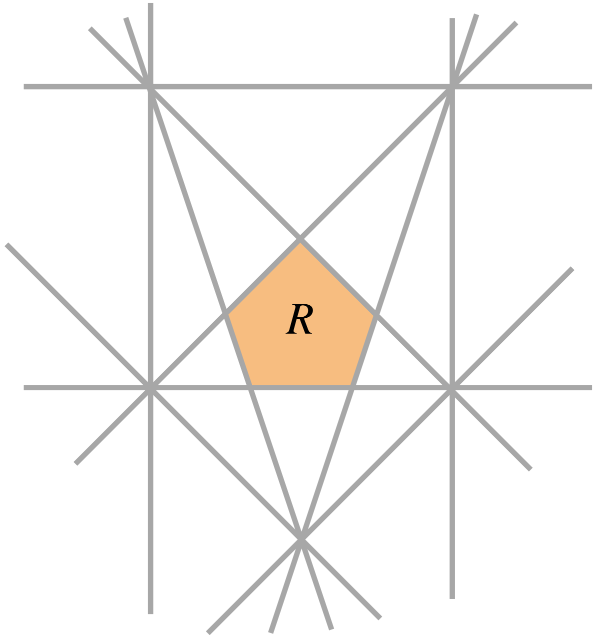
$(t_1, t_2) \in R \iff$

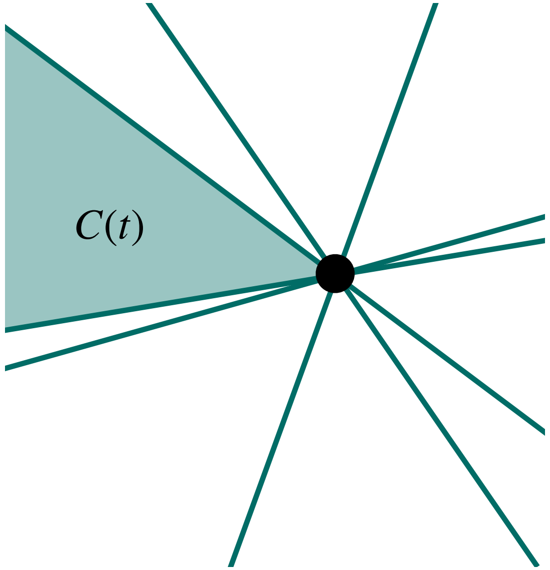
$-t_1 - t_2 \geq 0, \quad t_1 - t_2 \geq 0$

$-3t_1 + t_2 \geq -2, \; 3t_1 + t_2 \geq -2$

$t_2 \geq -1$

Rotational Approach





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If $(t_1, t_2) \in R$ then $(u_1, u_2) \in C(t) \iff$

$2u_2 + t_1u_1 + t_2u_2 \geq 0$

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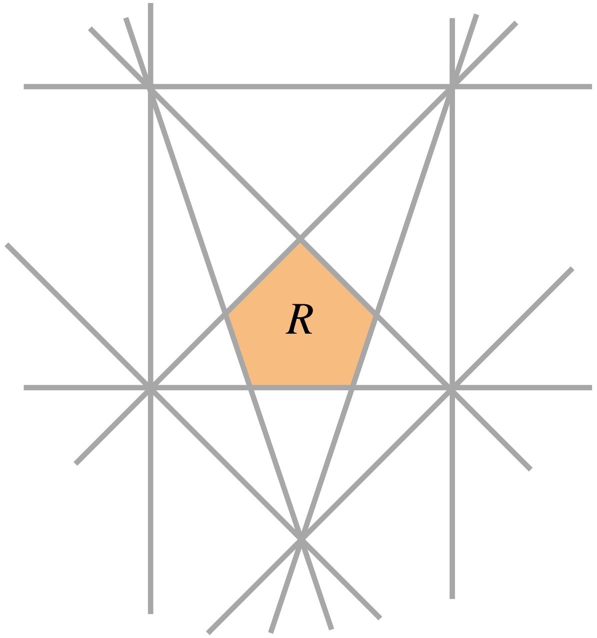
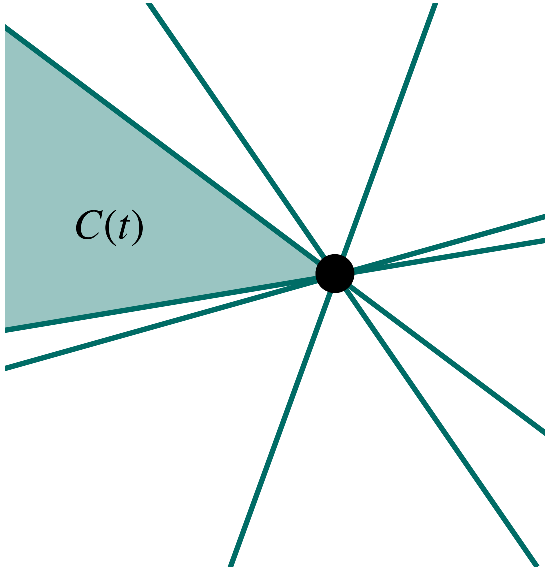
Rotational Approach





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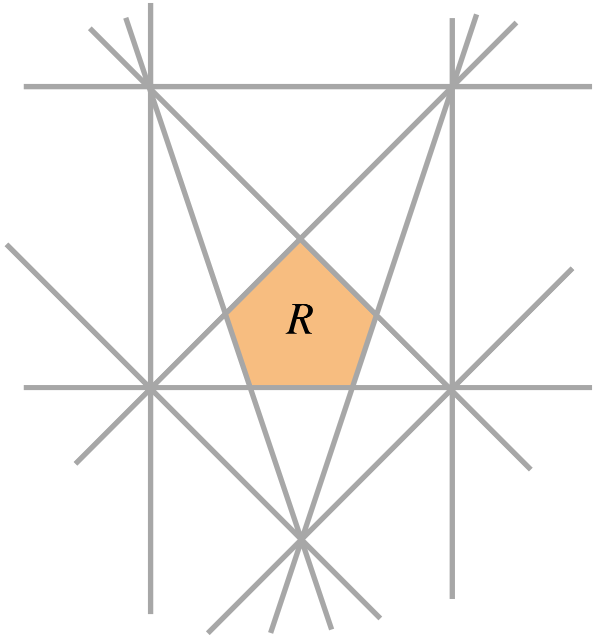
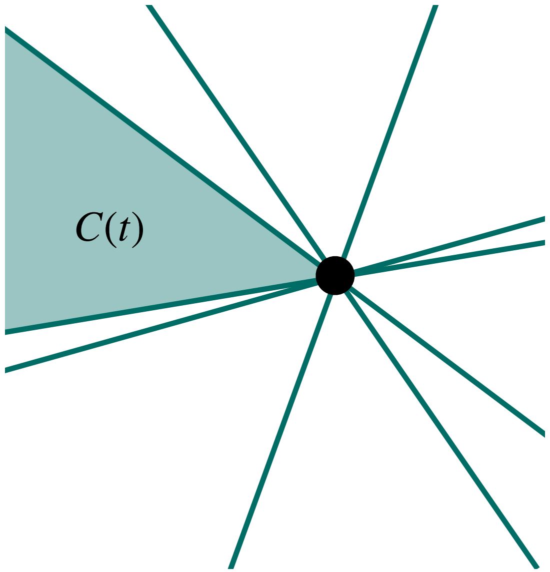
$$\text{vol}((P+t) \cap u^{\perp}) = \int_{(P+t)\cap u^{\perp}} 1 \mathrm{d}x = \frac{-(t_1u_1 + t_2u_2 + 3u_1 - u_2)}{u_1(u_1 - u_2)}$$

Rotational Approach





Let the computer find the biggest slice:

maximize	$\dfrac{-(t_1u_1 + t_2u_2 + u_1 - u_2)}{u_1(u_1 - u_2)}$
s.t	$(t_1, t_2) \in R$
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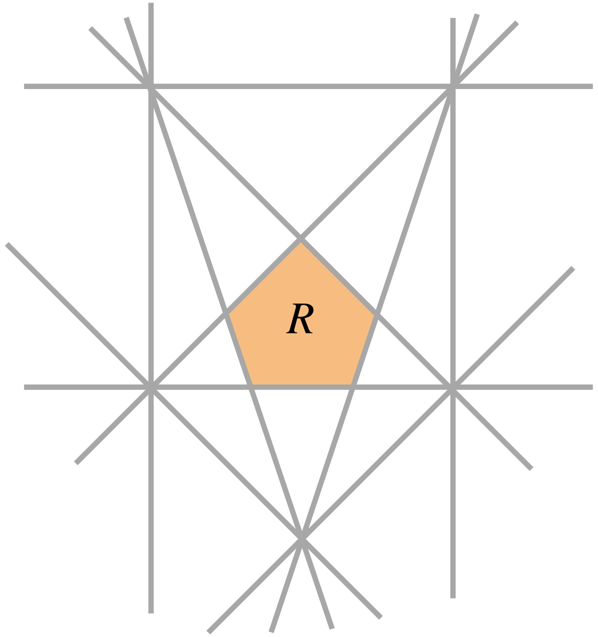
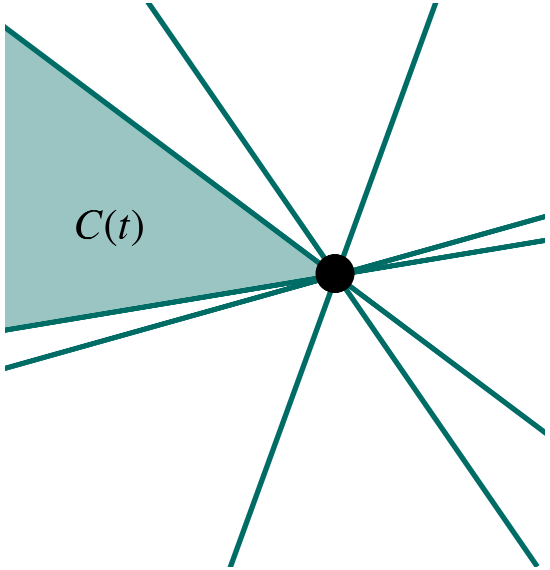
Rotational Approach





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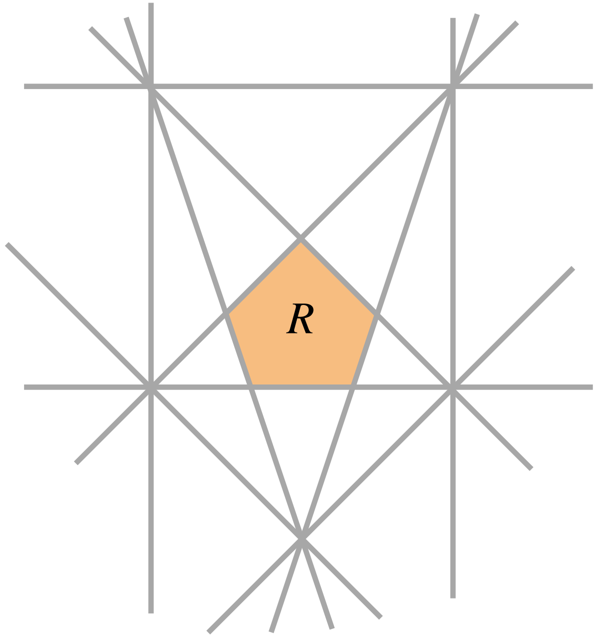
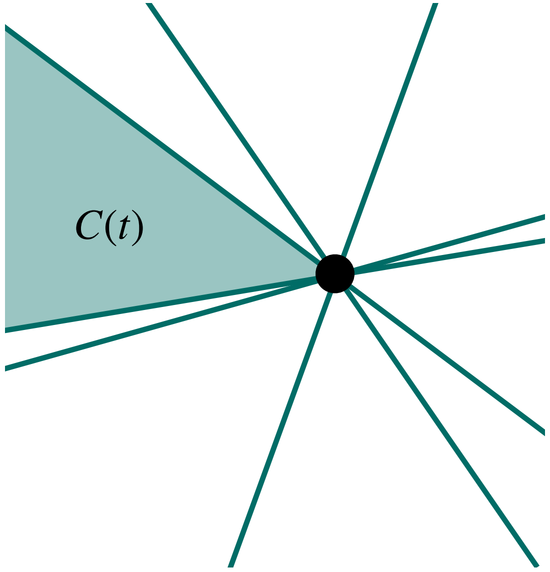
Rotational Approach

Let the computer find the biggest slice:





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Compute this for all regions $R \in \mathscr{R}_{\circlearrowleft}(P)$
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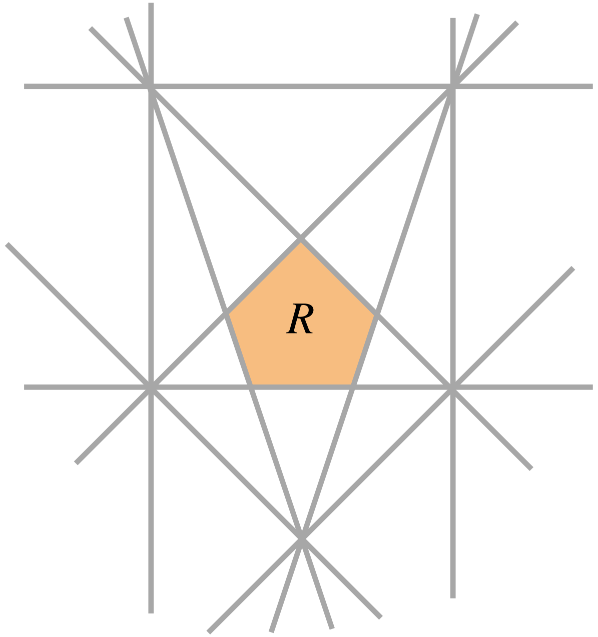
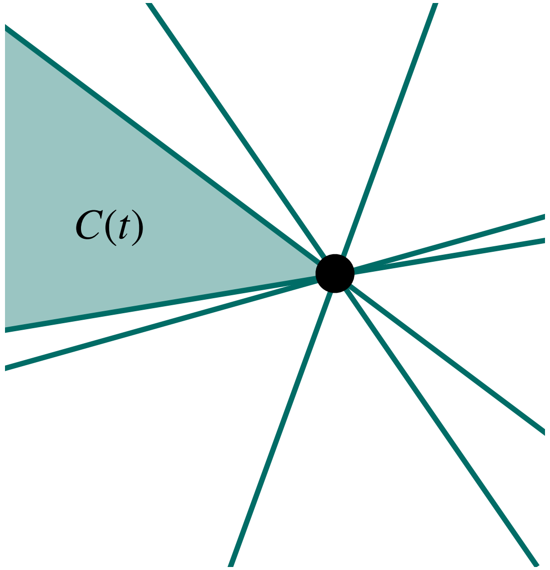
$\longrightarrow$ largest slice!

Rotational Approach

Let the computer find the biggest slice:





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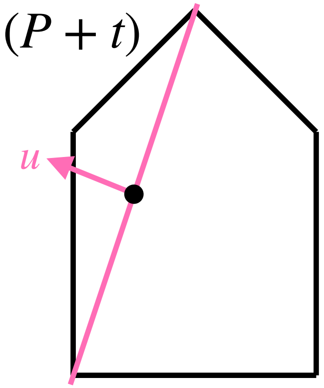

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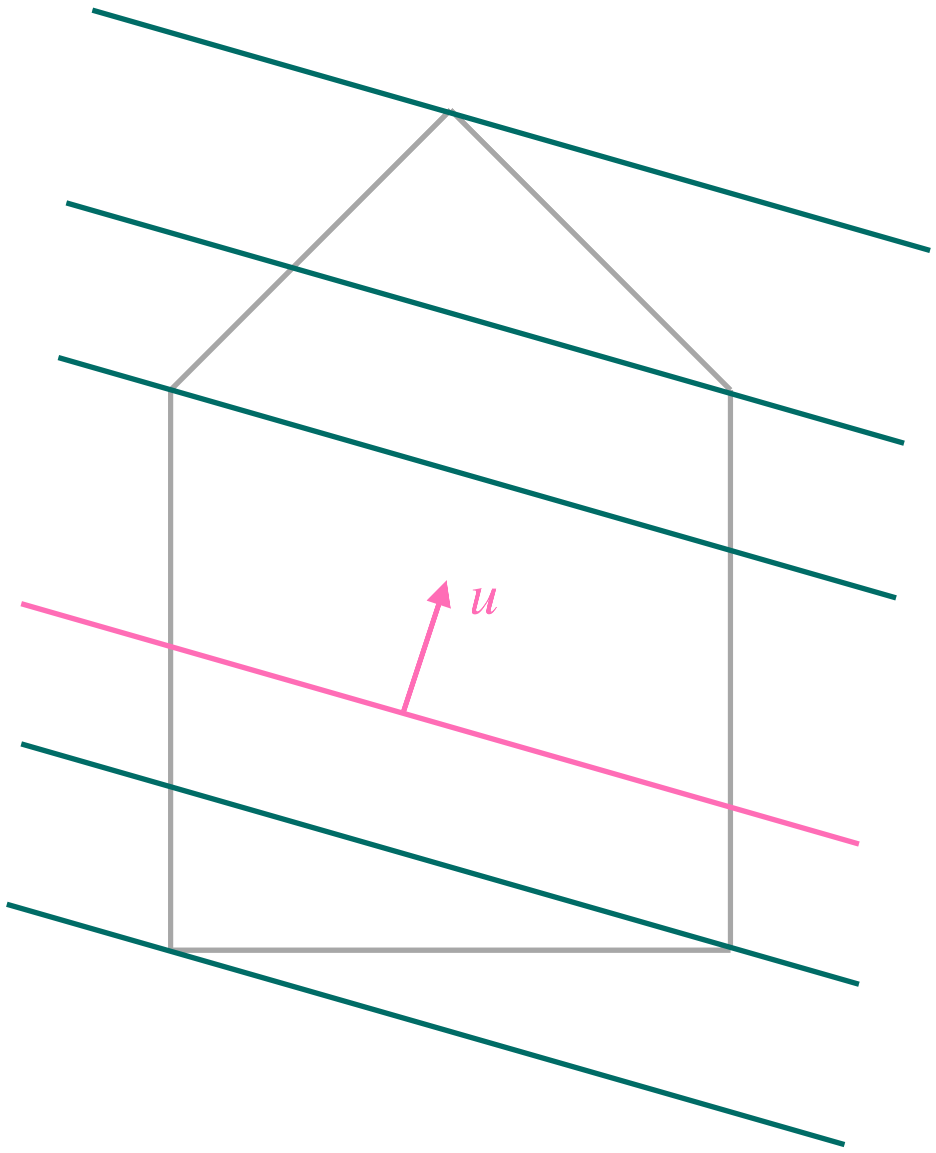
TRANSLATIONAL APPROACH

Translational Approach

Fix a normal direction $u \in S^{d-1}$

$H(\beta) = \{x \in \mathbb{R}^d \mid \langle x, u \rangle = \beta\}$

hyperplane parallel to $u^\perp$



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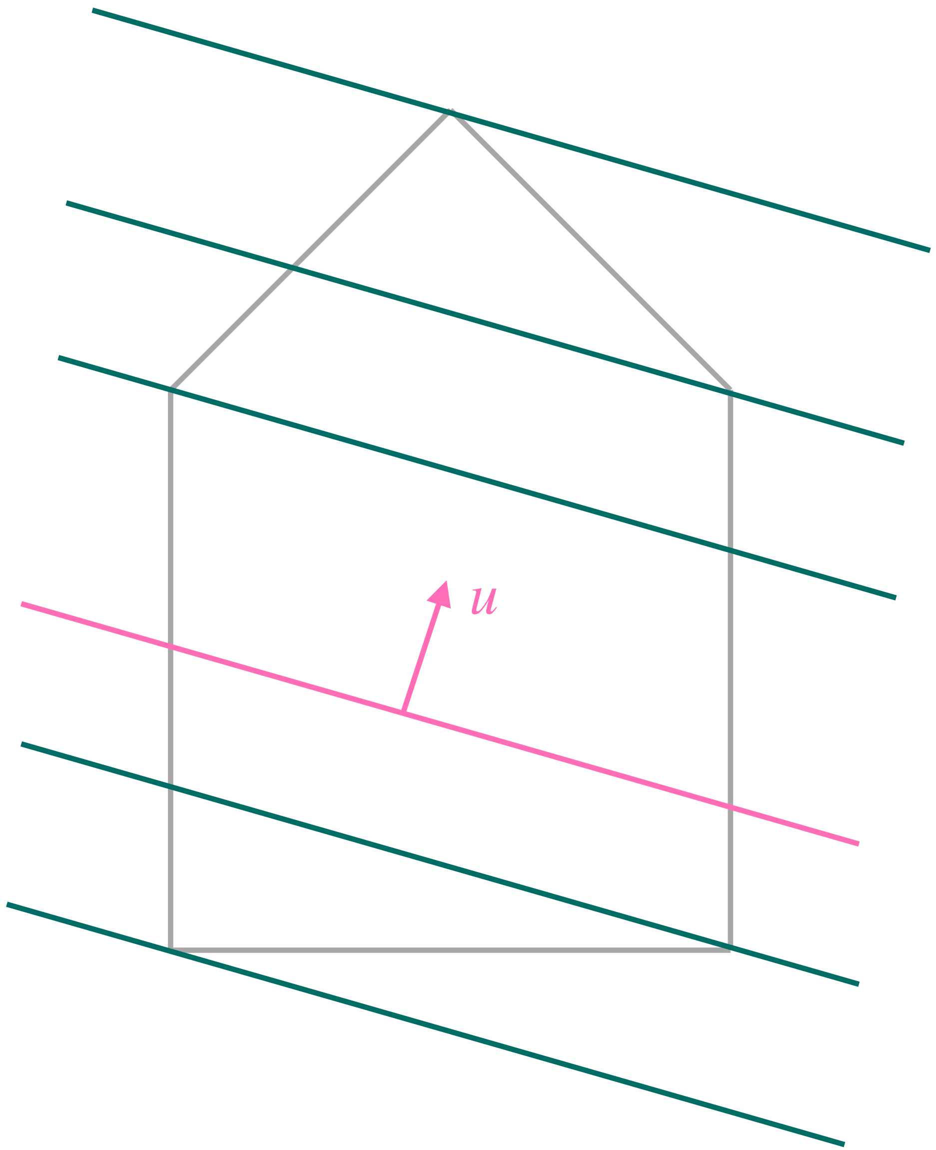
Consider the parallel hyperplane arrangement

$$\mathscr{C}^u_\uparrow(P) = \{H(\langle v, u \rangle \mid v \text{ is a vertex of } P\}.$$



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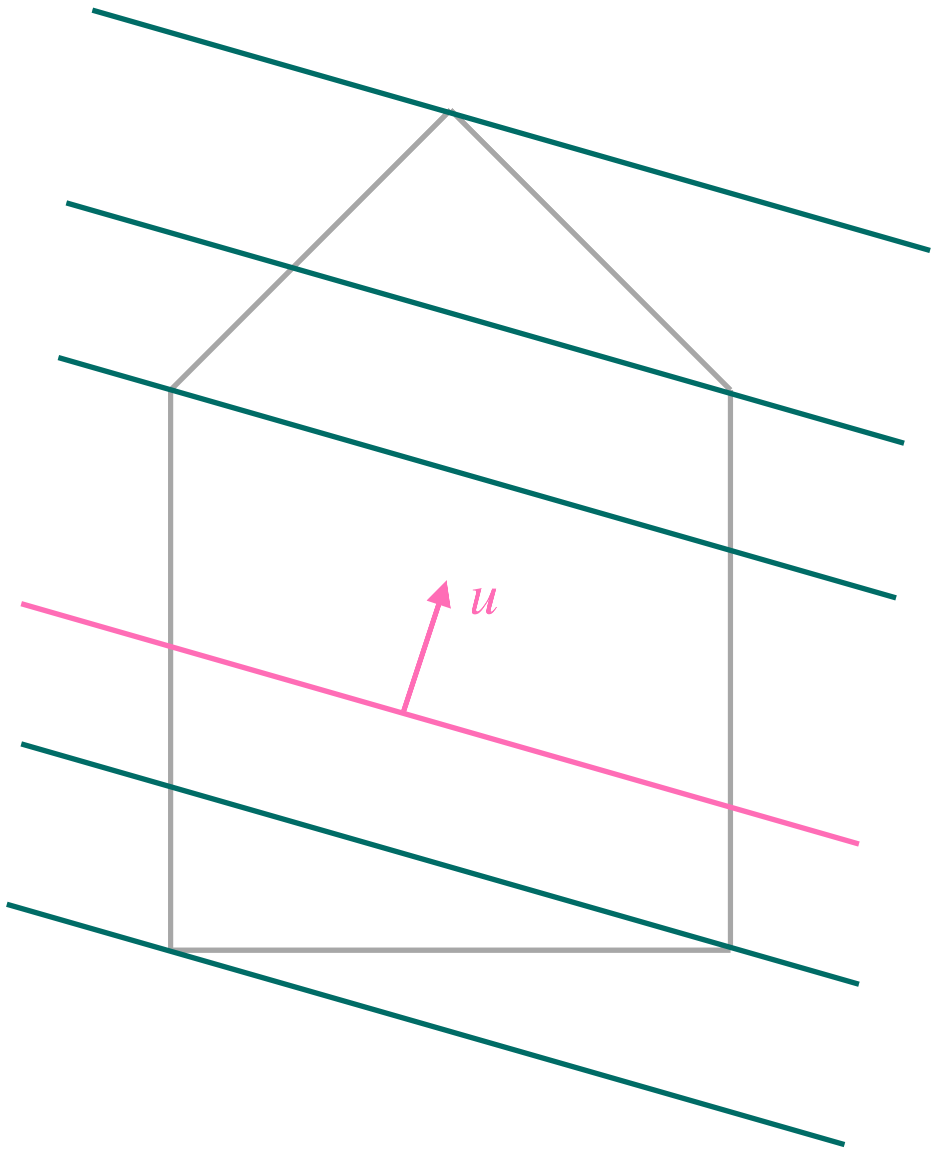
$\longrightarrow$ The combinatorial type of $P \cap H(\beta)$ is constant in each cell of $\mathscr{C}^u_\uparrow(P)$.



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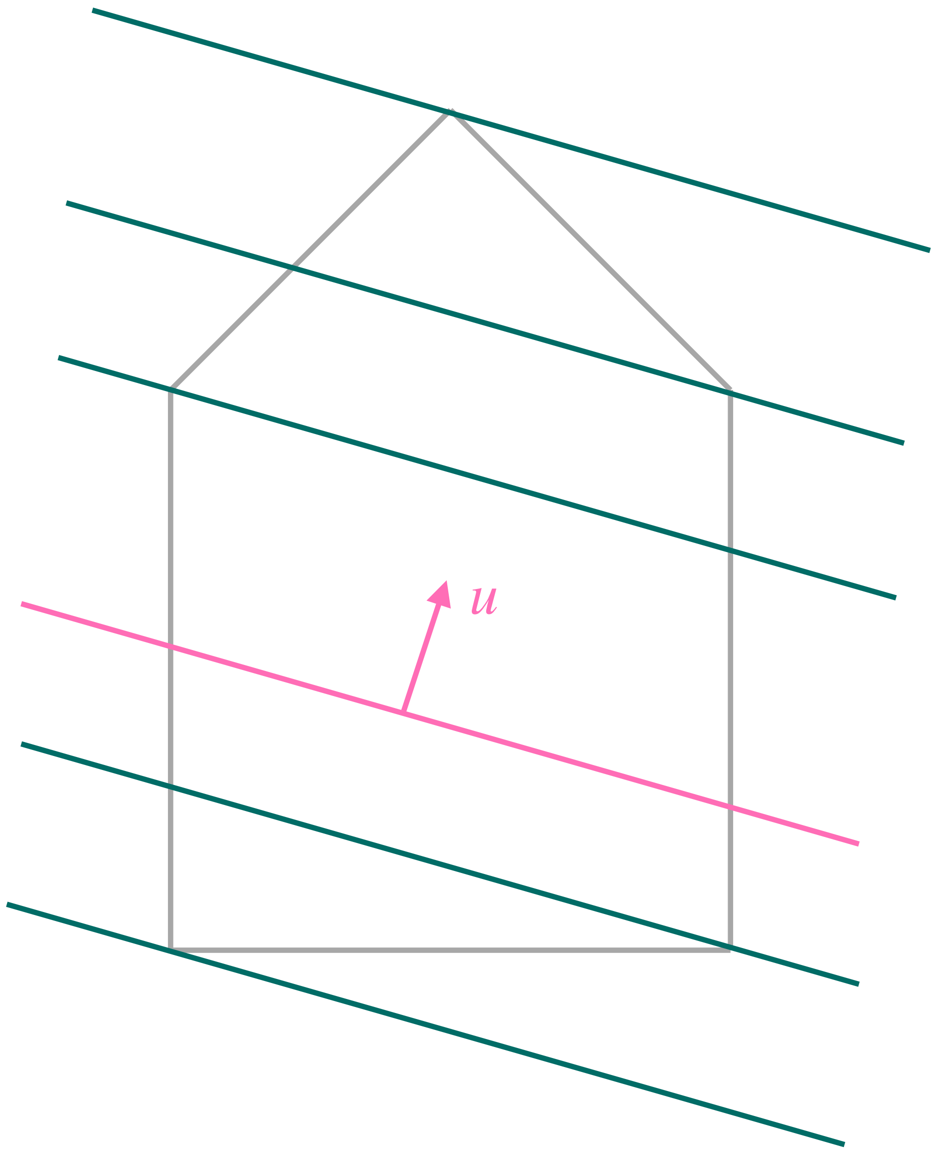
We refer to the maximal cells of $\mathscr{C}^u_\uparrow$ as chambers.



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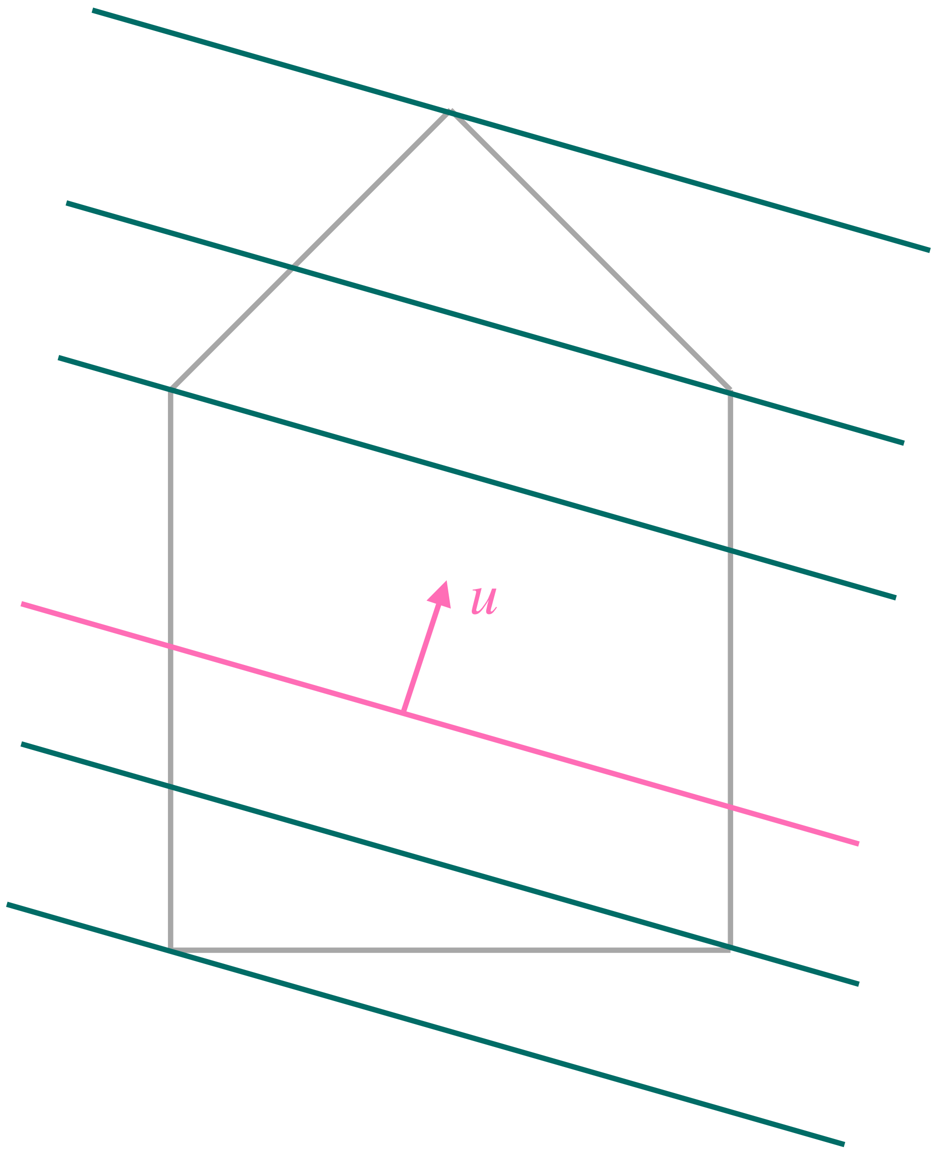
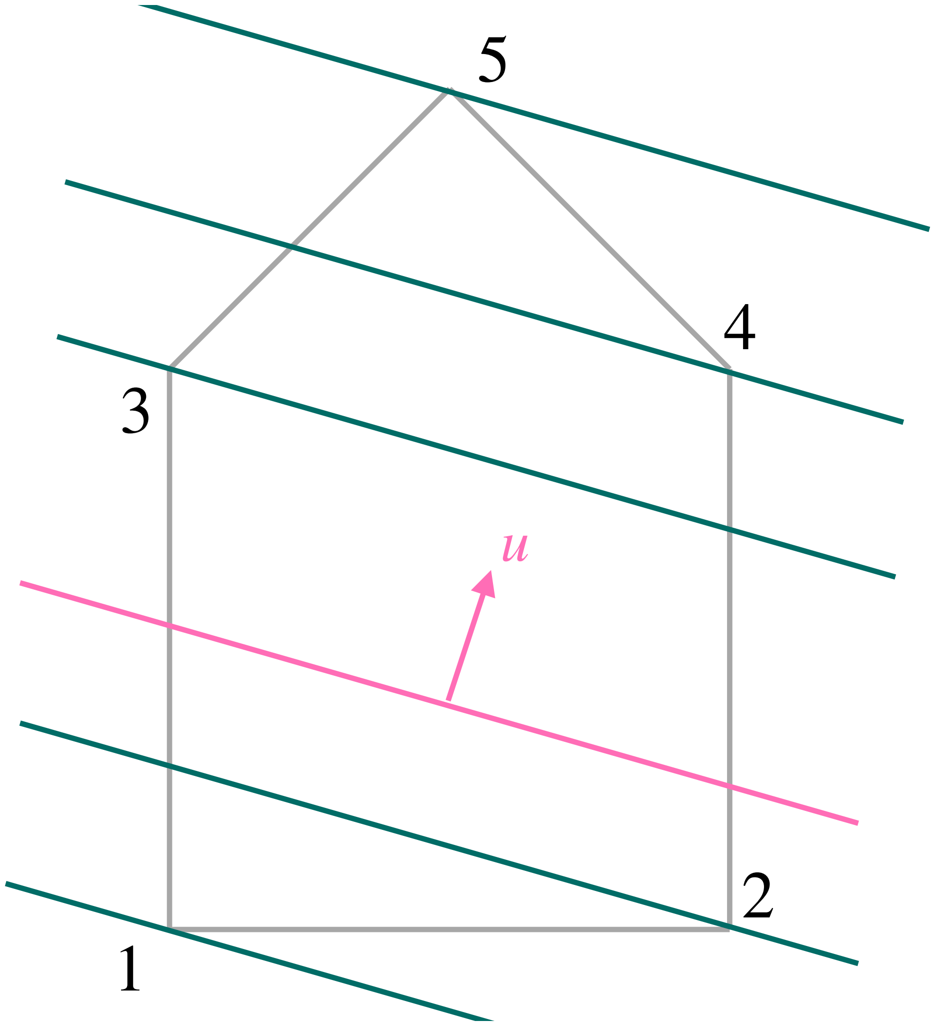
What happens if we vary the direction $u \in S^{d-1}$?



Translational Approach



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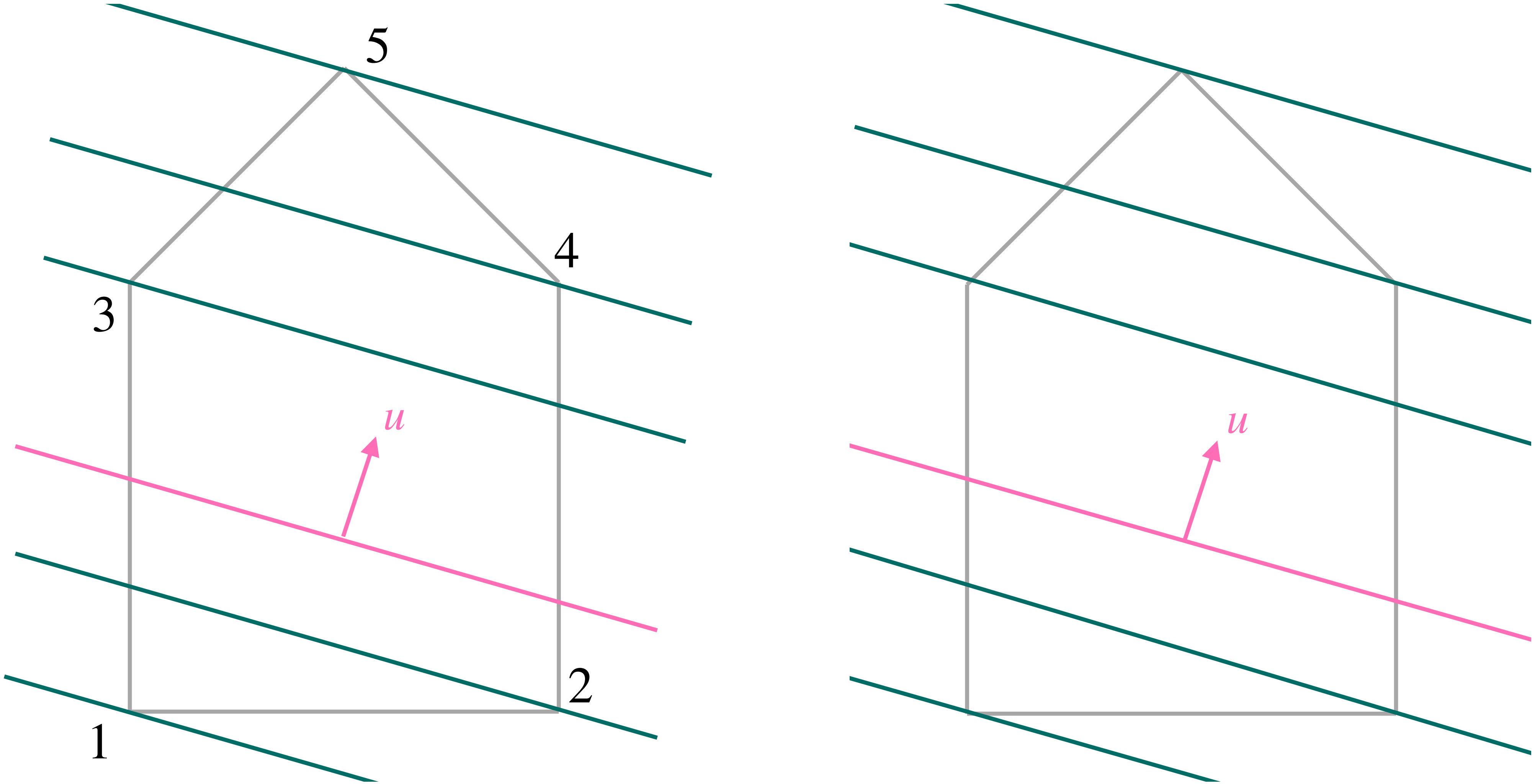

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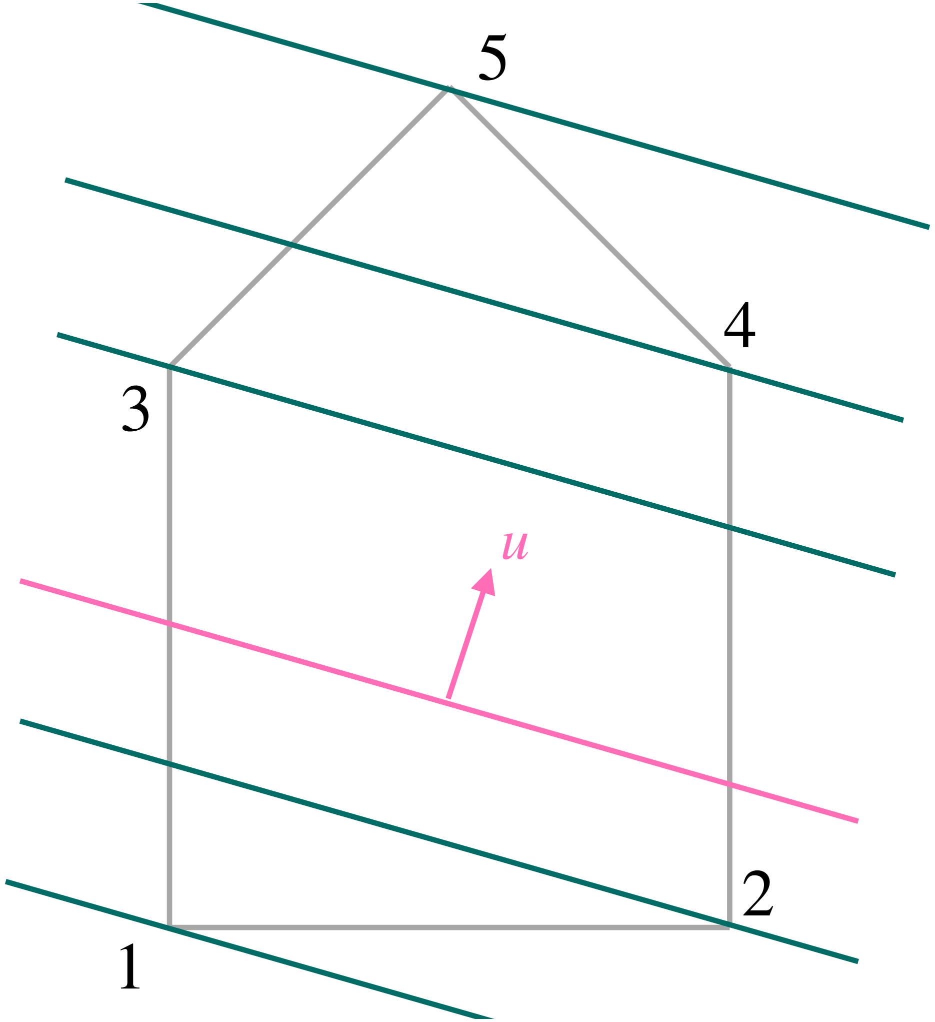
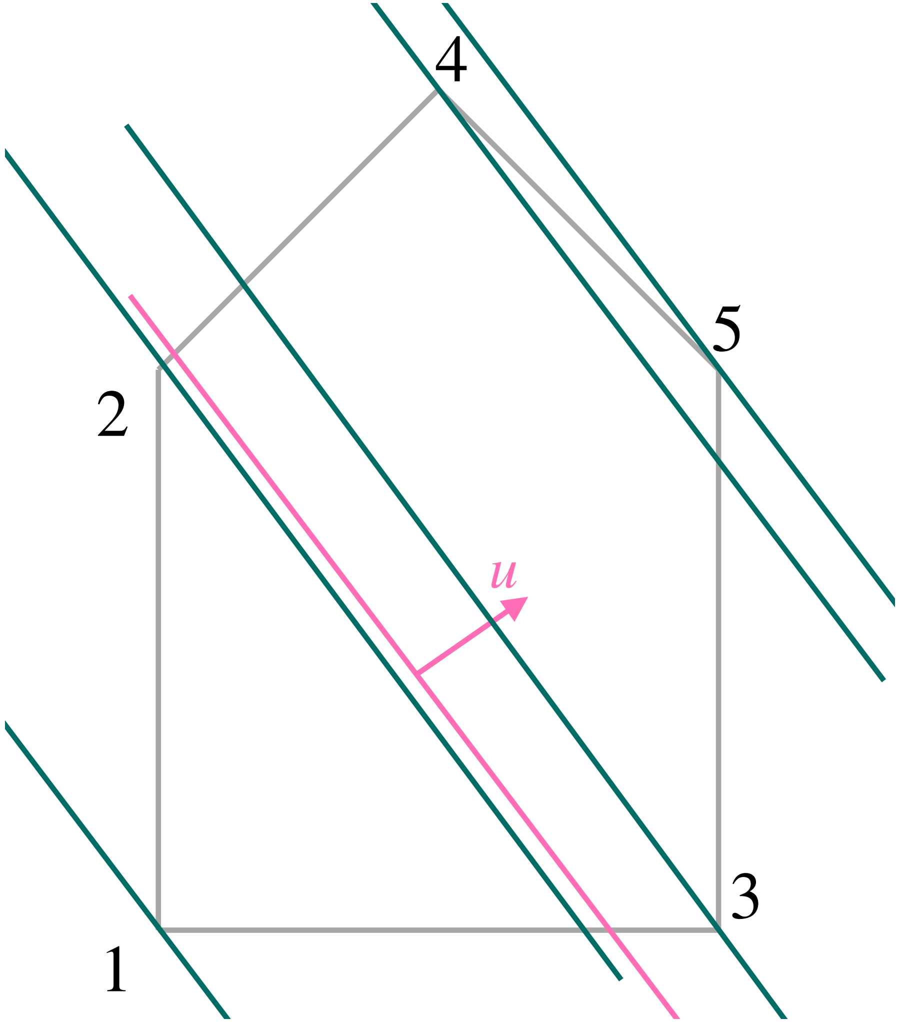




$$\mathscr{C}^u_{\uparrow}(P) = \{H(\langle v, u\rangle \mid v \text{ is a vertex of } P\}$$

Translational Approach

For which $u \in S^{d-1}$ *does* $\mathscr{C}^u_\uparrow(P)$ *induce the same ordering of the vertices?*

Translational Approach

For which $u \in S^{d-1}$ does $\mathscr{C}^u_\uparrow(P)$ induce the same ordering of the vertices?

Consider the sweep arrangement

$$\mathscr{R}_\uparrow(P) = \{(v_i - v_j)^\perp \mid v_i, v_j \text{ are vertices of } P\}$$

Translational Approach

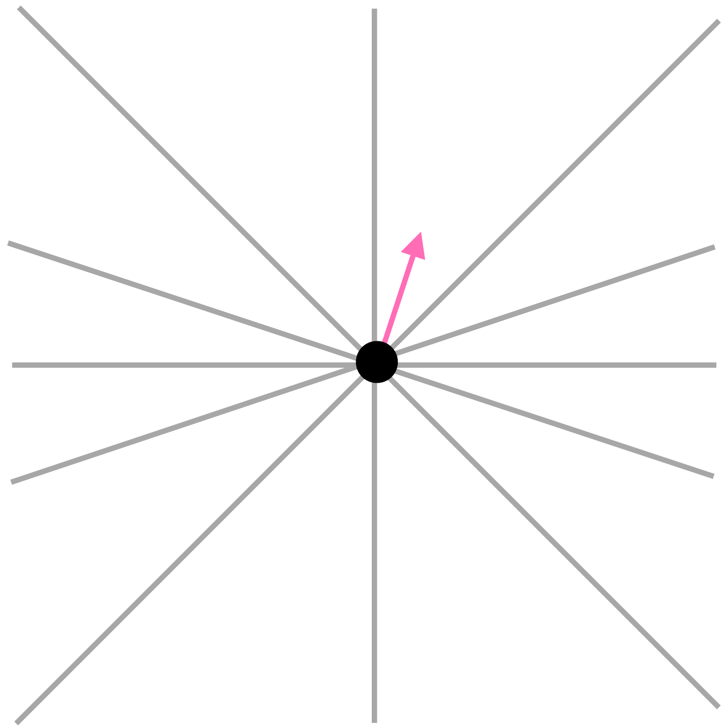
For which $u \in S^{d-1}$ does $\mathscr{C}^u_\uparrow(P)$ induce the same ordering of the vertices?

Consider the sweep arrangement

$$\mathscr{R}_\uparrow(P) = \{(v_i - v_j)^\perp \mid v_i, v_j \text{ are vertices of } P\}$$

$\longrightarrow$ with each region of $\mathscr{R}_\uparrow(P)$ the induced ordering given by $\mathscr{C}^u_\uparrow(P)$ is fixed

TRANSLATIONAL APPROACH



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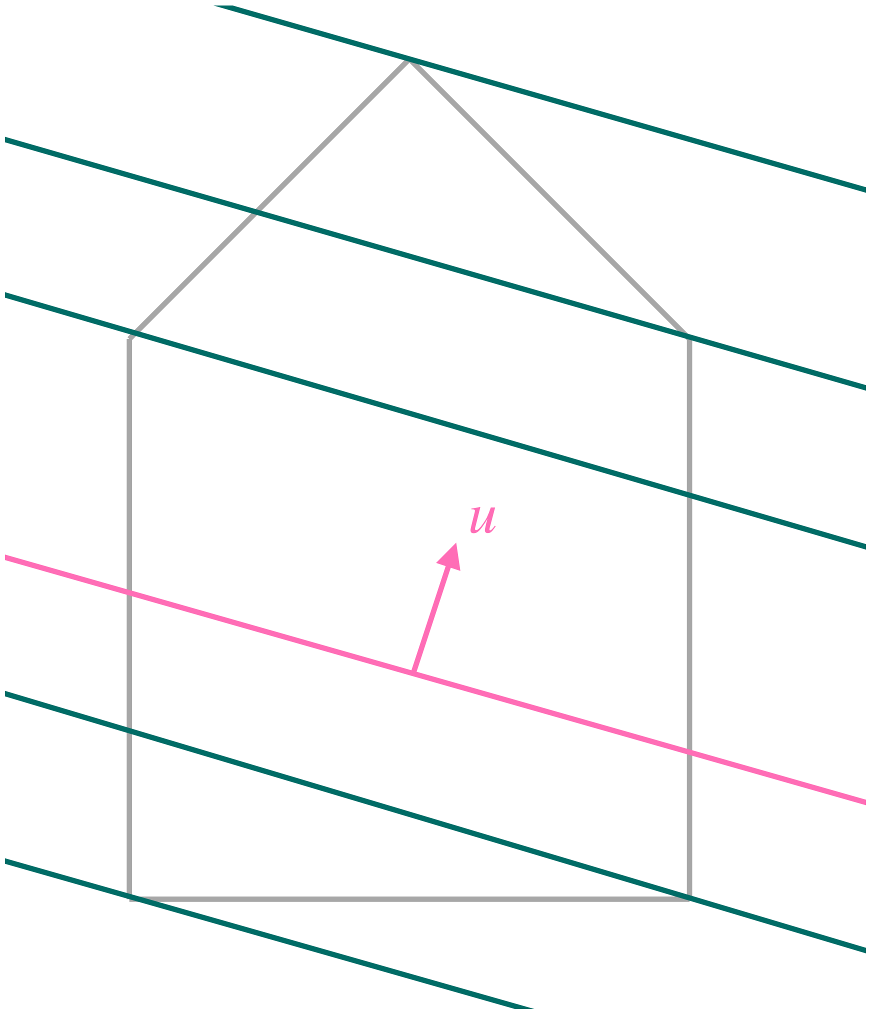

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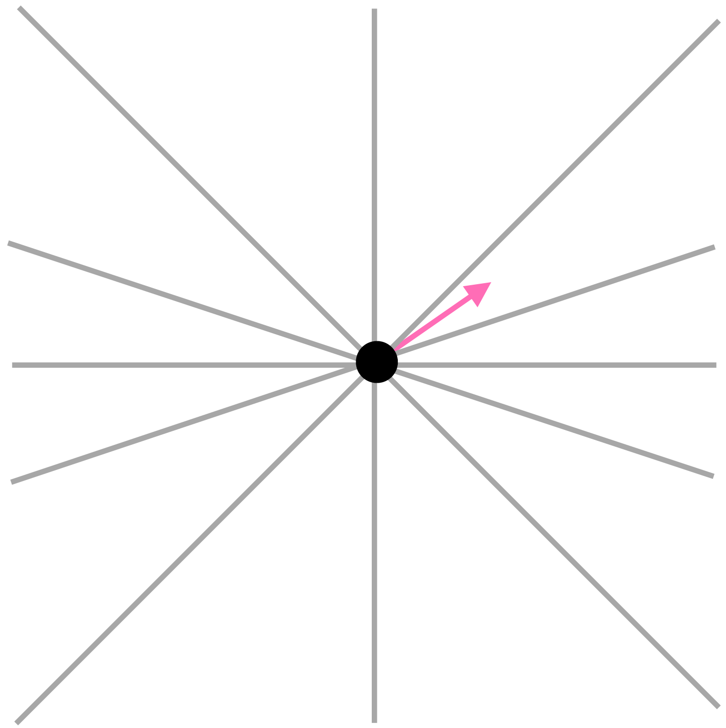
Translational Approach



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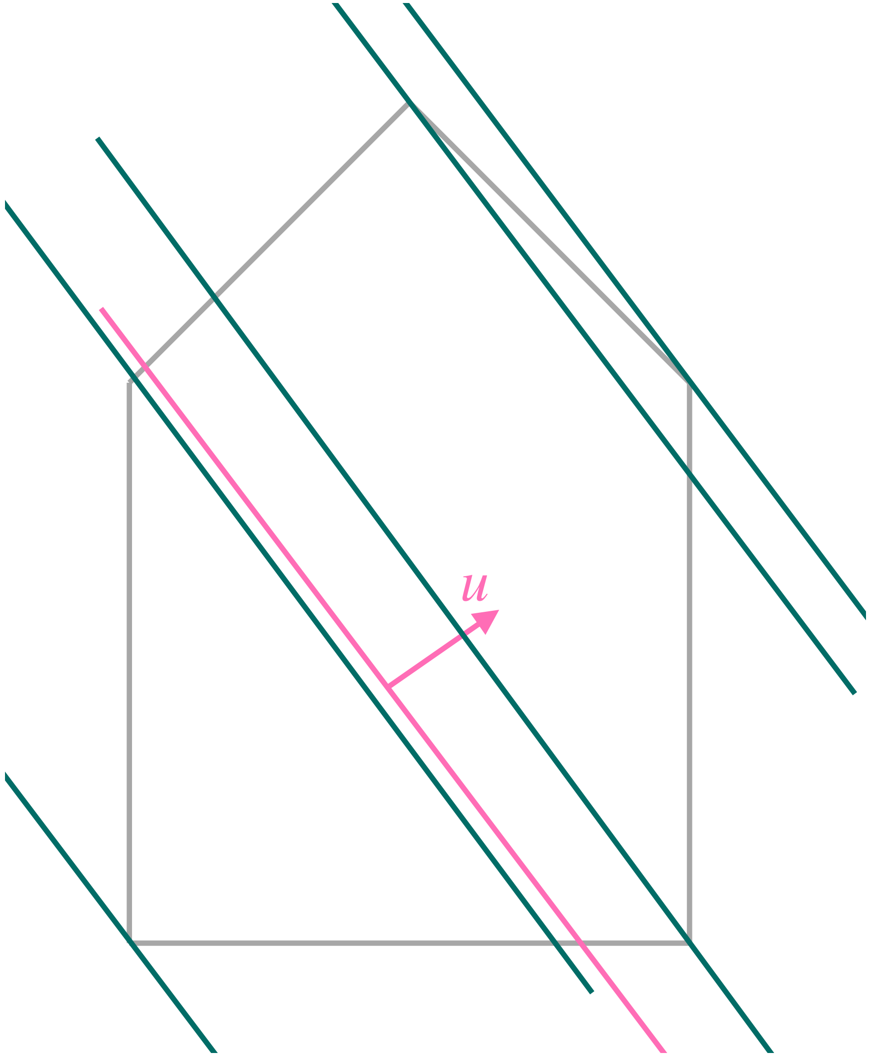


$\mathscr{C}^{u}_{\uparrow}(P) = \{H(\langle v, u\rangle \mid v \text{ is a vertex of } P\}$

Translational Approach

Theorem (B.-Meroni-De Loera ’23):

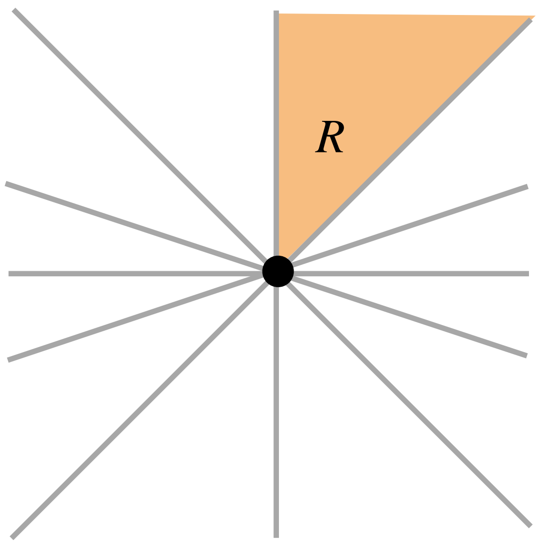
Let $P \subseteq \mathbb{R}^d$ be a polytope and $f(x) = \sum_\alpha c_\alpha x^\alpha$ be a polynomial in variables $x_1, \ldots, x_d$. Fix a region $R \in \mathscr{R}_\uparrow(P)$ of the sweep arrangement, a unit direction $u \in R \cap S^{d-1}$ and a chamber $C(u) \in \mathscr{C}^u_\uparrow(P)$ of the parallel arrangement.

Restricted to $u \in R \cap S^{d-1}$ and $H(\beta) \in C(u)$, the integral

$$\int_{P \cap H(\beta)} f(x)\ \mathrm{d}x$$

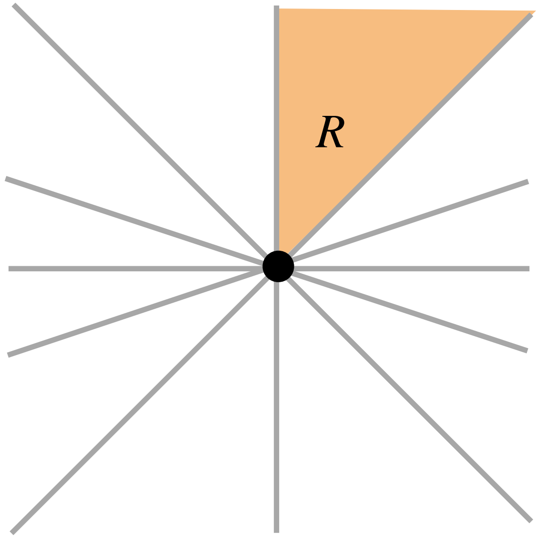
is a rational function in variables $u_1, \ldots, u_d, \beta$.

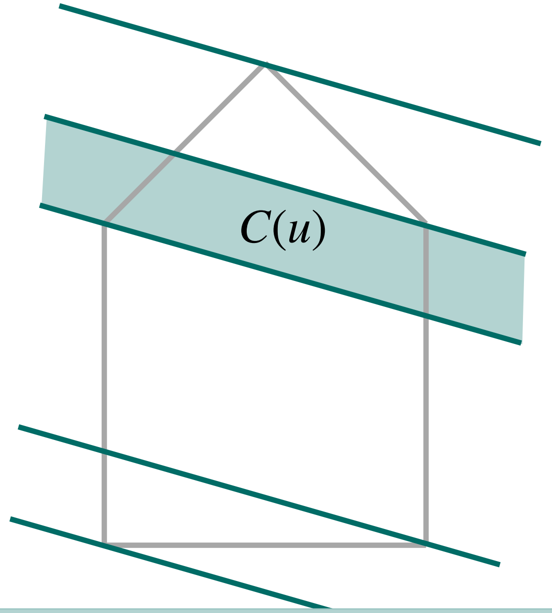
Translational Approach



$$(u_1, u_2) \in R \iff \begin{array}{r} u_1 \geq 0 \\ u_1 - u_2 \leq 0 \end{array}$$

Translational Approach





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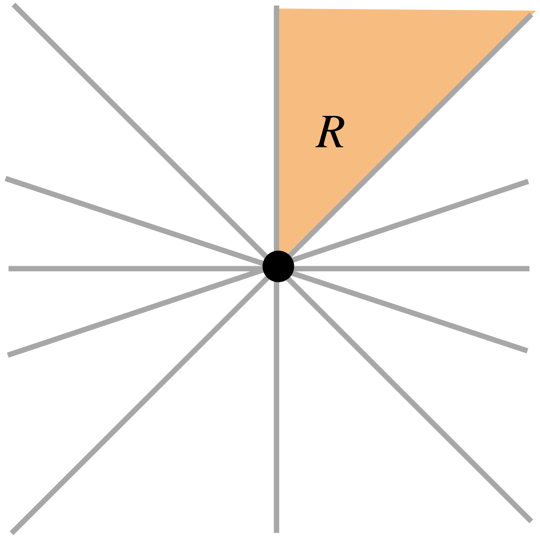
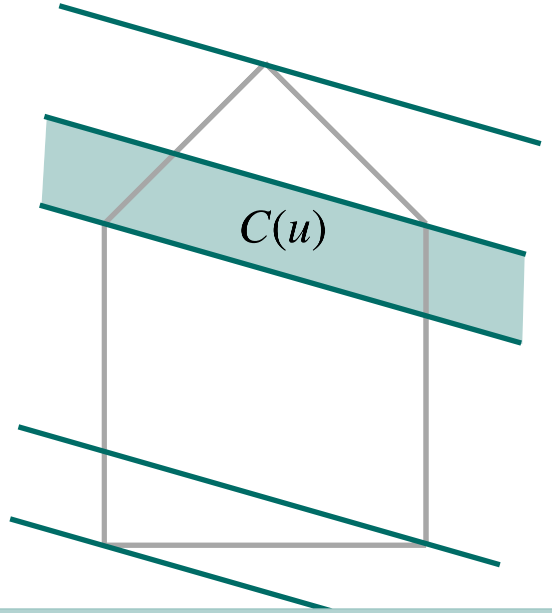
$u_1 - u_2 \leq \beta \leq -u_1 + u_2$

Translational Approach





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Let the computer find the biggest slice:

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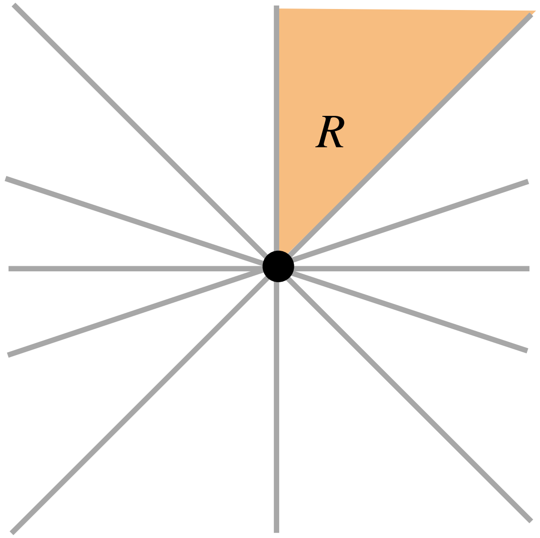

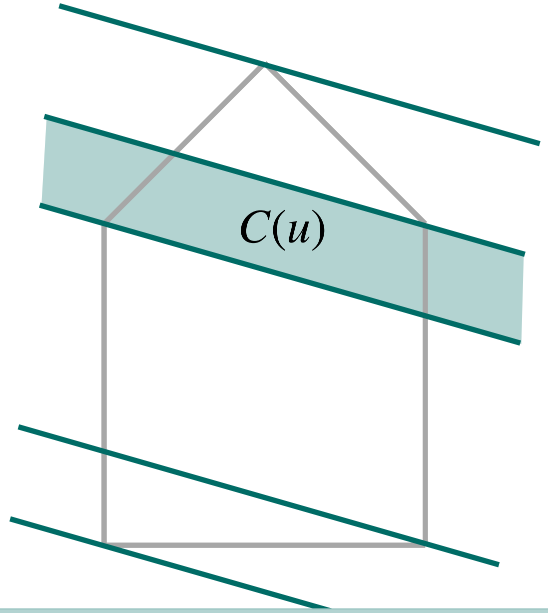

$(u_1, u_2) \in R \iff u_1 \geq 0$, $u_1 - u_2 \leq 0$

If $(u_1, u_2) \in R \cap S^{d-1}$ then $\beta \in C(u) \iff$ $u_1 - u_2 \leq \beta \leq -u_1 + u_2$

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Translational Approach

Let the computer find the biggest slice:

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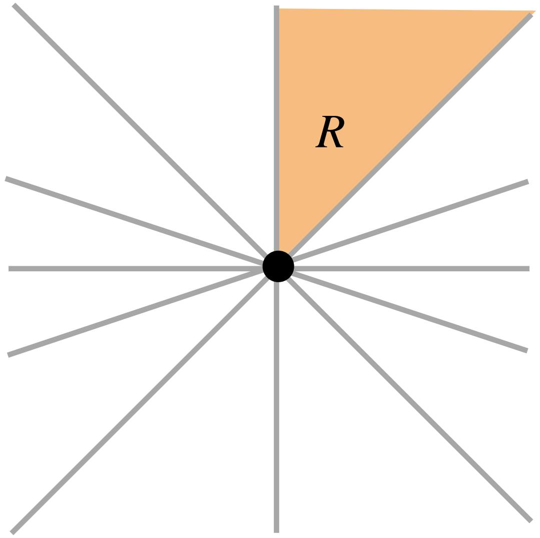

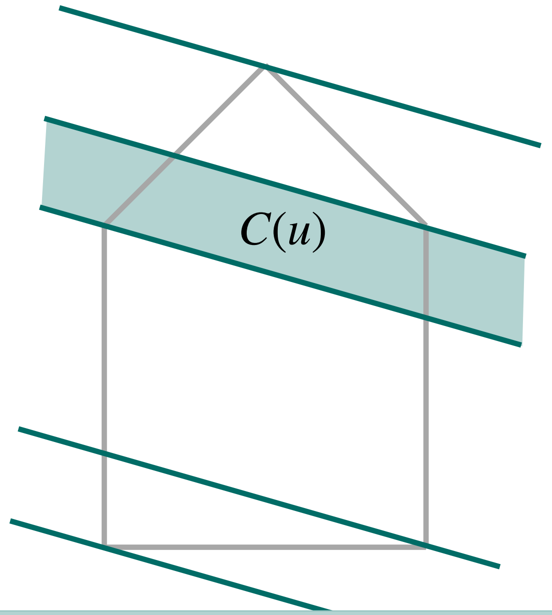

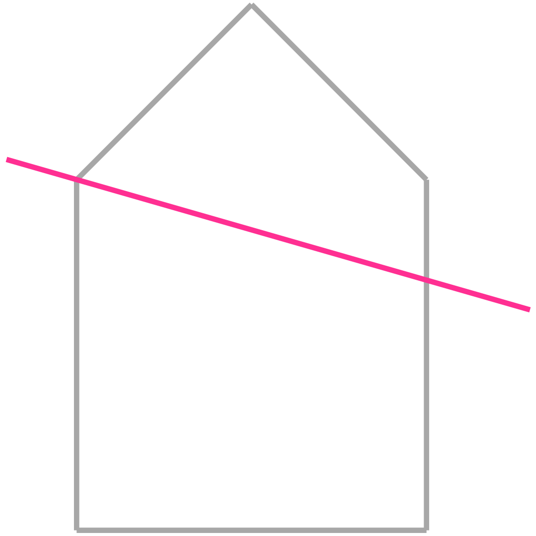

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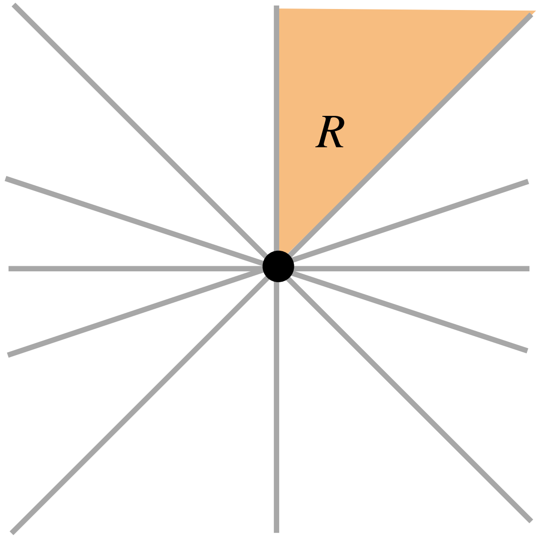

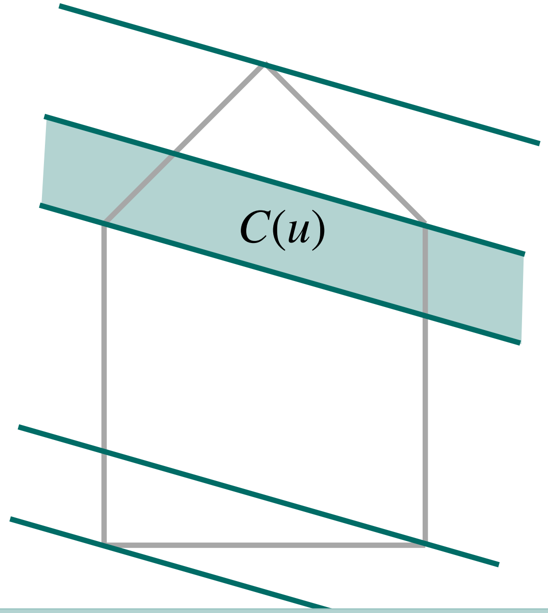

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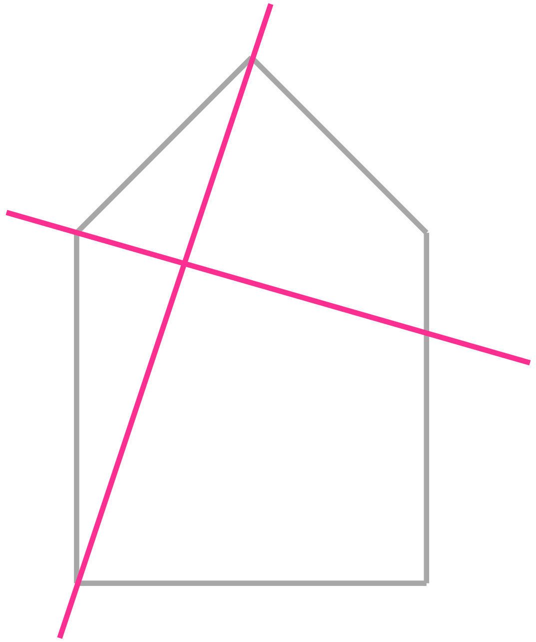

ROTATION VS TRANSLATION

COMPARISON OF THE APPROACHES

Comparison

Running time of the algorithm $\longleftrightarrow$ number of chambers in the arrangements

$n =$ # vertices of P

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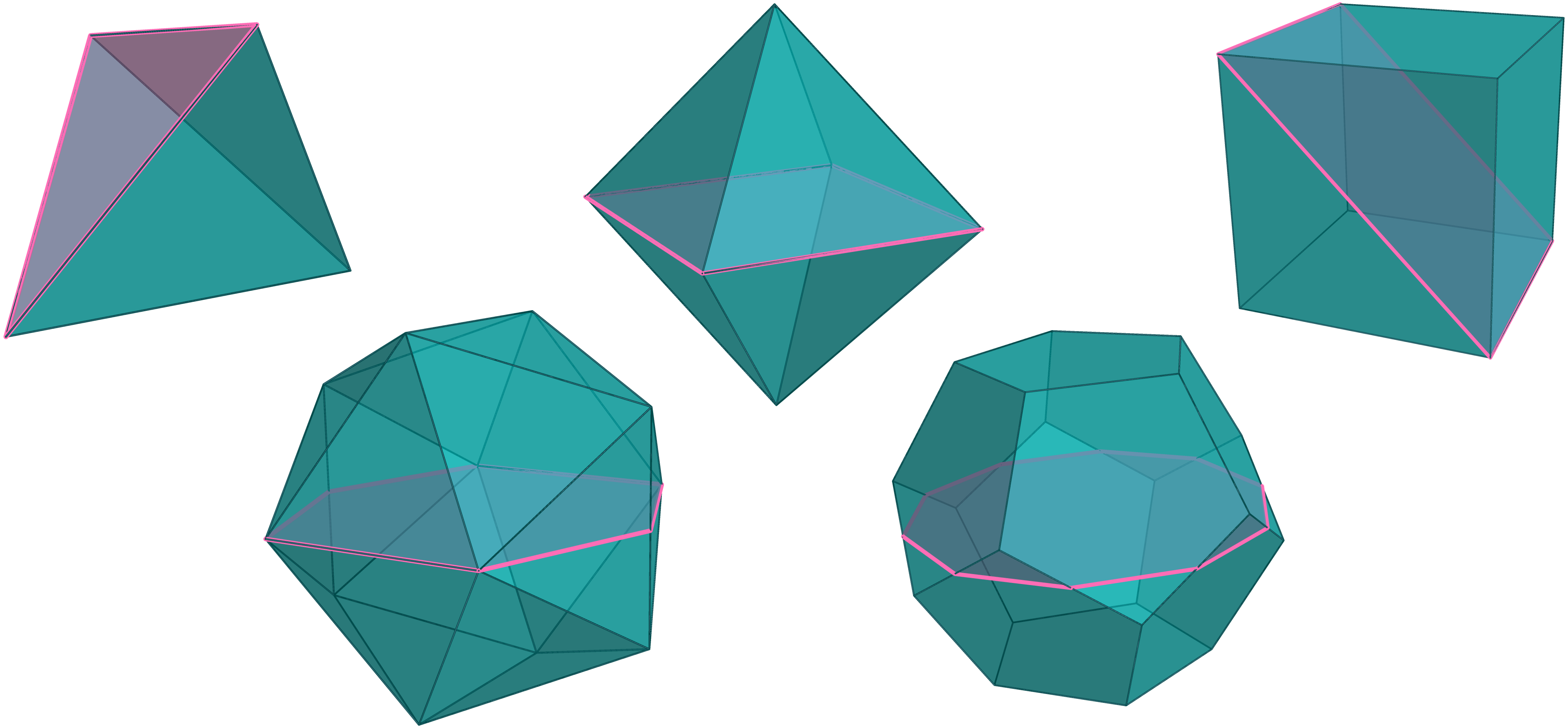
$\longrightarrow$ both approaches yield algorithms in polynomial running time

$\longrightarrow$ Translational approach runs much faster

Maximum Volume Slices of Platonic solids



VARIATIONS

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WITH THE SAME METHODS WE CAN UNDERSTAND…

- Intersections with half-spaces
- Projections onto hyperplanes
- Combinatorial types

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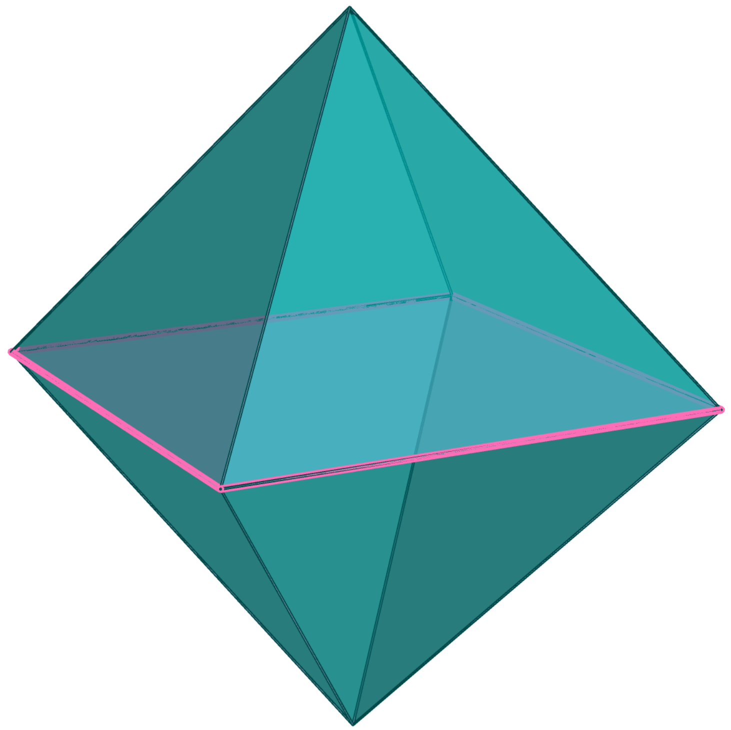
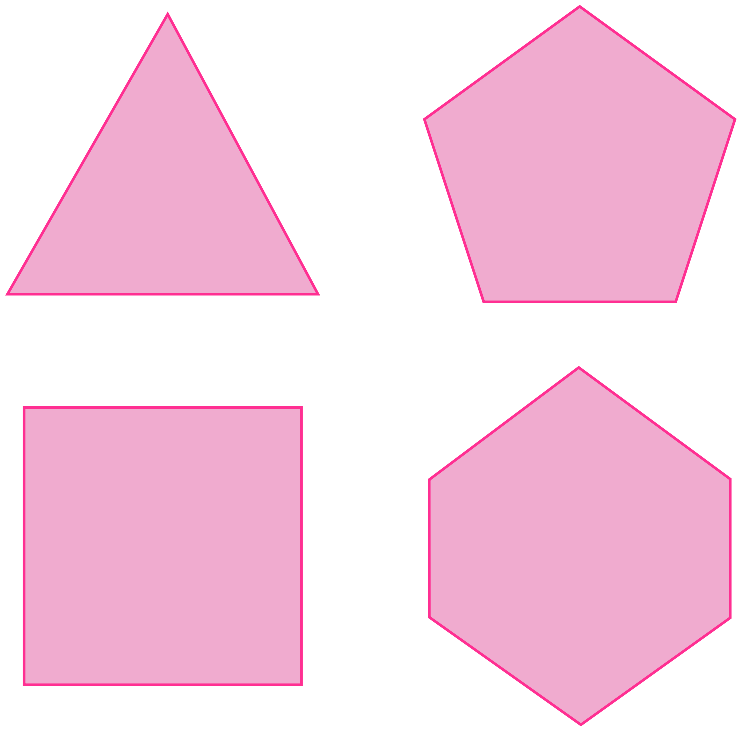
Combinatorial Types of Sections of the Cross-Polytope

$P = \text{conv}(\pm e_i \mid i \in [d])$

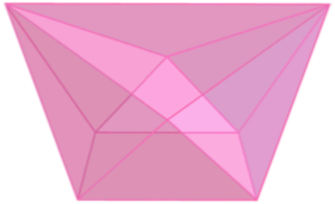
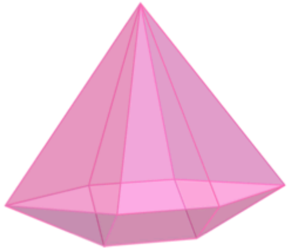
$d = 3$





Combinatorial Types of Sections of the Cross-Polytope

$P = \text{conv}(\pm e_i \mid i \in [d])$ $d = 4$

$P \cap H$					
f-vector	$(4, 6, 4)$	$(6, 12, 8)$	$(8, 18, 12)$	$(8, 17, 11)$	$(9, 19, 12)$
H	$x_1 + x_2 + x_3 + x_4 = 1$	$2x_1 = 1$	$x_1 + x_2 + x_3 = 0$	$2x_1 + 2x_2 + x_3 + x_4 = 1$	$2x_1 + 2x_2 + x_3 = 1$

$P \cap H$				
f-vector	$(8, 18, 12)$	$(10, 21, 13)$	$(12, 24, 14)$	$(12, 24, 14)$
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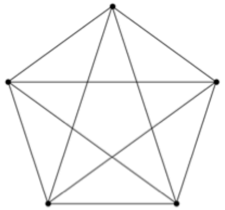
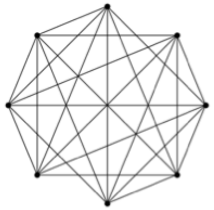
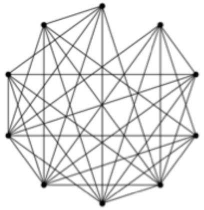
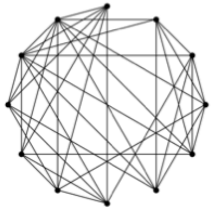
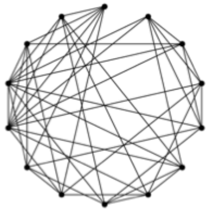
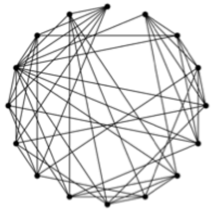
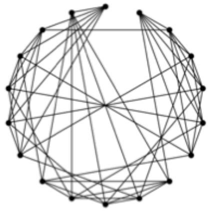
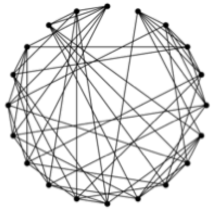
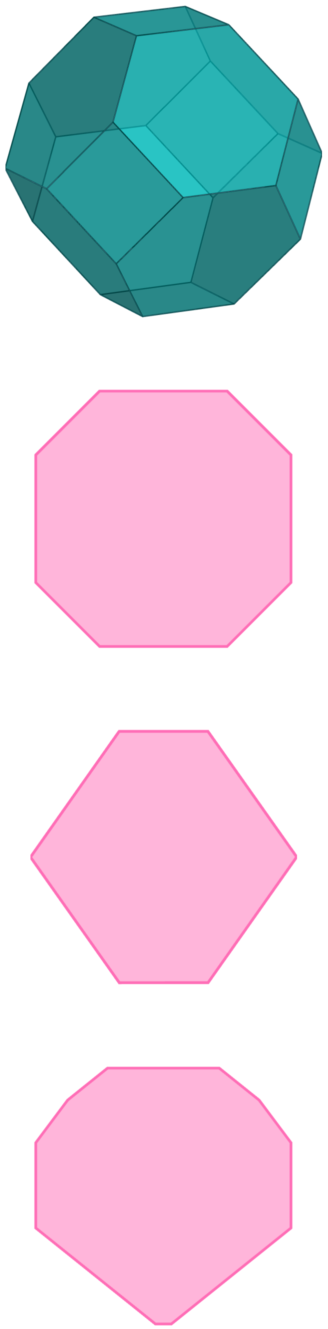
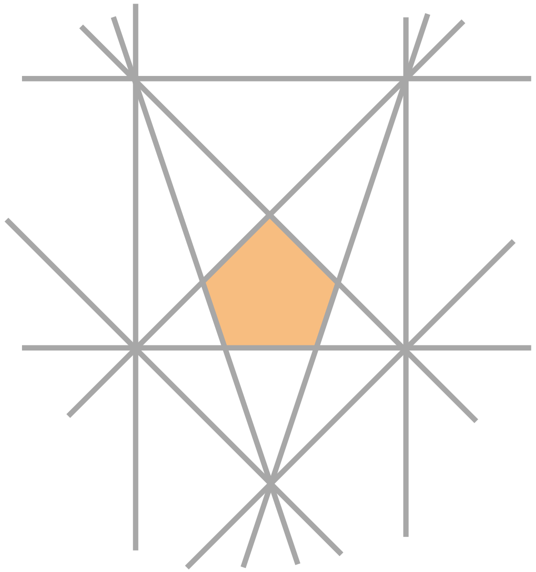
Combinatorial Types of Sections of the Cross-Polytope

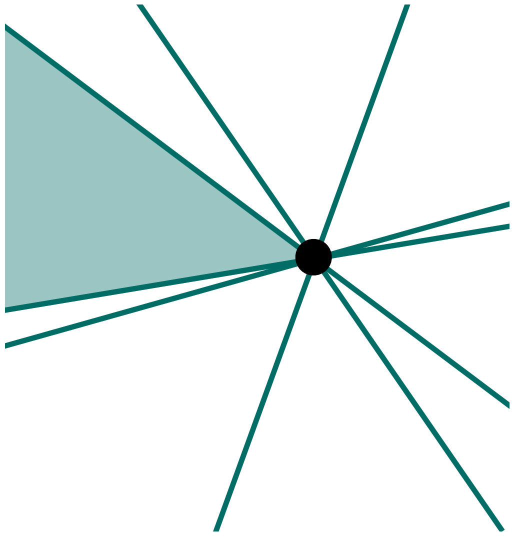
$P = \text{conv}(\pm e_i \mid i \in [d])$ $\qquad$ $d = 5$

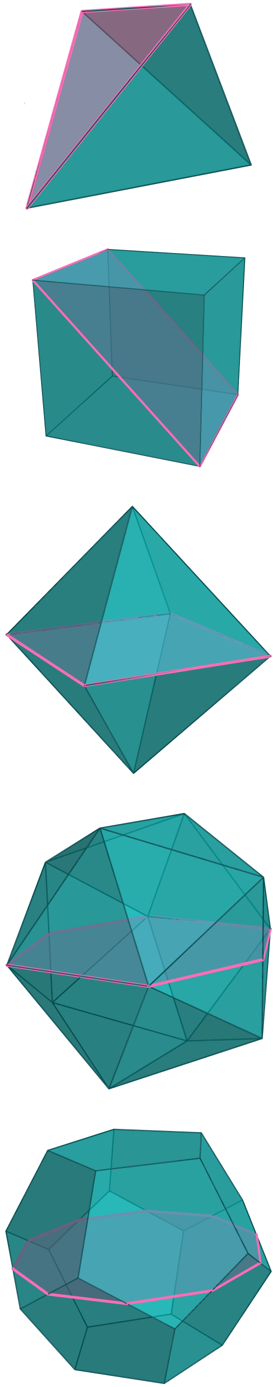
$P \cap H$								
f-vector	$(5, 10, 10, 5)$	$(8, 24, 32, 16)$	$(10, 34, 48, 24)$	$(11, 36, 48, 23)$	$(12, 39, 51, 24)$	$(13, 41, 52, 24)$	$(14, 42, 52, 24)$	$(14, 48, 62, 28)$
H	$x_1 + x_2 + x_3$ $+x_4 + x_5 = 1$	$2x_1 = 1$	$x_1 + x_2$ $+x_3 = 0$	$2x_1 + 2x_2 + x_3$ $+x_4 + x_5 = 1$	$2x_1 + 2x_2$ $+x_3 + x_4 = 1$	$2x_1 + 2x_2$ $+x_3 = 1$	$2x_1 + 2x_2 = 1$	$x_1 + x_2$ $+x_3 + x_4 = 0$

$P \cap H$						
f-vector	$(14, 46, 59, 27)$	$(16, 51, 63, 28)$	$(17, 54, 66, 29)$	$(18, 54, 64, 28)$	$(20, 60, 70, 30)$	$(20, 60, 70, 30)$
H	$2x_1 + 2x_2 + 2x_3$ $+x_4 + x_5 = 1$	$2x_1 + 2x_2$ $+2x_3 + x_4 = 1$	$2x_1 + 2x_2 + 2x_3$ $+2x_4 + x_5 = 1$	$2x_1 + 2x_2$ $+2x_3 = 1$	$2x_1 + 2x_2 + 2x_3 + 2x_4 = 1$	$x_1 + x_2 + x_3 + x_4 + x_5 = 0$









THANK YOU!