MAX PLANCK INSTITUTE
FOR MATHEMATICS IN THE SCIENCES

The Best Ways to Slice a Polytope

joint work with Chiara Meroni and Jesús A. De Loera.
arXiv: 2304.14239

Marie-Charlotte Brandenburg

Nonlinear Algebra Seminar
MPI MiS Leipzig
June 05, 2023

Joint work with

Chiara Meroni

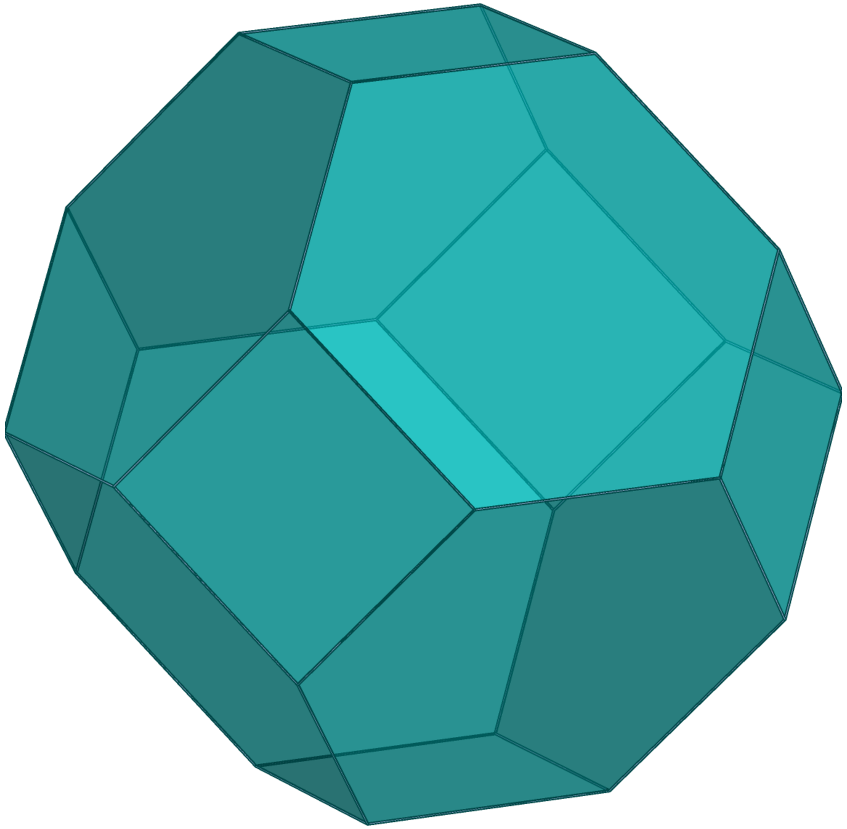
ICERM
MPI MiS Leipzig

Jesús A. De Loera

UC Davis

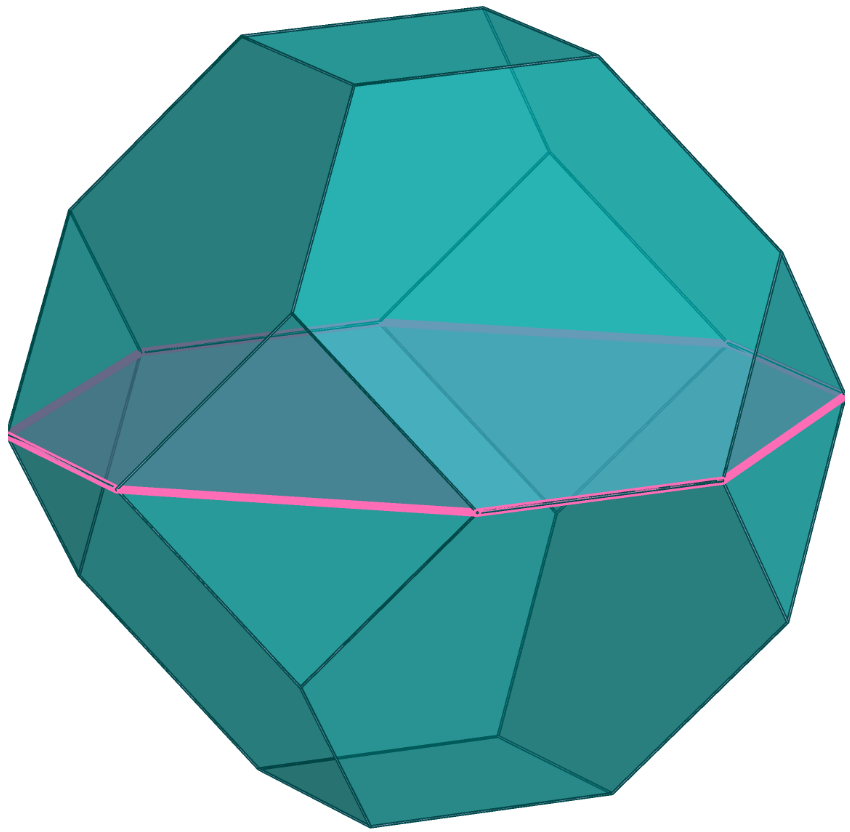
MB, Chiara Meroni, and Jesús A. De Loera. *The Best Ways to Slice a Polytope*. 2023. arXiv: 2304.14239

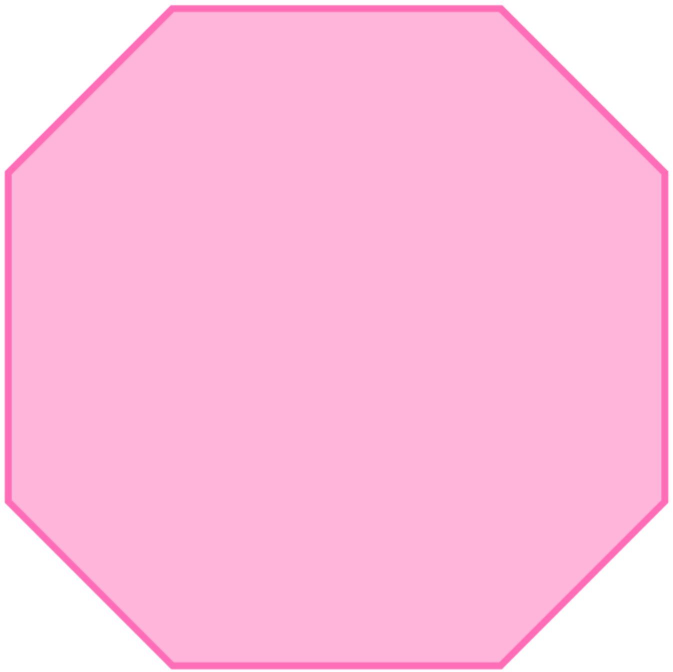
Slices of the Permutohedron



$$P = \text{conv}(\ (\sigma(1), \sigma(2), \sigma(3), \sigma(4))\ \mid \sigma \in S_4)$$
$$= \text{conv}(\ (1,2,3,4),\ (1,2,4,3),\ \ldots,\ (4,3,2,1)\ )$$

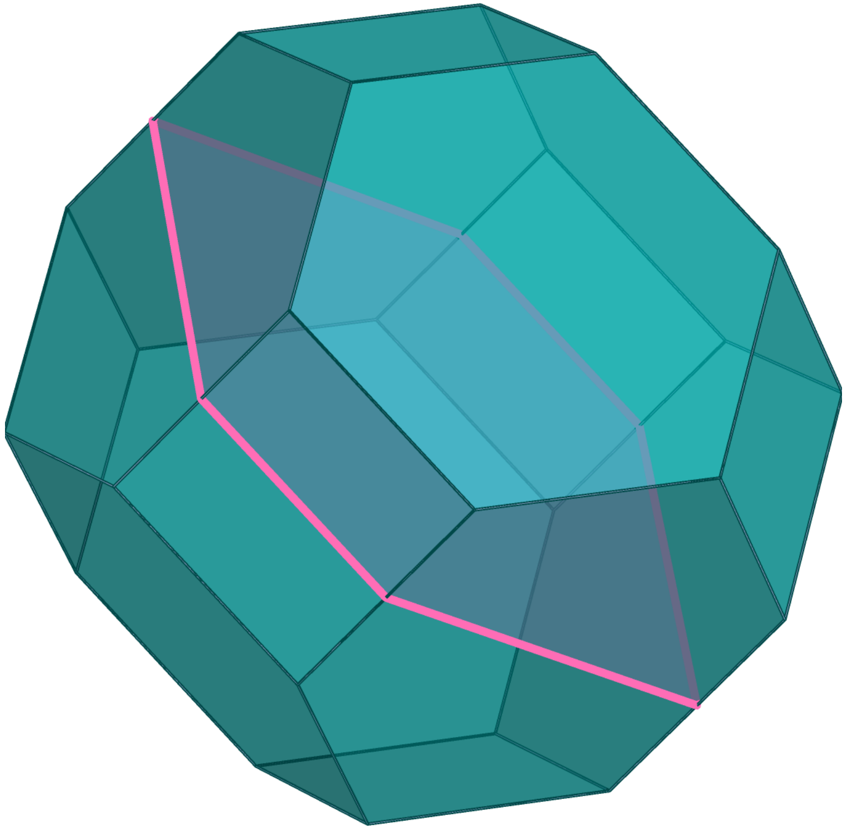
Slices of the Permutohedron

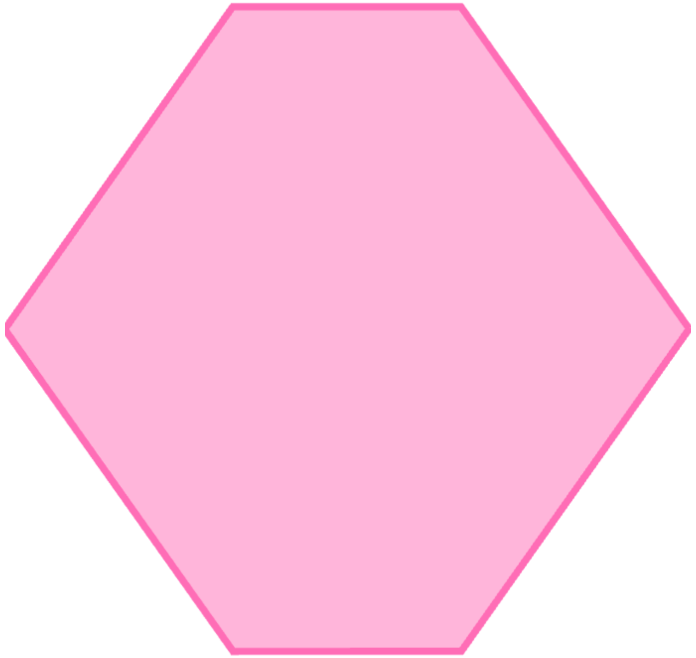




Affine slice of maximum volume

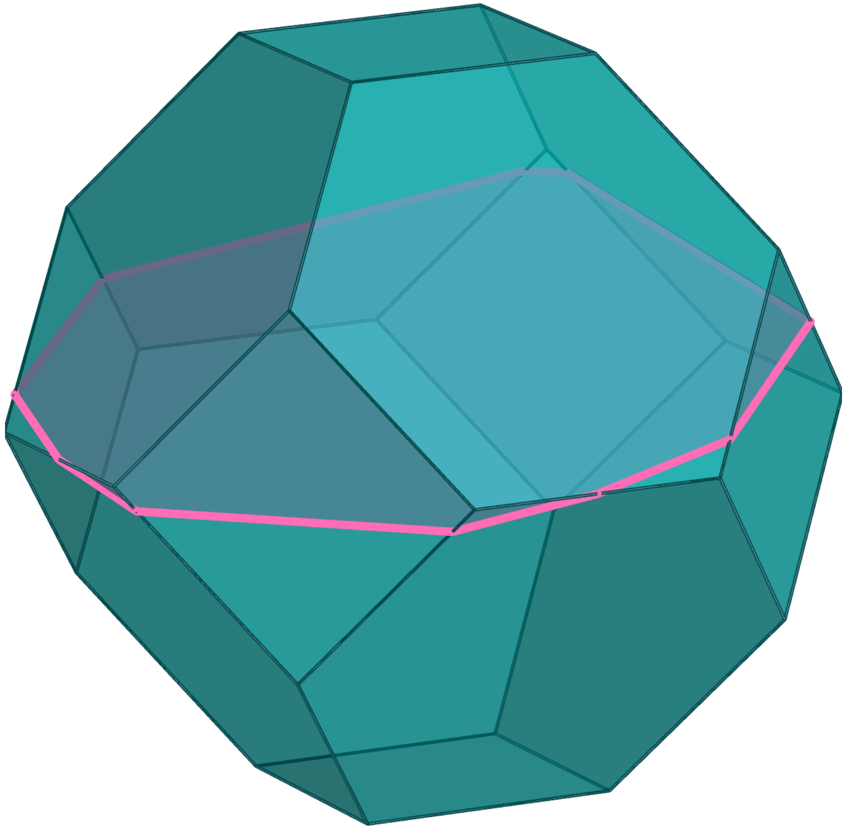
Slices of the Permutohedron

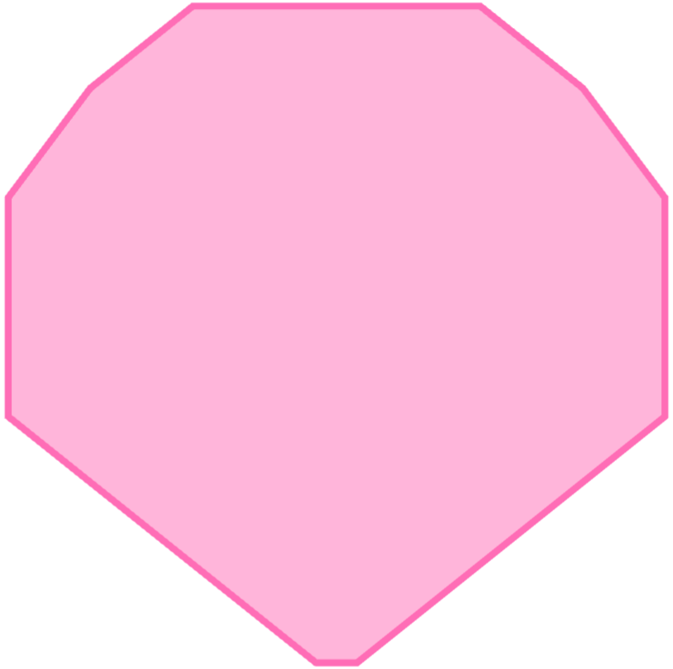




Central slice of minimum volume

Slices of the Permutohedron





Affine slice with maximum number of vertices

WHY DO WE WANT TO COMPUTE (EXTREMAL) SLICES OF POLYTOPES?

Motivation

- **Maximal volume slice**: What is the slice of P with maximal volume?

  [Ball '89, Meyer-Pajor '88, Pournin '22, Webb '96, …]

Motivation

- **Maximal volume slice**: What is the slice of P with maximal volume?

  [Ball '89, Meyer-Pajor '88, Pournin '22, Webb '96, …]

- **Bourgain's slicing problem**: Does there exist $c > 0$ such that for any convex body $K \subset \mathbb{R}^d$ with vol$(K) = 1$ there exist a hyperplane H such that vol$(K \cap H) > c$?

  [Bourgain '84, Klartag '23, Koldobsky-Milman '22,…]

Motivation

- **Maximal volume slice**: What is the slice of P with maximal volume?
  [Ball '89, Meyer-Pajor '88, Pournin '22, Webb '96, …]

- **Bourgain's slicing problem**: Does there exist $c > 0$ such that for any convex body $K \subset \mathbb{R}^d$ with vol$(K) = 1$ there exist a hyperplane H such that vol$(K \cap H) > c$?
  [Bourgain '84, Klartag '23, Koldobsky-Milman '22,…]

- How does the **h-vector** of P compare with the h-vector of a generic hyperplane section of P?
  [Khovanskii '06]

Motivation

- **Maximal volume slice**: What is the slice of P with maximal volume?
  [Ball '89, Meyer-Pajor '88, Pournin '22, Webb '96, …]

- **Bourgain's slicing problem**: Does there exist $c > 0$ such that for any convex body $K \subset \mathbb{R}^d$ with vol$(K) = 1$ there exist a hyperplane H such that vol$(K \cap H) > c$?
  [Bourgain '84, Klartag '23, Koldobsky-Milman '22,…]

- How does the **h-vector** of P compare with the h-vector of a generic hyperplane section of P?
  [Khovanskii '06]

- **Densest hemisphere problem**: Given points on the sphere, how can we find the hemisphere with the most points?
  [Johnson-Preparata '78,…]

Motivation

- **Maximal volume slice**: What is the slice of P with maximal volume?

  [Ball '89, Meyer-Pajor '88, Pournin '22, Webb '96, …]

- **Bourgain's slicing problem**: Does there exist $c > 0$ such that for any convex body $K \subset \mathbb{R}^d$ with vol$(K) = 1$ there exist a hyperplane H such that vol$(K \cap H) > c$?

  [Bourgain '84, Klartag '23, Koldobsky-Milman '22,…]

- How does the **h-vector** of P compare with the h-vector of a generic hyperplane section of P?

  [Khovanskii '06]

- **Densest hemisphere problem**: Given points on the sphere, how can we find the hemisphere with the most points?

  [Johnson-Preparata '78,…]

- **Slices of the permutahedron** fixed under a certain group action

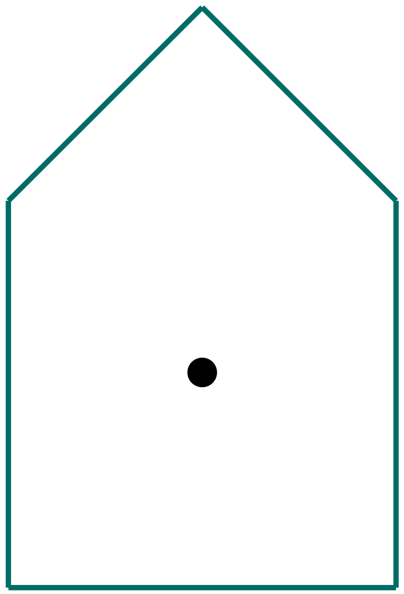
  [Ardila-Schindler-(Vindas-Meléndez) '21,…]

HOW CAN WE COMPUTE THESE “EXTREMAL” SLICES?

2 Approaches

Rotational Approach

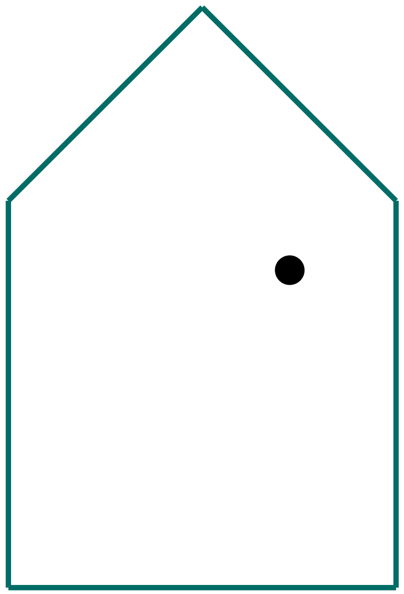
1. Choose a position of the origin



2 Approaches

Rotational Approach

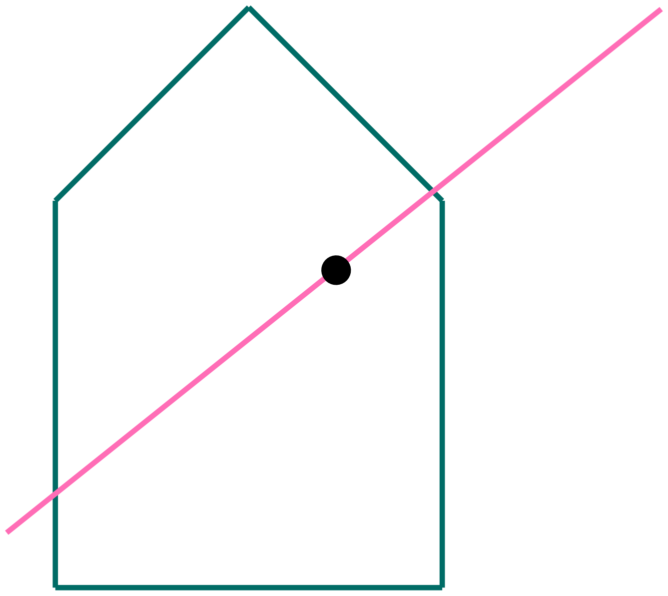
1. Choose a position of the origin



2 Approaches

Rotational Approach

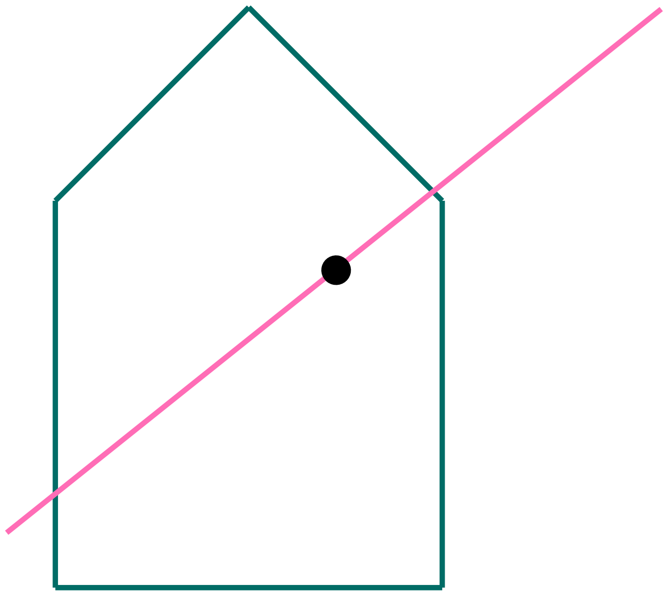
1. Choose a position of the origin
2. Consider all hyperplanes through the origin



2 Approaches

Rotational Approach

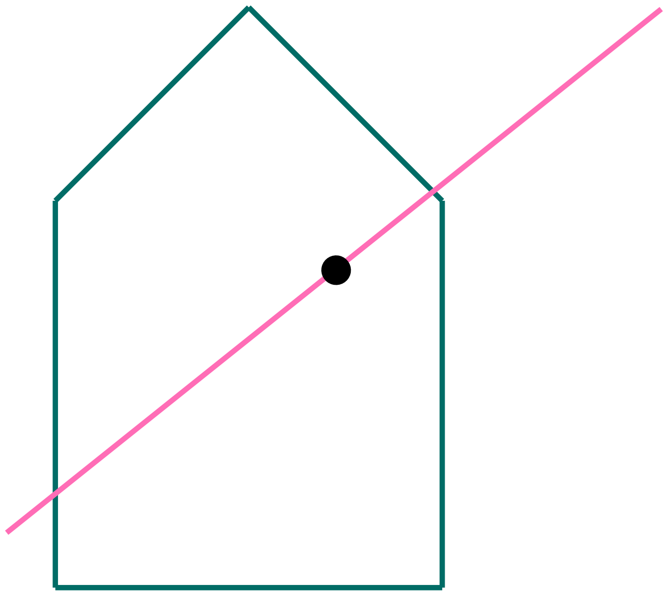
1. Choose a position of the origin
2. Consider all hyperplanes through the origin



2 Approaches

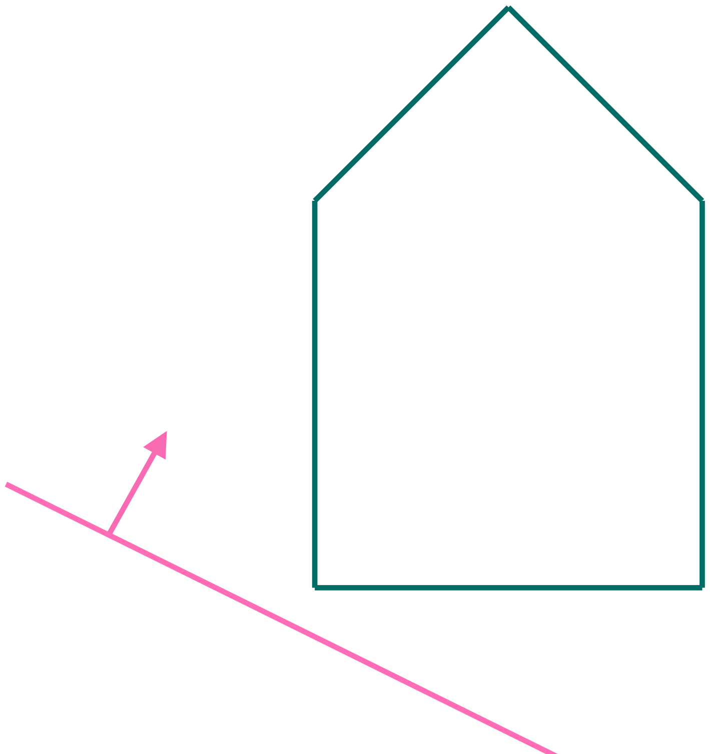
Rotational Approach

1. Choose a position of the origin
2. Consider all hyperplanes through the origin



Translational Approach

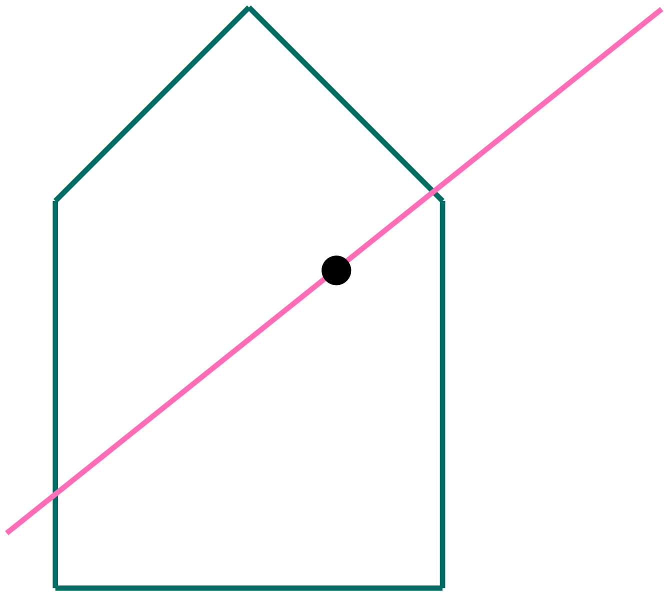
1. Choose a normal direction



2 Approaches

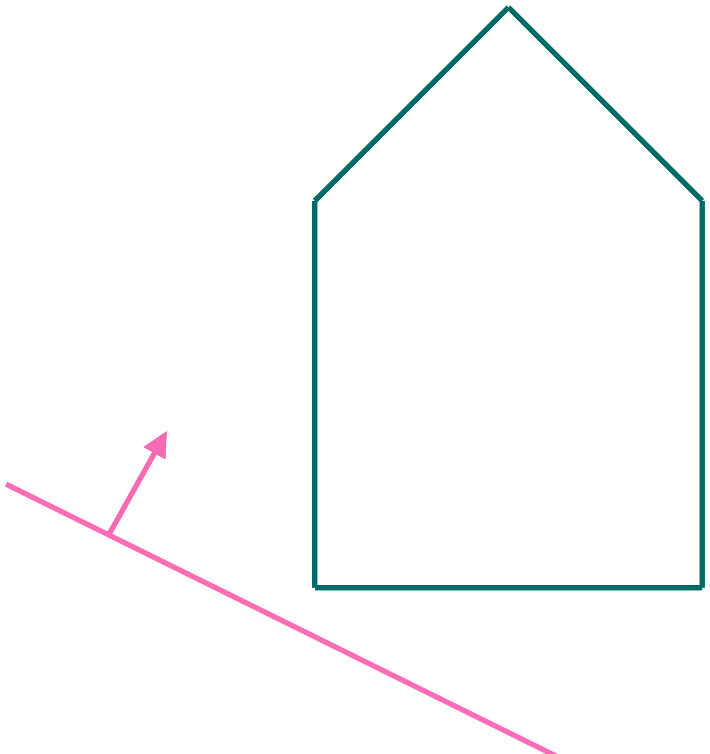
Rotational Approach

1. Choose a position of the origin
2. Consider all hyperplanes through the origin



Translational Approach

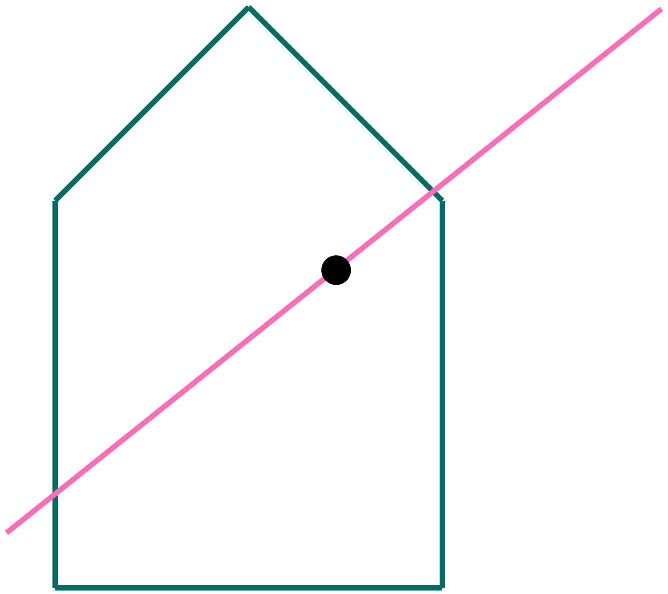
1. Choose a normal direction
2. Consider all affine translates of the orthogonal hyperplane



2 Approaches

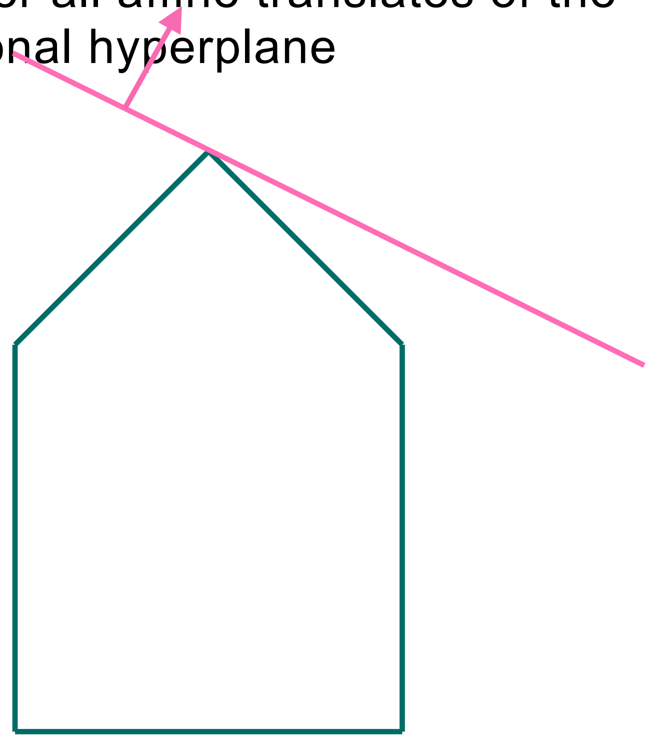
Rotational Approach

1. Choose a position of the origin
2. Consider all hyperplanes through the origin



Translational Approach

1. Choose a normal direction
2. Consider all affine translates of the orthogonal hyperplane



2 Approaches

Key Observations:

- H generic hyperplane $\Longrightarrow$ vertices of $P \cap H$ = intersections of H with edges of P

2 Approaches

Key Observations:

- H generic hyperplane $\Longrightarrow$ vertices of $P \cap H$ = intersections of H with edges of P
- H, H' intersect the same set of edges of P
  $\Longrightarrow$ $P \cap H, P \cap H'$ have the same combinatorial type

2 Approaches

Key Observations:

- H generic hyperplane $\implies$ vertices of $P \cap H$ = intersections of H with edges of P
- H, H' intersect the same set of edges of P
  $\implies P \cap H, P \cap H'$ have the same combinatorial type

Main Idea for both approaches:

Collect all hyperplanes which intersect P in the same set of edges

$\longrightarrow$ regions of hyperplane arrangements

2 Approaches

Key Observations:

- H generic hyperplane $\Longrightarrow$ vertices of $P \cap H$ = intersections of H with edges of P
- H, H' intersect the same set of edges of P
  $\Longrightarrow P \cap H, P \cap H'$ have the same combinatorial type

Main Idea for both approaches:

Collect all hyperplanes which intersect P in the same set of edges

$\longrightarrow$ regions of hyperplane arrangements

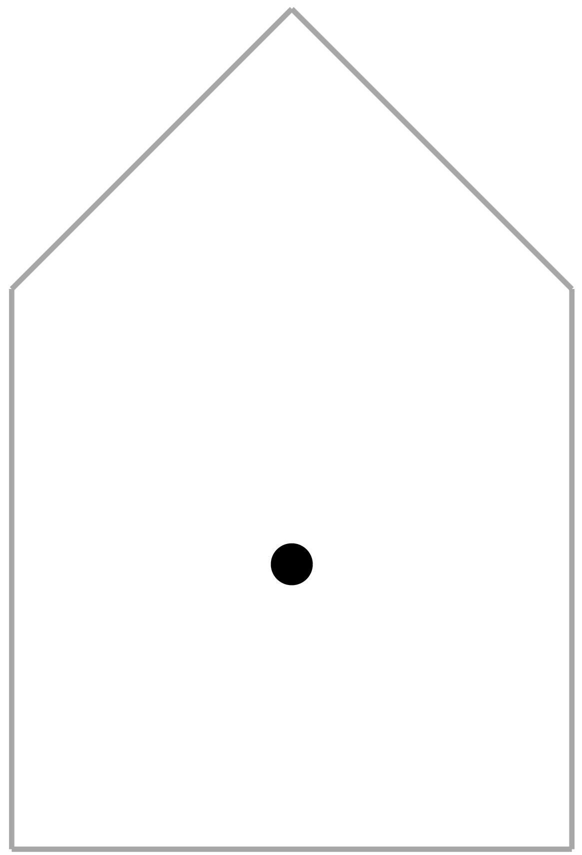
	Hyperplane Arrangement	**Notation**	**Reference Object**
$\circlearrowleft$	central arrangement	$\mathcal{C}_{\circlearrowleft}$	intersection body
	cocircuit arrangement	$\mathcal{R}_{\circlearrowleft}$	oriented matroid
$\uparrow$	parallel arrangement	$\mathcal{C}^{\mathbf{u}}_{\uparrow}$	fiber polytope
	sweep arrangement	$\mathcal{R}_{\uparrow}$	sweep polytope

ROTATIONAL APPROACH

Rotational Approach

Fix the position of the origin.

$u^\perp$ = central hyperplane orthogonal to u



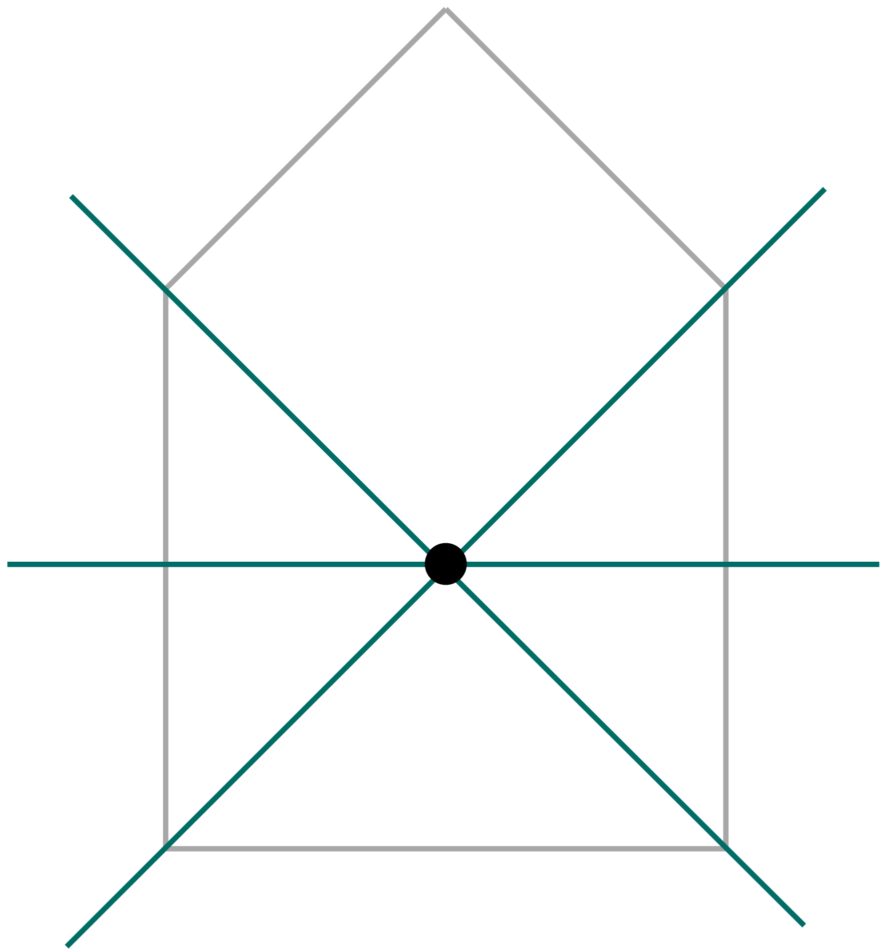
Rotational Approach

Fix the position of the origin.

$u^{\perp}$ = central hyperplane orthogonal to u

Consider the central hyperplane arrangement

$$\mathscr{C}_{\circlearrowleft}(P) = \{v^{\perp} \mid v \text{ is a vertex of } P\}.$$



Rotational Approach

Fix the position of the origin.

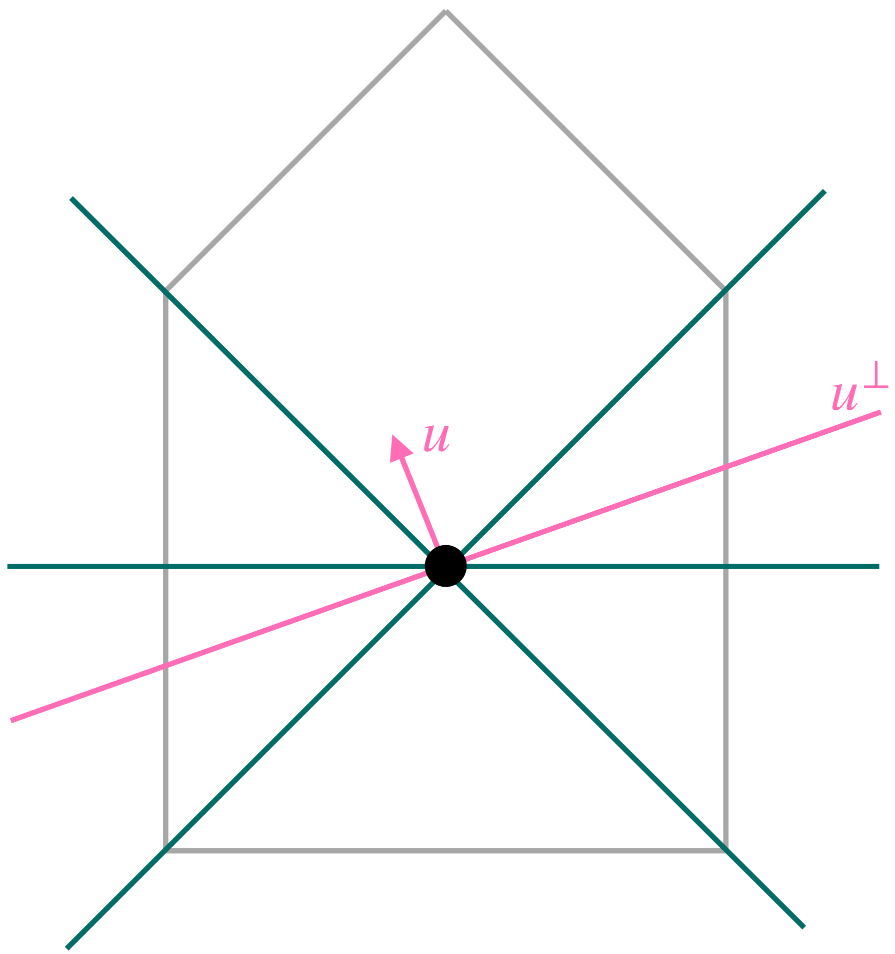
$u^\perp$ = central hyperplane orthogonal to u

Consider the central hyperplane arrangement

$$\mathscr{C}_{\circlearrowleft}(P) = \{v^\perp \mid v \text{ is a vertex of } P\}.$$

$\longrightarrow$ The combinatorial type of $P \cap u^\perp$ is constant in each cell of $\mathscr{C}_{\circlearrowleft}(P)$.

We refer to the maximal cells of $\mathscr{C}_{\circlearrowleft}(P)$ as chambers.



Rotational Approach

Fix the position of the origin.

$u^\perp$ = central hyperplane orthogonal to u

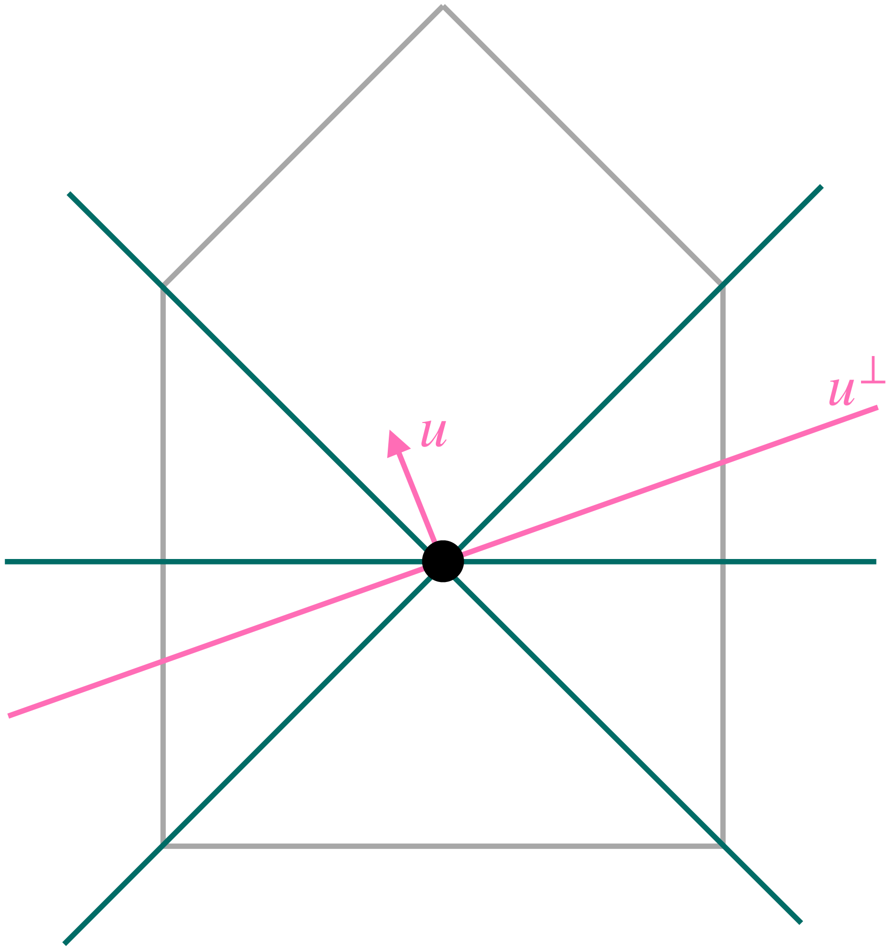
Consider the central hyperplane arrangement

$$\mathscr{C}_{\circlearrowleft}(P) = \{v^\perp \mid v \text{ is a vertex of } P\}.$$

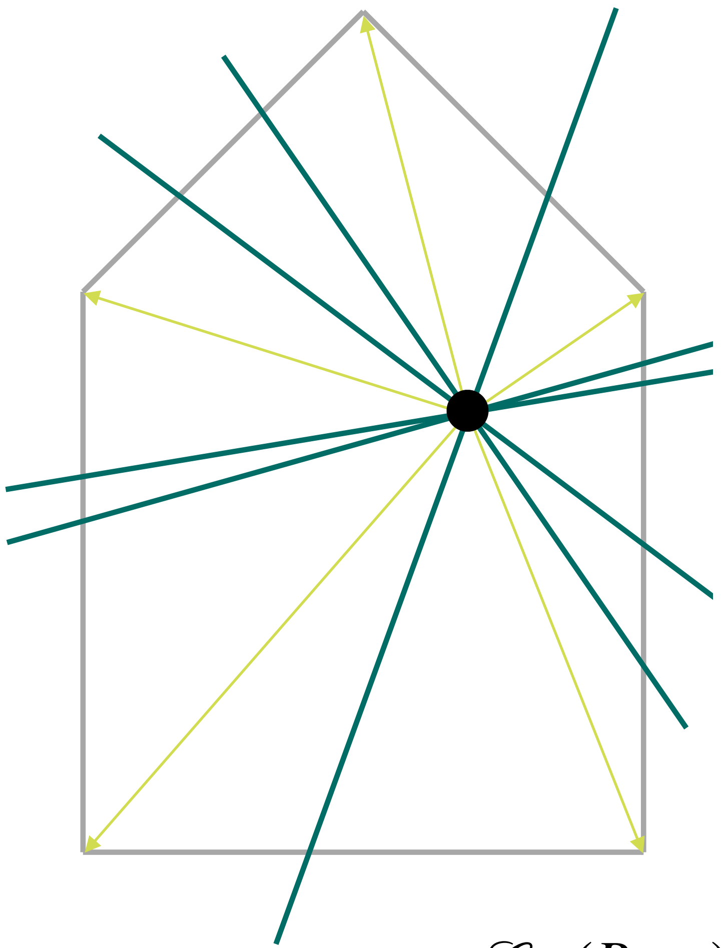
$\longrightarrow$ The combinatorial type of $P \cap u^\perp$ is constant in each cell of $\mathscr{C}_{\circlearrowleft}(P)$.

We refer to the maximal cells of $\mathscr{C}_{\circlearrowleft}(P)$ as chambers.

What happens if we translate P, i.e. vary the position of the origin?

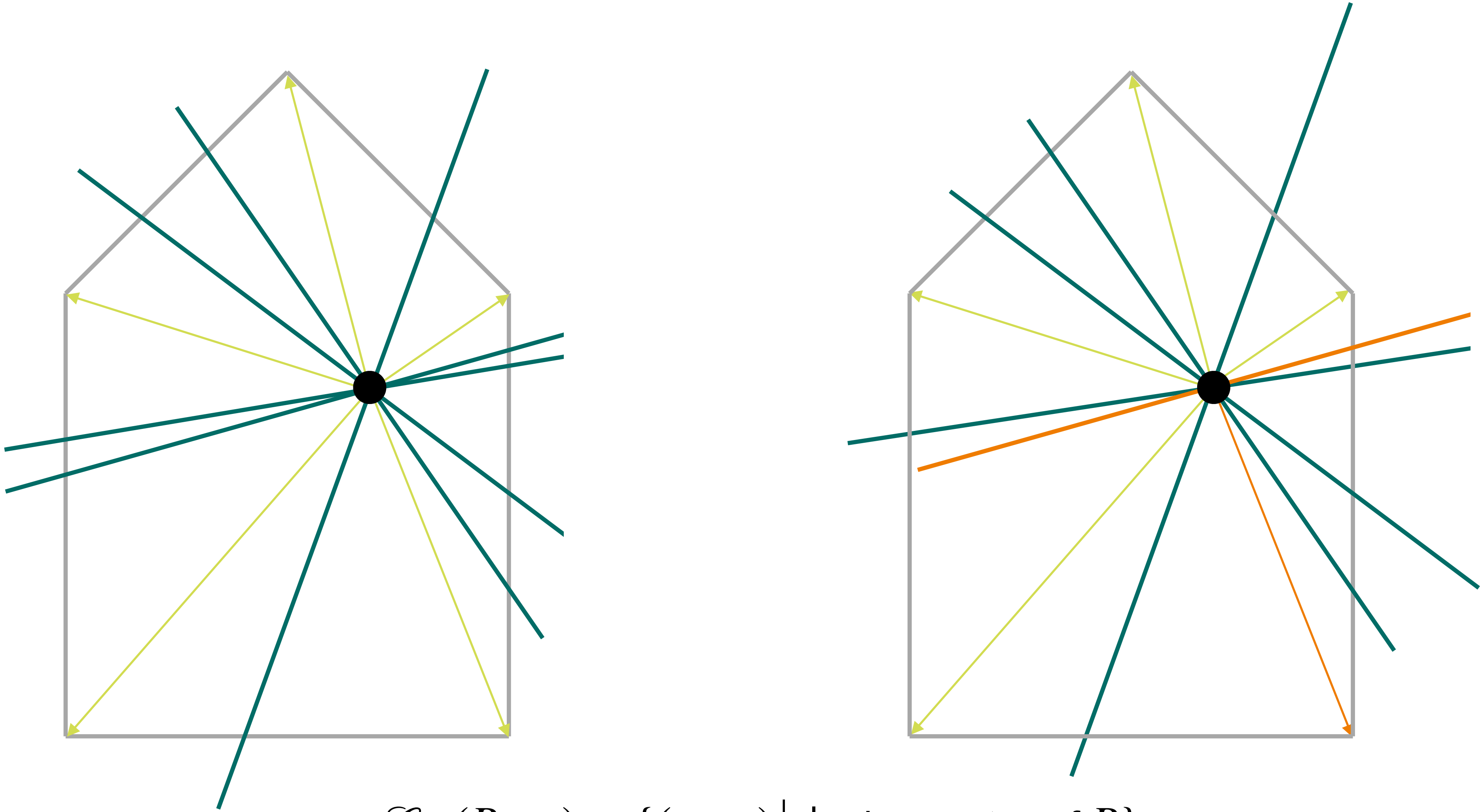


Rotational Approach



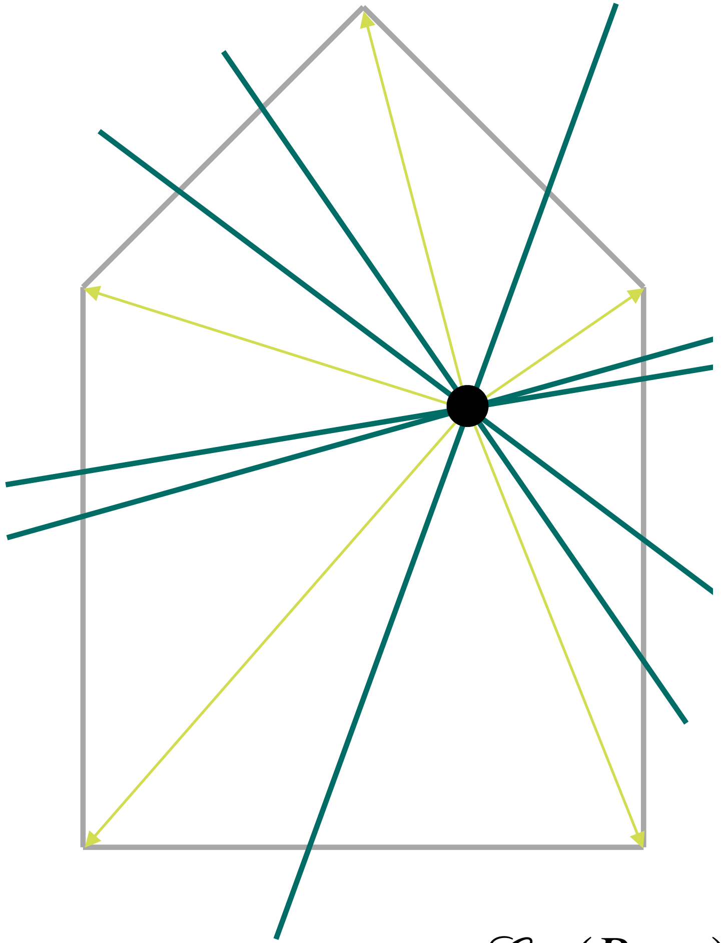
$$\mathscr{C}_{\circlearrowleft}(P+t) = \{(v+t)^{\perp} \mid v \text{ is a vertex of } P\}$$

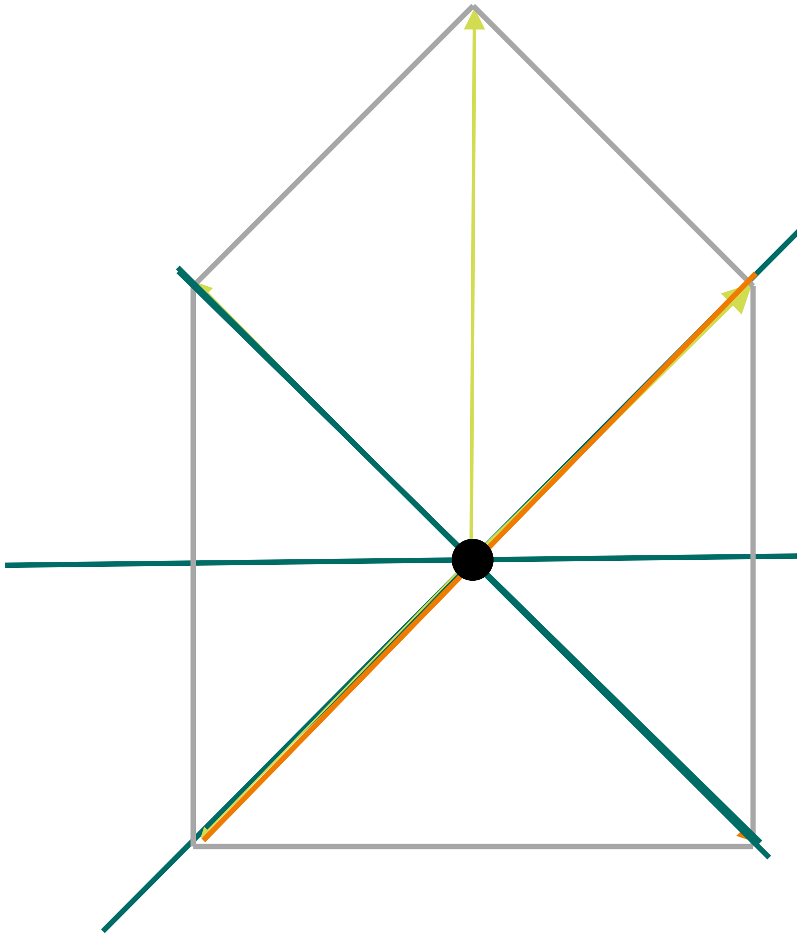
Rotational Approach



$$\mathscr{C}_{\circlearrowleft}(P + t) = \{(v + t)^{\perp} \mid v \text{ is a vertex of } P\}$$

Rotational Approach





$$\mathscr{C}_{\circlearrowleft}(P + t) = \{(v + t)^{\perp} \mid v \text{ is a vertex of } P\}$$

Rotational Approach

Translation $P + t \longleftrightarrow$ rotation of hyperplanes $(v + t)^{\perp}$ in central arrangement $\mathscr{C}_{\circlearrowleft}(P + t)$

Rotational Approach

Translation $P + t \longleftrightarrow$ rotation of hyperplanes $(v + t)^{\perp}$ in central arrangement $\mathscr{C}_{\circlearrowleft}(P + t)$

For which $t \in \mathbb{R}^d$ does $\mathscr{C}_{\circlearrowleft}(P + t)$ have the same combinatorics?
(i.e. the same oriented matroid)

Rotational Approach

Translation $P + t \longleftrightarrow$ rotation of hyperplanes $(v+t)^{\perp}$ in central arrangement $\mathscr{C}_{\circlearrowleft}(P+t)$

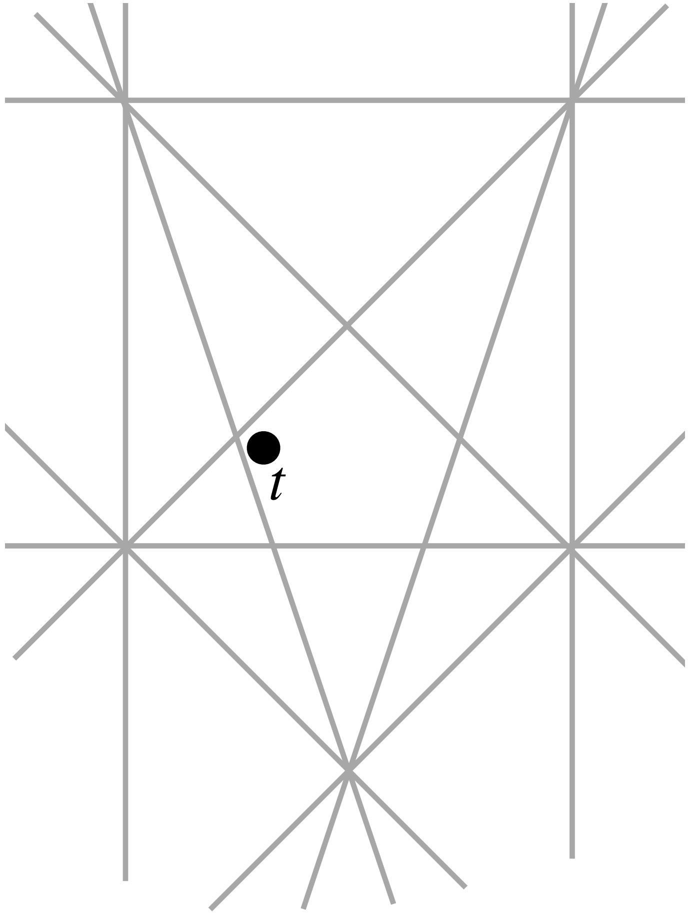
For which $t \in \mathbb{R}^d$ does $\mathscr{C}_{\circlearrowleft}(P+t)$ have the same combinatorics?
(i.e. the same oriented matroid)

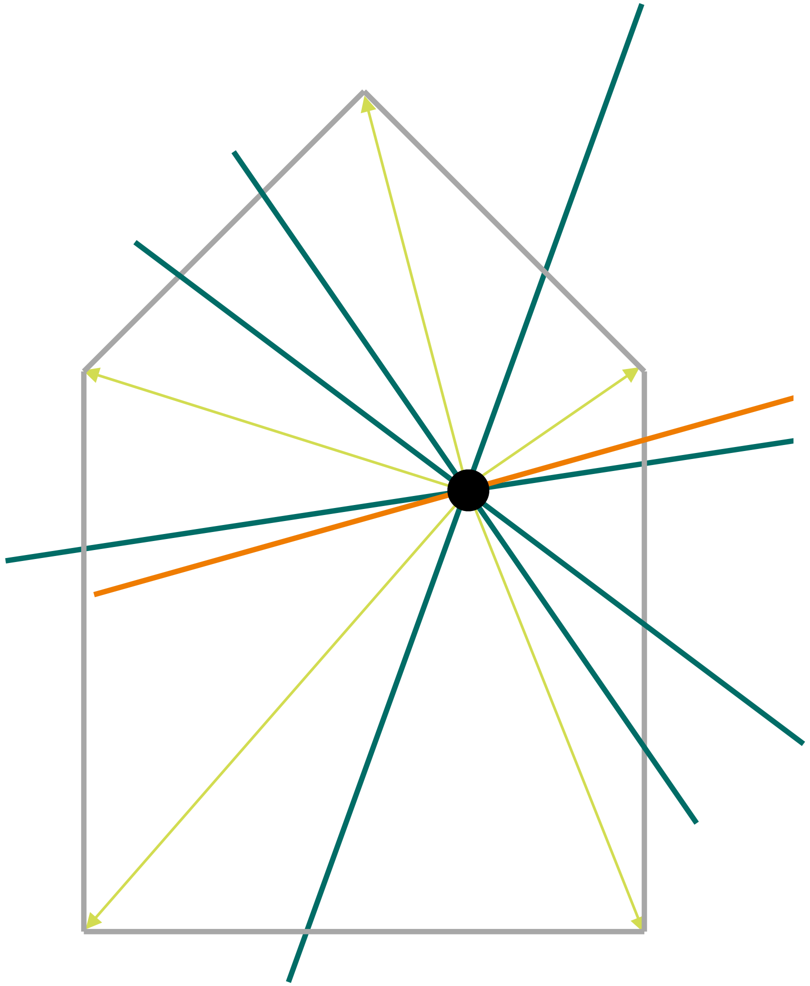
Consider the affine hyperplane arrangement (called cocircuit arrangement)

$$\mathscr{R}_{\circlearrowleft}(P) = \{\text{aff}(-v_1, \ldots, -v_d) \mid v_k \text{ are vertices of } P\}$$

$\longrightarrow$ with each region of $\mathscr{R}_{\circlearrowleft}(P)$ the combinatorics of $\mathscr{C}_{\circlearrowleft}(P+t)$ are fixed

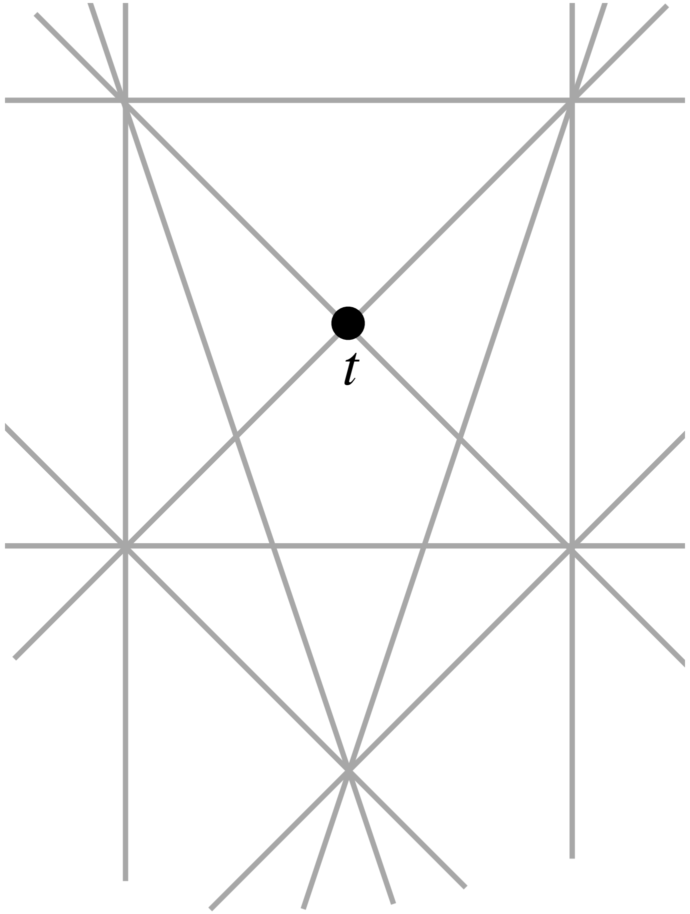
Rotational Approach

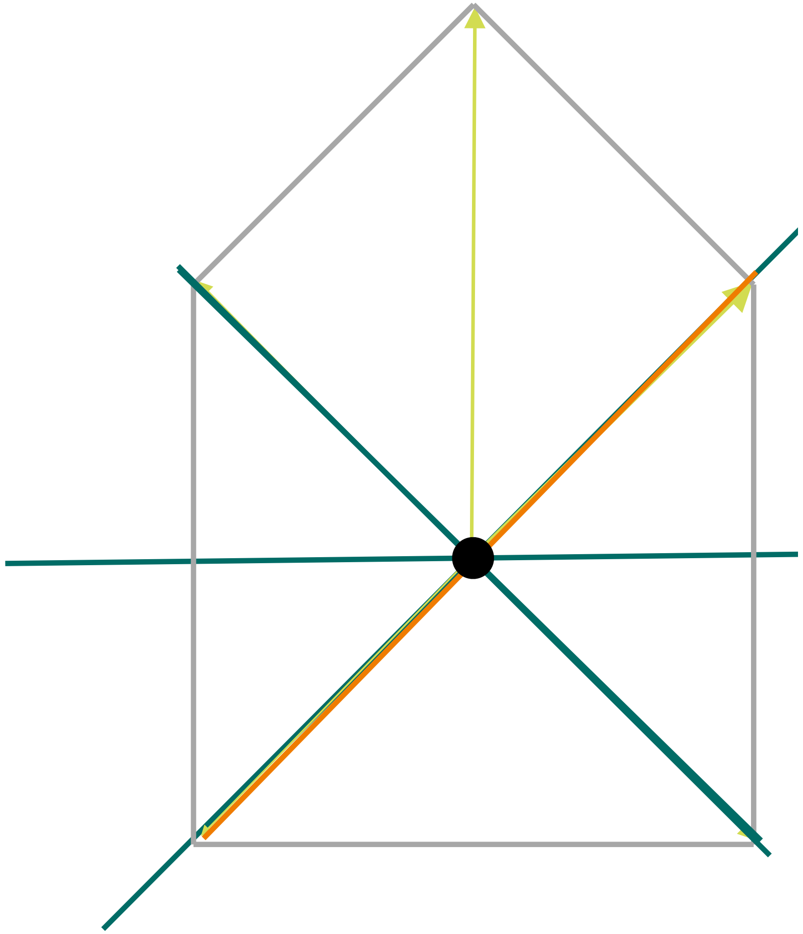




$\mathscr{R}_{\circlearrowleft}(P) = \{\text{aff}(-v_1, \ldots, -v_d) \mid v_k \text{ are vertices of } P\}$ $\mathscr{C}_{\circlearrowleft}(P+t) = \{(v+t)^{\perp} \mid v \text{ is a vertex of } P\}$

Rotational Approach





$\mathscr{R}_{\circlearrowleft}(P) = \{\text{aff}(-v_1, \ldots, -v_d) \mid v_k \text{ are vertices of } P\}$

$\mathscr{C}_{\circlearrowleft}(P + t) = \{(v + t)^{\perp} \mid v \text{ is a vertex of } P\}$

Rotational Approach

Theorem (B.-Meroni-De Loera '23):

Let $P \subseteq \mathbb{R}^d$ be a polytope, and $f(x) = \sum_\alpha c_\alpha x^\alpha$ be a polynomial in variables $x_1, \ldots, x_d$.

Rotational Approach

Theorem (B.-Meroni-De Loera '23):

Let $P \subseteq \mathbb{R}^d$ be a polytope, and $f(x) = \sum_\alpha c_\alpha x^\alpha$ be a polynomial in variables $x_1, \ldots, x_d$.
Fix a region $R \in \mathscr{R}_{\circlearrowleft}(P)$ of the cocircuit arrangement, a translation $t \in R$ and a chamber $C(t) \in \mathscr{C}_{\circlearrowleft}(P + t)$ of the central arrangement.

Rotational Approach

Theorem (B.-Meroni-De Loera '23):

Let $P \subseteq \mathbb{R}^d$ be a polytope, and $f(x) = \sum_\alpha c_\alpha x^\alpha$ be a polynomial in variables $x_1, \ldots, x_d$.
Fix a region $R \in \mathscr{R}_{\circlearrowleft}(P)$ of the cocircuit arrangement, a translation $t \in R$ and a chamber $C(t) \in \mathscr{C}_{\circlearrowleft}(P + t)$ of the central arrangement.

Restricted to $t \in R$ and $u \in C(t) \cap S^{d-1}$, the integral

$$\int_{(P+t)\cap u^\perp} f(x) \, \mathrm{d}x$$

is a rational function in variables $t_1, \ldots, t_d, u_1, \ldots, u_d$.

Rotational Approach

Theorem (B.-Meroni-De Loera '23):

Let $P \subseteq \mathbb{R}^d$ be a polytope, and $f(x) = \sum_\alpha c_\alpha x^\alpha$ be a polynomial in variables $x_1, \ldots, x_d$.
Fix a region $R \in \mathscr{R}_{\circlearrowleft}(P)$ of the cocircuit arrangement, a translation $t \in R$ and a chamber $C(t) \in \mathscr{C}_{\circlearrowleft}(P + t)$ of the central arrangement.

Restricted to $t \in R$ and $u \in C(t) \cap S^{d-1}$, the integral

$$\int_{(P+t)\cap u^\perp} f(x)\ \mathrm{d}x$$

is a rational function in variables $t_1, \ldots, t_d, u_1, \ldots, u_d$.
(and we have an algorithm to compute it)

Rotational Approach

Theorem (B.-Meroni-De Loera '23):

Let $P \subseteq \mathbb{R}^d$ be a polytope, and $f(x) = \sum_\alpha c_\alpha x^\alpha$ be a polynomial in variables $x_1, \ldots, x_d$.
Fix a region $R \in \mathscr{R}_{\circlearrowleft}(P)$ of the cocircuit arrangement, a translation $t \in R$ and a chamber $C(t) \in \mathscr{C}_{\circlearrowleft}(P+t)$ of the central arrangement.

Restricted to $t \in R$ and $u \in C(t) \cap S^{d-1}$, the integral

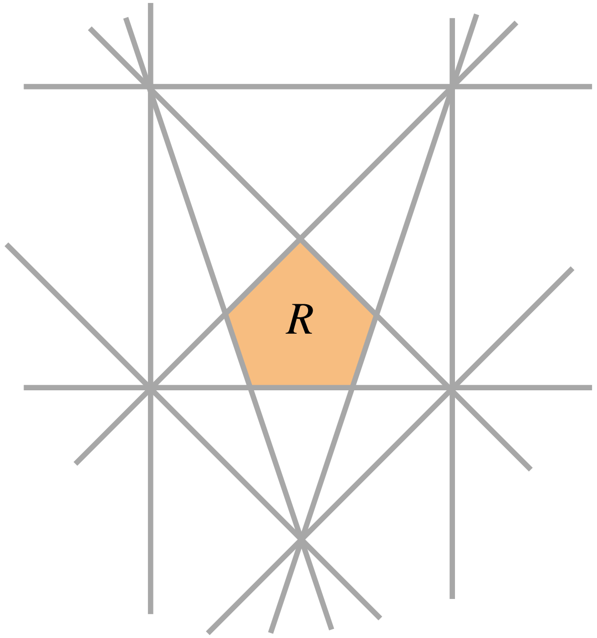
$$\int_{(P+t)\cap u^\perp} f(x)\ \mathrm{d}x$$

is a rational function in variables $t_1, \ldots, t_d, u_1, \ldots, u_d$.
(and we have an algorithm to compute it)

Note:

If $f(x) = 1$ then $\int_{(P+t)\cap u^\perp} f(x)\ \mathrm{d}x = \mathrm{vol}((P+t) \cap u^\perp)$.

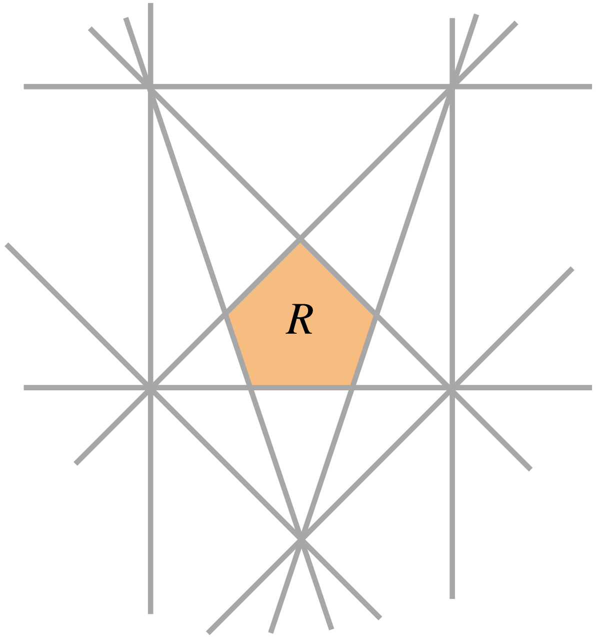
Rotational Approach

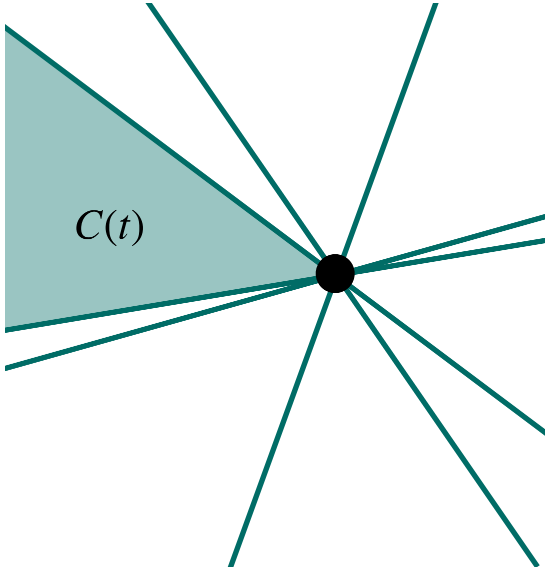


$(t_1, t_2) \in R \iff$

$$-t_1 - t_2 \geq 0, \quad t_1 - t_2 \geq 0$$
$$-3t_1 + t_2 \geq -2, \; 3t_1 + t_2 \geq -2$$
$$t_2 \geq -1$$

Rotational Approach

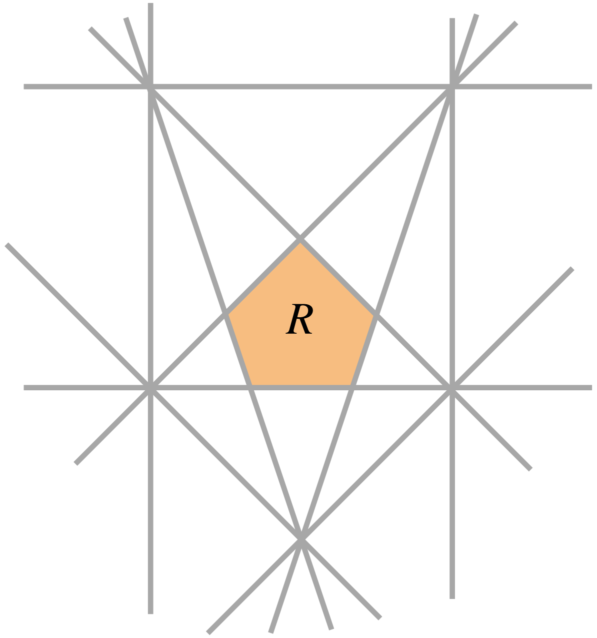


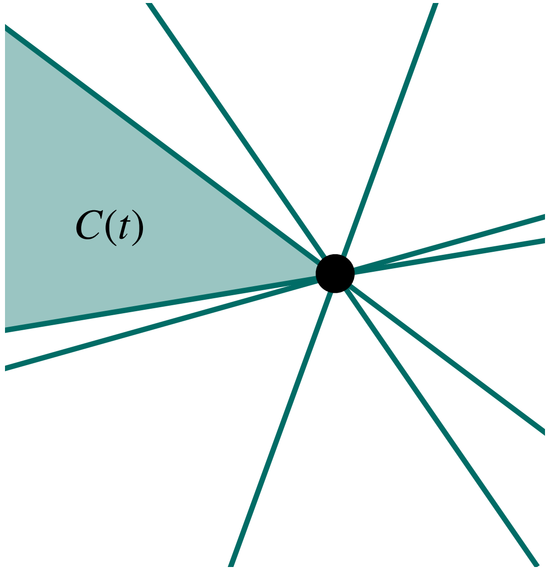


$(t_1, t_2) \in R \iff$
$-t_1 - t_2 \geq 0, \quad t_1 - t_2 \geq 0$
$-3t_1 + t_2 \geq -2, \; 3t_1 + t_2 \geq -2$
$t_2 \geq -1$

If $(t_1, t_2) \in R$ then $(u_1, u_2) \in C(t) \iff$
$2u_2 + t_1u_1 + t_2u_2 \geq 0$
$-u_1 - u_2 + t_1u_1 + t_2u_2 \geq 0$

Rotational Approach





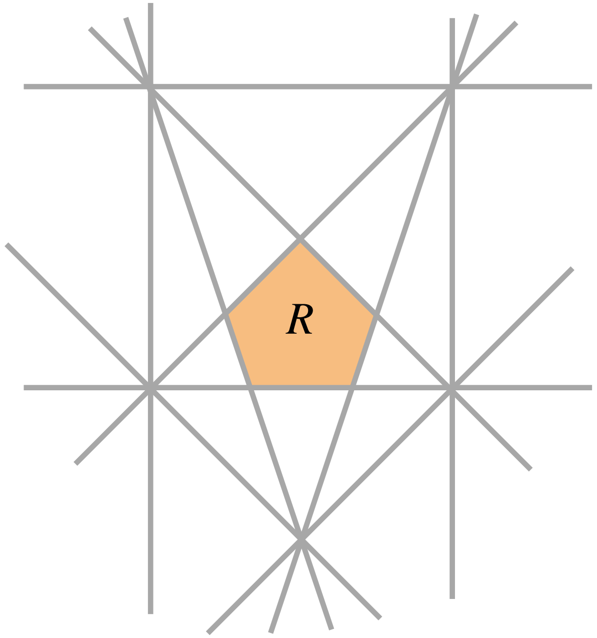
$(t_1, t_2) \in R \iff$
$-t_1 - t_2 \geq 0, \quad t_1 - t_2 \geq 0$
$-3t_1 + t_2 \geq -2, \; 3t_1 + t_2 \geq -2$
$t_2 \geq -1$

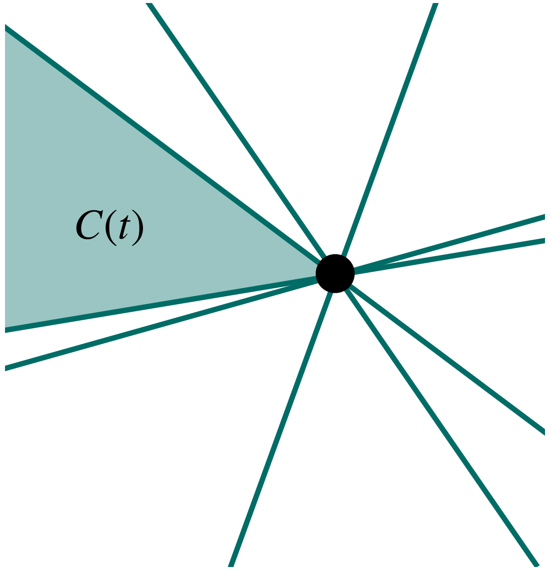
If $(t_1, t_2) \in R$ then $(u_1, u_2) \in C(t) \iff$
$2u_2 + t_1u_1 + t_2u_2 \geq 0$
$-u_1 - u_2 + t_1u_1 + t_2u_2 \geq 0$

If $t \in R$ and $u \in C(t) \cap S^{d-1}$ then

$$\mathrm{vol}((P+t) \cap u^{\perp}) = \int_{(P+t)\cap u^{\perp}} 1 \mathrm{d}x = \frac{-(t_1u_1 + t_2u_2 + 3u_1 - u_2)}{u_1(u_1 - u_2)}$$

Rotational Approach





Let the computer find the biggest slice:

maximize $\dfrac{-(t_1u_1 + t_2u_2 + u_1 - u_2)}{u_1(u_1 - u_2)}$

s.t $(t_1, t_2) \in R$

$(u_1, u_2) \in C(t) \cap S^{d-1}$

$(t_1, t_2) \in R \iff$

$$-t_1 - t_2 \geq 0, \quad t_1 - t_2 \geq 0$$
$$-3t_1 + t_2 \geq -2, \; 3t_1 + t_2 \geq -2$$
$$t_2 \geq -1$$

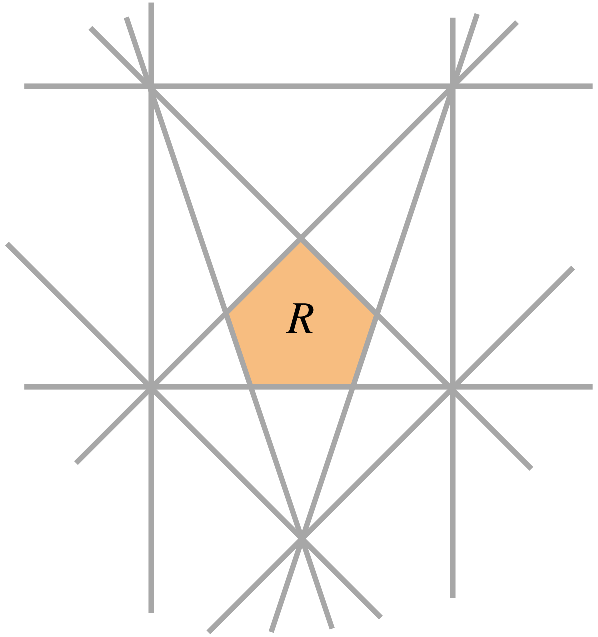
If $(t_1, t_2) \in R$ then $(u_1, u_2) \in C(t) \iff$

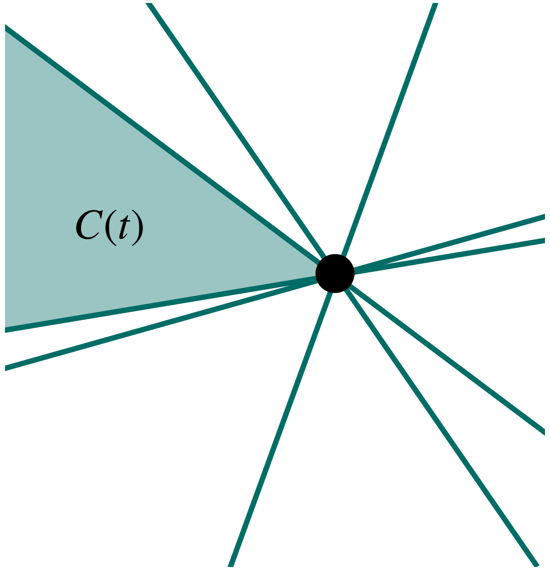
$$2u_2 + t_1u_1 + t_2u_2 \geq 0$$
$$-u_1 - u_2 + t_1u_1 + t_2u_2 \geq 0$$

If $t \in R$ and $u \in C(t) \cap S^{d-1}$ then

$$\text{vol}((P+t) \cap u^{\perp}) = \int_{(P+t)\cap u^{\perp}} 1 \mathrm{d}x = \frac{-(t_1u_1 + t_2u_2 + 3u_1 - u_2)}{u_1(u_1 - u_2)}$$

Rotational Approach





Let the computer find the biggest slice:

maximize $\dfrac{-(t_1u_1 + t_2u_2 + u_1 - u_2)}{u_1(u_1 - u_2)}$

s.t

$$-t_1 - t_2 \geq 0,$$
$$t_1 - t_2 \geq 0$$
$$-3t_1 + t_2 \geq -2,$$
$$3t_1 + t_2 \geq -2$$
$$t_2 \geq -1$$

$$2u_2 + t_1u_1 + t_2u_2 \geq 0$$
$$-u_1 - u_2 + t_1u_1 + t_2u_2 \geq 0$$
$$u_1^2 + u_2^2 + u_3^2 = 1$$

$(t_1, t_2) \in R \iff$

$$-t_1 - t_2 \geq 0, \quad t_1 - t_2 \geq 0$$
$$-3t_1 + t_2 \geq -2, \; 3t_1 + t_2 \geq -2$$
$$t_2 \geq -1$$

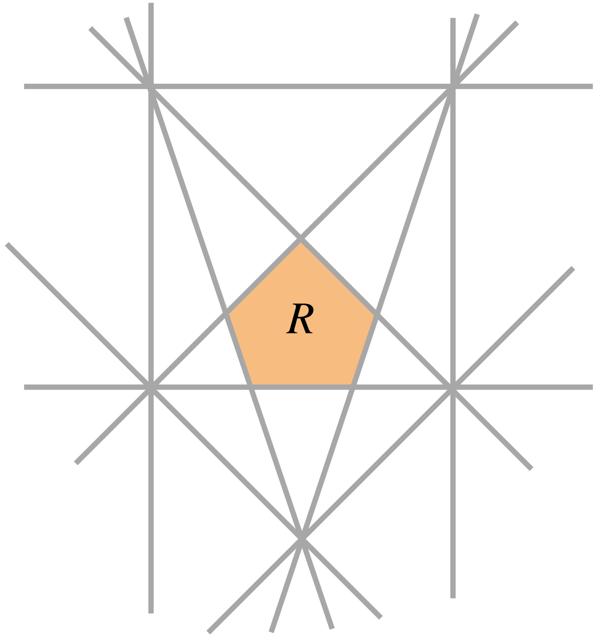
If $(t_1, t_2) \in R$ then $(u_1, u_2) \in C(t) \iff$

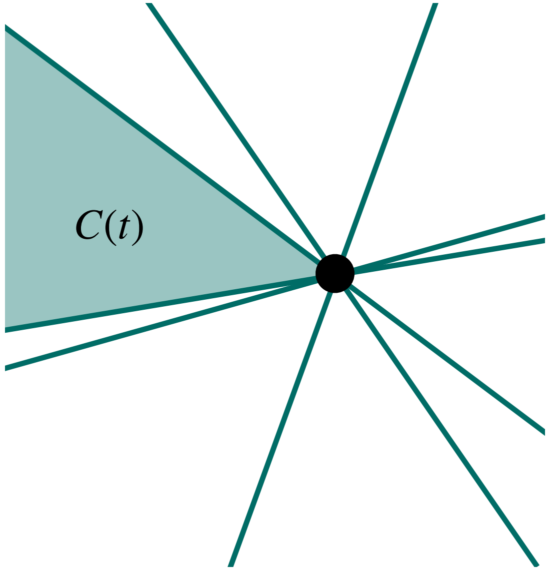
$$2u_2 + t_1u_1 + t_2u_2 \geq 0$$
$$-u_1 - u_2 + t_1u_1 + t_2u_2 \geq 0$$

If $t \in R$ and $u \in C(t) \cap S^{d-1}$ then

$$\text{vol}((P + t) \cap u^\perp) = \int_{(P+t)\cap u^\perp} 1 \mathrm{d}x = \frac{-(t_1u_1 + t_2u_2 + 3u_1 - u_2)}{u_1(u_1 - u_2)}$$

Rotational Approach





Let the computer find the biggest slice:

maximize $\dfrac{-(t_1u_1 + t_2u_2 + u_1 - u_2)}{u_1(u_1 - u_2)}$

s.t
$$-t_1 - t_2 \geq 0,$$
$$t_1 - t_2 \geq 0$$
$$-3t_1 + t_2 \geq -2,$$
$$3t_1 + t_2 \geq -2$$
$$t_2 \geq -1$$

$$2u_2 + t_1u_1 + t_2u_2 \geq 0$$
$$-u_1 - u_2 + t_1u_1 + t_2u_2 \geq 0$$
$$u_1^2 + u_2^2 + u_3^2 = 1$$

$(t_1, t_2) \in R \iff$
$$-t_1 - t_2 \geq 0, \quad t_1 - t_2 \geq 0$$
$$-3t_1 + t_2 \geq -2, \; 3t_1 + t_2 \geq -2$$
$$t_2 \geq -1$$

If $(t_1, t_2) \in R$ *then* $(u_1, u_2) \in C(t) \iff$
$$2u_2 + t_1u_1 + t_2u_2 \geq 0$$
$$-u_1 - u_2 + t_1u_1 + t_2u_2 \geq 0$$

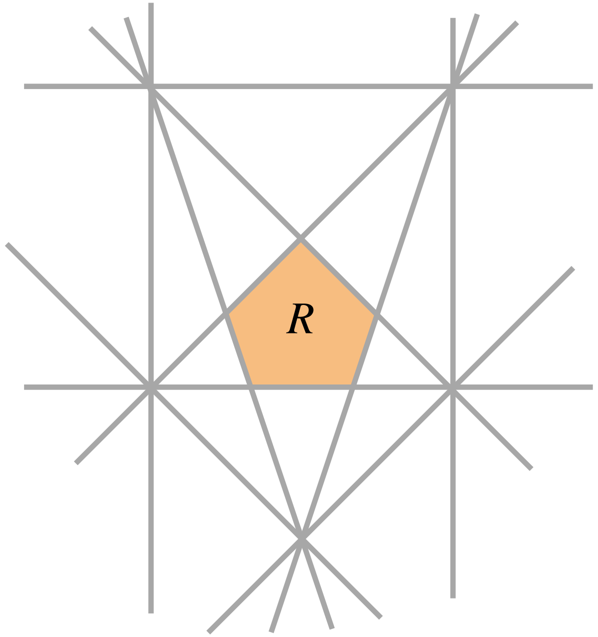
If $t \in R$ and $u \in C(t) \cap S^{d-1}$ then

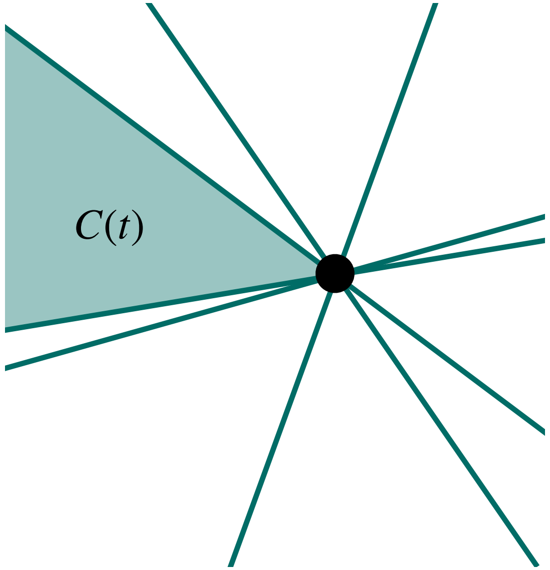
$$\text{vol}((P + t) \cap u^\perp) = \int_{(P+t)\cap u^\perp} 1\mathrm{d}x = \frac{-(t_1u_1 + t_2u_2 + 3u_1 - u_2)}{u_1(u_1 - u_2)}$$

Compute this for all regions $R \in \mathscr{R}_{\circlearrowleft}(P)$ and chambers $C(t) \in \mathscr{C}_{\circlearrowleft}(P + t)$

$\longrightarrow$ largest slice!

ROTATIONAL APPROACH





Let the computer find the biggest slice:

maximize $\dfrac{-(t_1u_1 + t_2u_2 + u_1 - u_2)}{u_1(u_1 - u_2)}$

s.t
$$-t_1 - t_2 \geq 0,$$
$$t_1 - t_2 \geq 0$$
$$-3t_1 + t_2 \geq -2,$$
$$3t_1 + t_2 \geq -2$$
$$t_2 \geq -1$$

$$2u_2 + t_1u_1 + t_2u_2 \geq 0$$
$$-u_1 - u_2 + t_1u_1 + t_2u_2 \geq 0$$
$$u_1^2 + u_2^2 + u_3^2 = 1$$

$(t_1, t_2) \in R \iff$

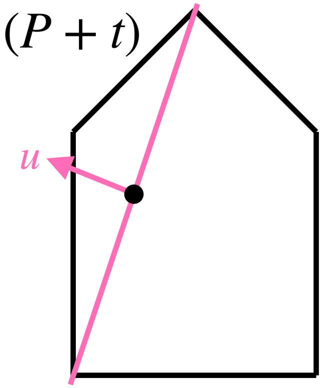
$$-t_1 - t_2 \geq 0, \quad t_1 - t_2 \geq 0$$
$$-3t_1 + t_2 \geq -2, \; 3t_1 + t_2 \geq -2$$
$$t_2 \geq -1$$

If $(t_1, t_2) \in R$ *then* $(u_1, u_2) \in C(t) \iff$

$$2u_2 + t_1u_1 + t_2u_2 \geq 0$$
$$-u_1 - u_2 + t_1u_1 + t_2u_2 \geq 0$$

Compute this for all regions $R \in \mathscr{R}_{\circlearrowleft}(P)$ and chambers $C(t) \in \mathscr{C}_{\circlearrowleft}(P + t)$

$\longrightarrow$ largest slice!



If $t \in R$ and $u \in C(t) \cap S^{d-1}$ then

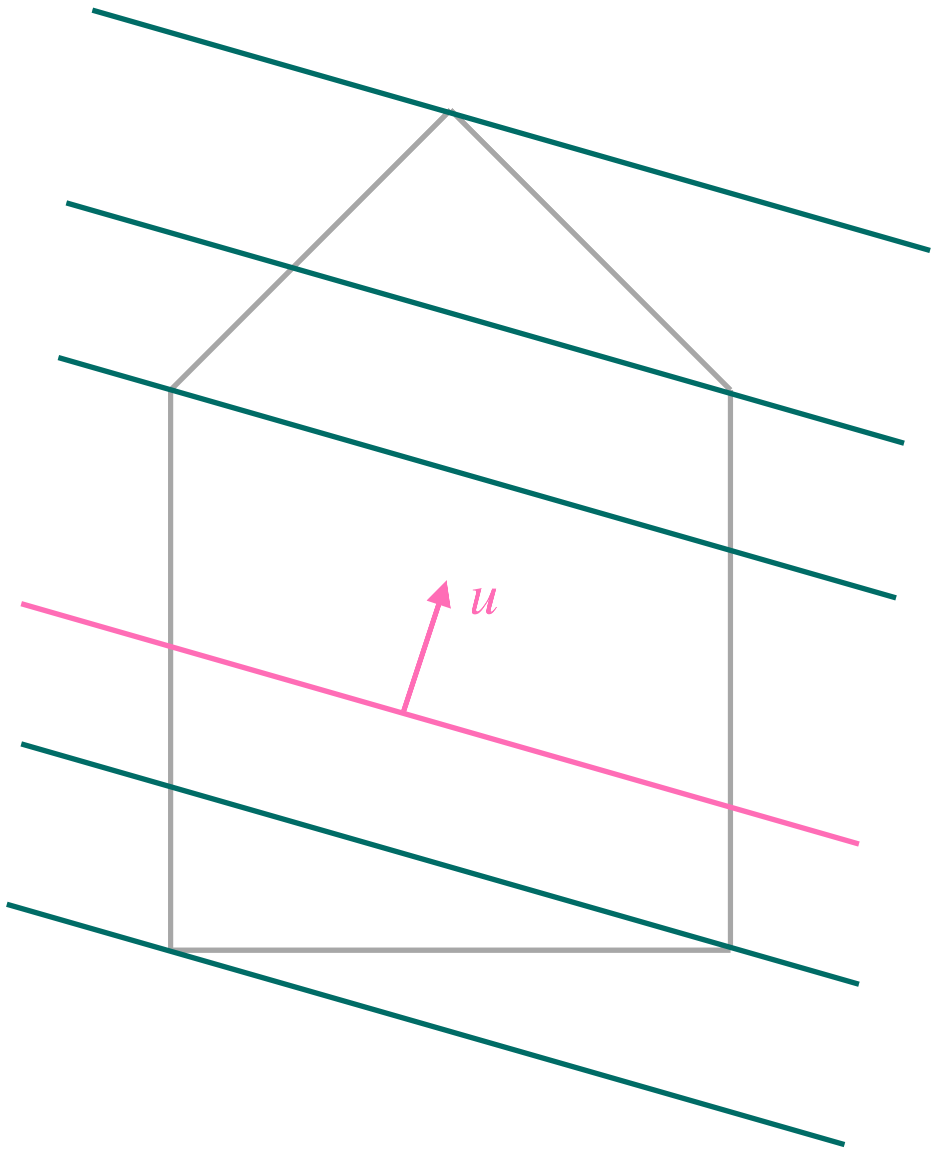
$$\text{vol}((P + t) \cap u^{\perp}) = \int_{(P+t)\cap u^{\perp}} 1\mathrm{d}x = \frac{-(t_1u_1 + t_2u_2 + 3u_1 - u_2)}{u_1(u_1 - u_2)}$$

TRANSLATIONAL APPROACH

Translational Approach

Fix a normal direction $u \in S^{d-1}$

$H(\beta) = \{x \in \mathbb{R}^d \mid \langle x, u \rangle = \beta\}$
hyperplane parallel to $u^\perp$



Translational Approach

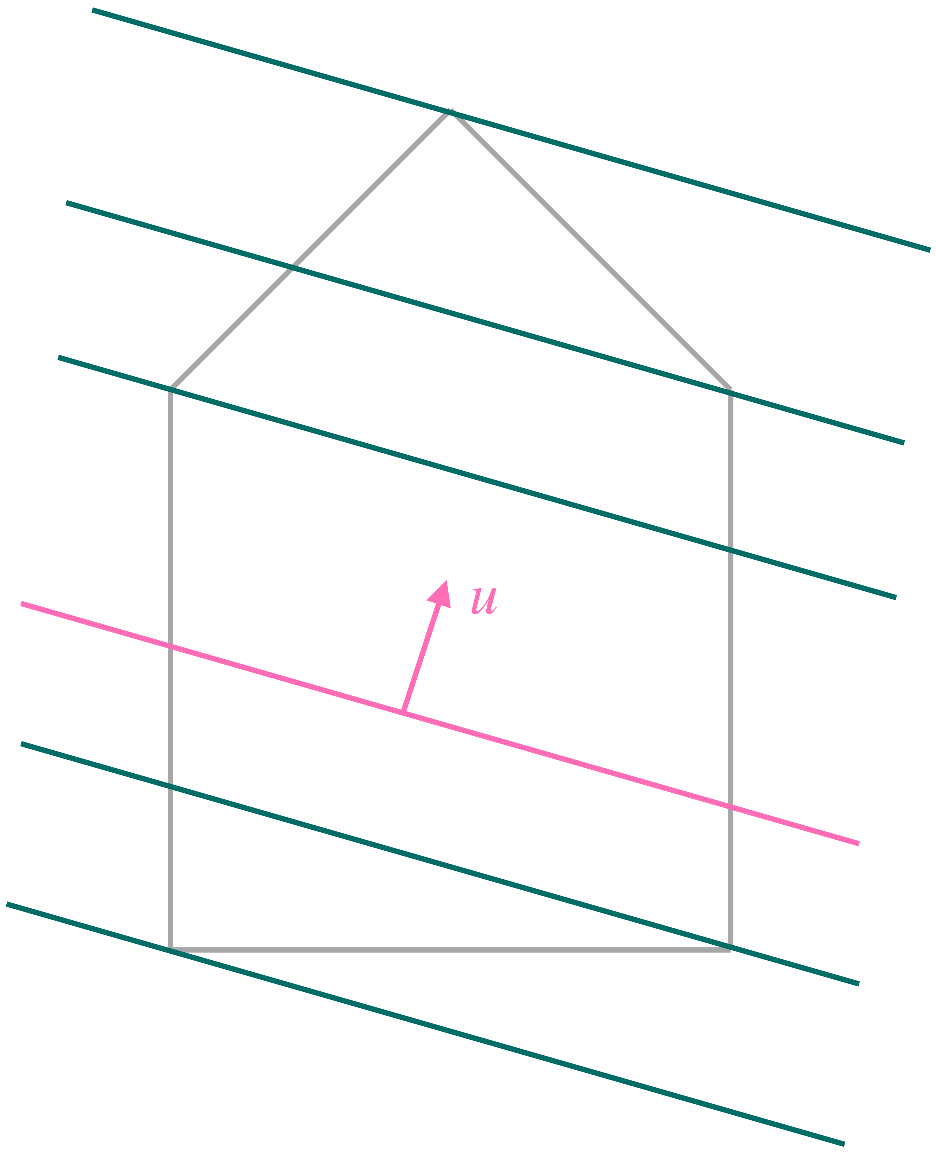
Fix a normal direction $u \in S^{d-1}$

$H(\beta) = \{x \in \mathbb{R}^d \mid \langle x, u \rangle = \beta\}$

hyperplane parallel to $u^\perp$

Consider the parallel hyperplane arrangement

$$\mathscr{C}^u_\uparrow(P) = \{H(\langle v, u \rangle) \mid v \text{ is a vertex of } P\}.$$



Translational Approach

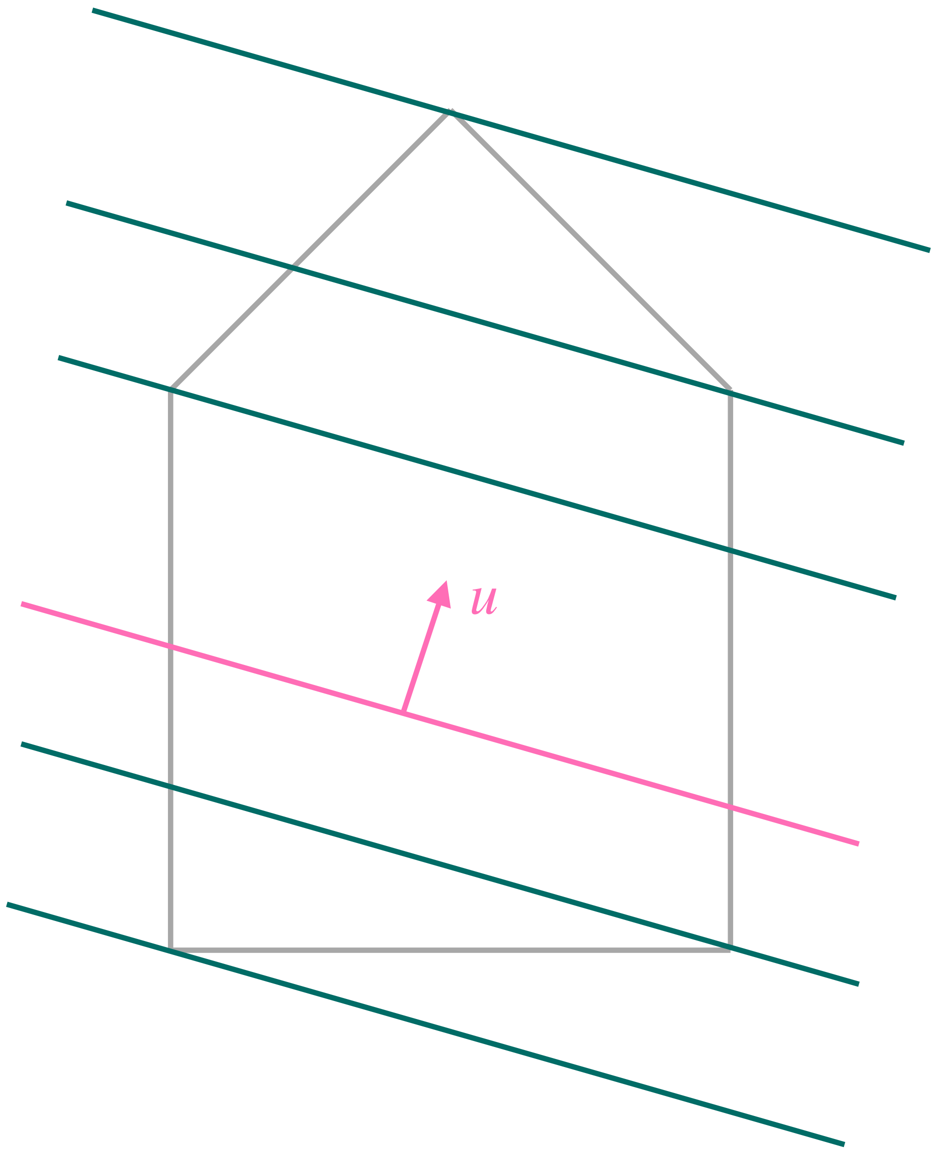
Fix a normal direction $u \in S^{d-1}$

$H(\beta) = \{x \in \mathbb{R}^d \mid \langle x, u \rangle = \beta\}$
hyperplane parallel to $u^\perp$

Consider the parallel hyperplane arrangement

$$\mathscr{C}^u_\uparrow(P) = \{H(\langle v, u \rangle \mid v \text{ is a vertex of } P\}.$$

$\longrightarrow$ The combinatorial type of $P \cap H(\beta)$ is constant in each cell of $\mathscr{C}^u_\uparrow(P)$.



TRANSLATIONAL APPROACH

Fix a normal direction $u \in S^{d-1}$

$H(\beta) = \{x \in \mathbb{R}^d \mid \langle x, u \rangle = \beta\}$

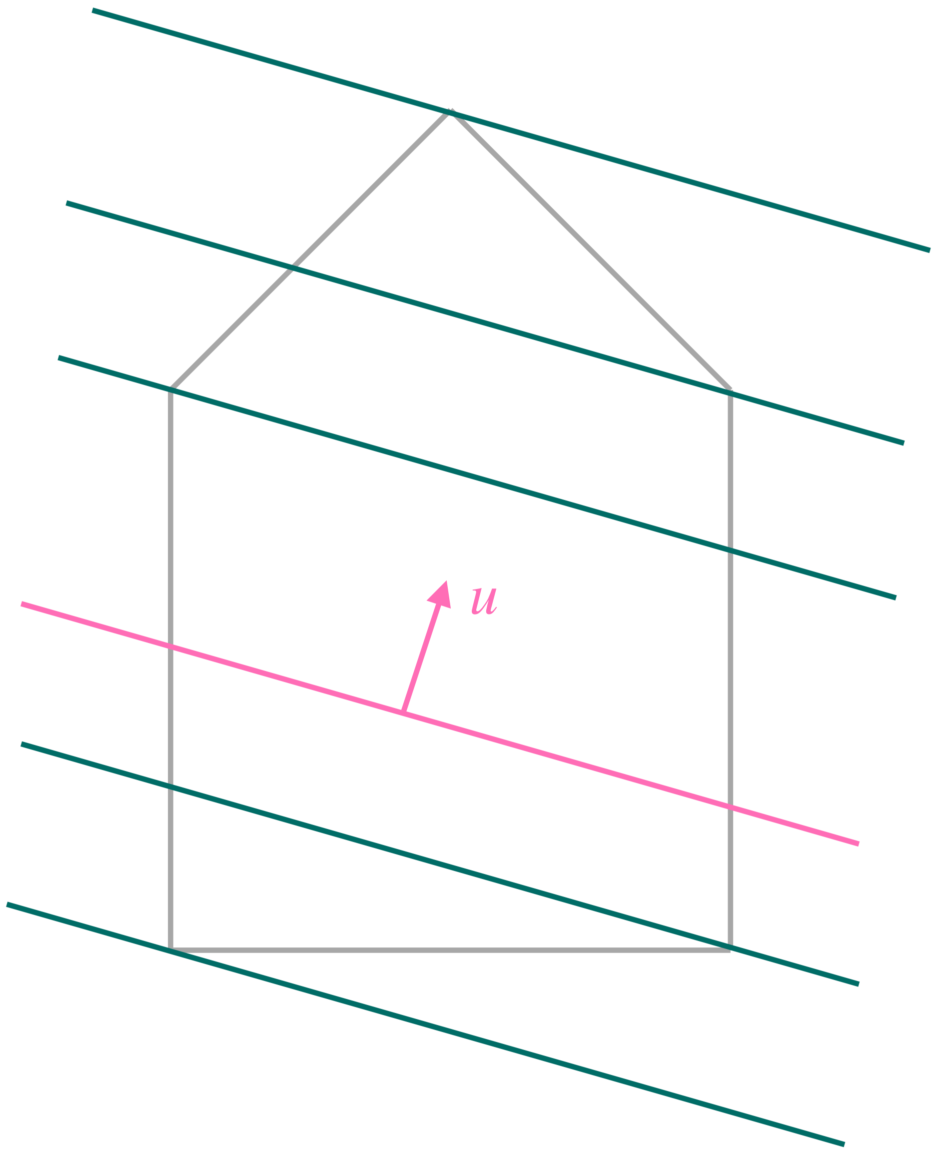
hyperplane parallel to $u^\perp$

Consider the parallel hyperplane arrangement

$$\mathscr{C}^u_\uparrow(P) = \{H(\langle v, u \rangle) \mid v \text{ is a vertex of } P\}.$$

$\longrightarrow$ The combinatorial type of $P \cap H(\beta)$ is constant in each cell of $\mathscr{C}^u_\uparrow(P)$.

We refer to the maximal cells of $\mathscr{C}^u_\uparrow$ as chambers.



Translational Approach

Fix a normal direction $u \in S^{d-1}$

$H(\beta) = \{x \in \mathbb{R}^d \mid \langle x, u \rangle = \beta\}$
hyperplane parallel to $u^\perp$

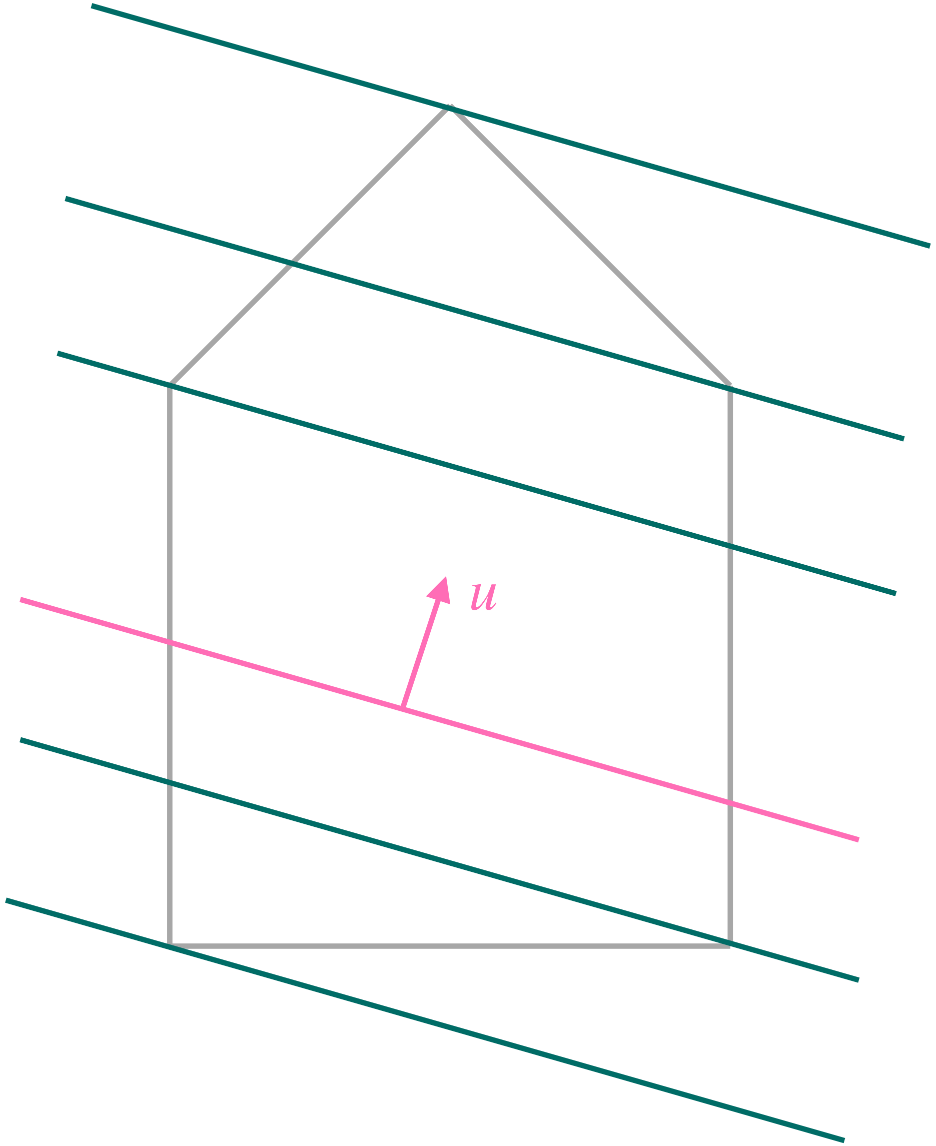
Consider the parallel hyperplane arrangement

$$\mathscr{C}^u_\uparrow(P) = \{H(\langle v, u \rangle \mid v \text{ is a vertex of } P\}.$$

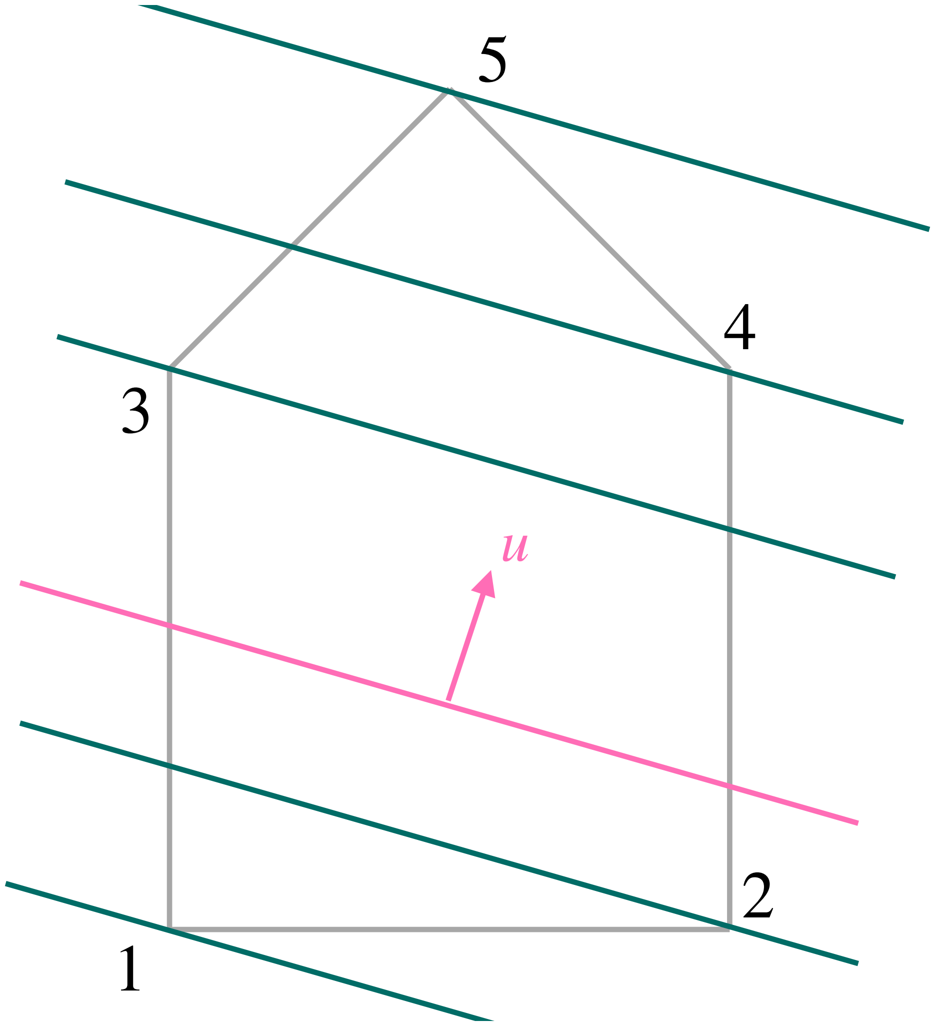
$\longrightarrow$ The combinatorial type of $P \cap H(\beta)$ is constant in each cell of $\mathscr{C}^u_\uparrow(P)$.

We refer to the maximal cells of $\mathscr{C}^u_\uparrow$ as chambers.

What happens if we vary the direction $u \in S^{d-1}$?

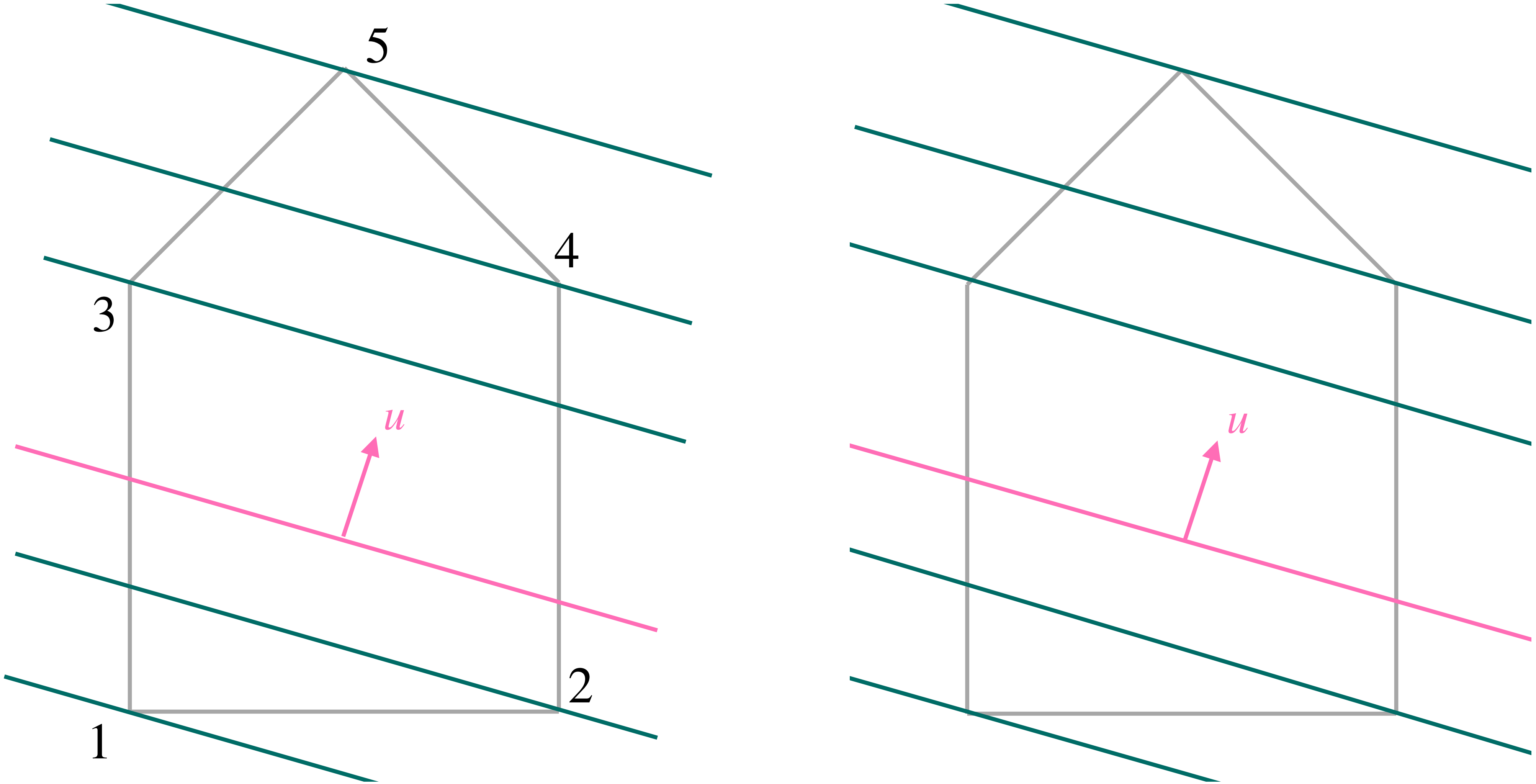


Translational Approach



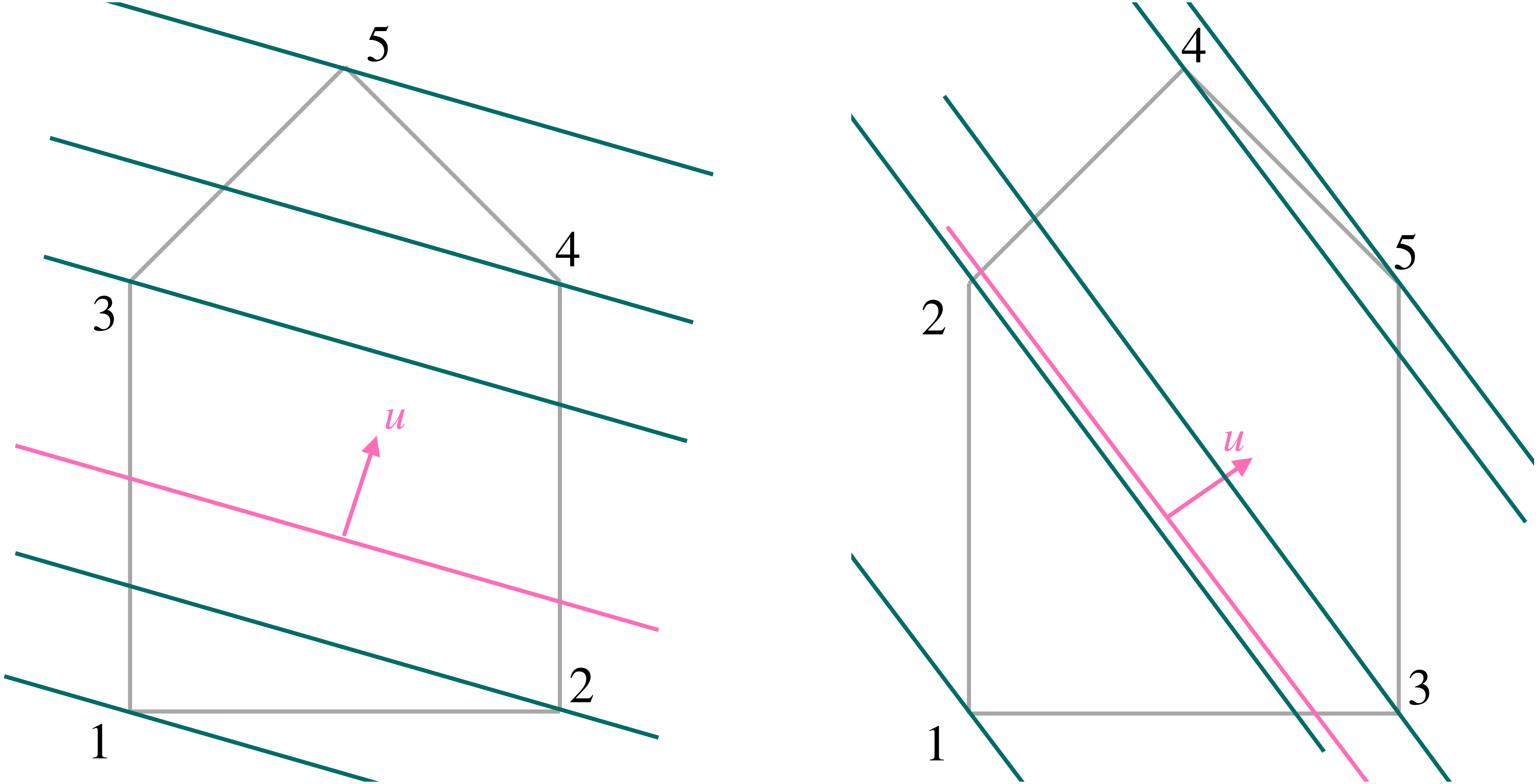
$$\mathscr{C}^u_\uparrow(P) = \{H(\langle v, u\rangle \mid v \text{ is a vertex of } P\}$$

Translational Approach



$$\mathscr{C}^{u}_{\uparrow}(P) = \{H(\langle v, u\rangle \mid v \text{ is a vertex of } P\}$$

Translational Approach



$$\mathscr{C}^u_\uparrow(P) = \{H(\langle v, u\rangle \mid v \text{ is a vertex of } P\}$$

Translational Approach

For which $u \in S^{d-1}$ does $\mathscr{C}^u_\uparrow(P)$ induce the same ordering of the vertices?

Translational Approach

For which $u \in S^{d-1}$ does $\mathscr{C}^u_\uparrow(P)$ induce the same ordering of the vertices?

Consider the central hyperplane arrangement (called sweep arrangement)

$$\mathscr{R}_\uparrow(P) = \{(v_i - v_j)^\perp \mid v_i, v_j \text{ are vertices of } P\}$$

Translational Approach

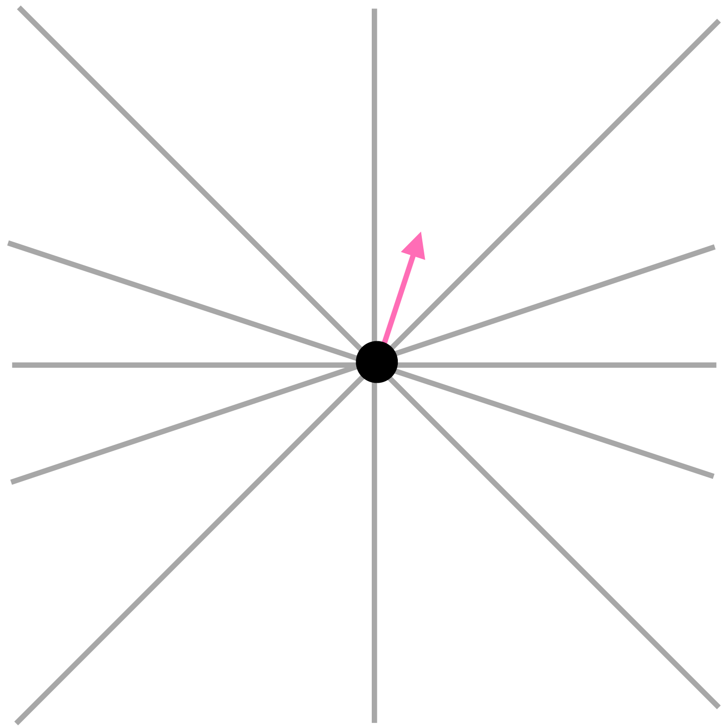
For which $u \in S^{d-1}$ does $\mathscr{C}^u_{\uparrow}(P)$ induce the same ordering of the vertices?

Consider the central hyperplane arrangement (called sweep arrangement)

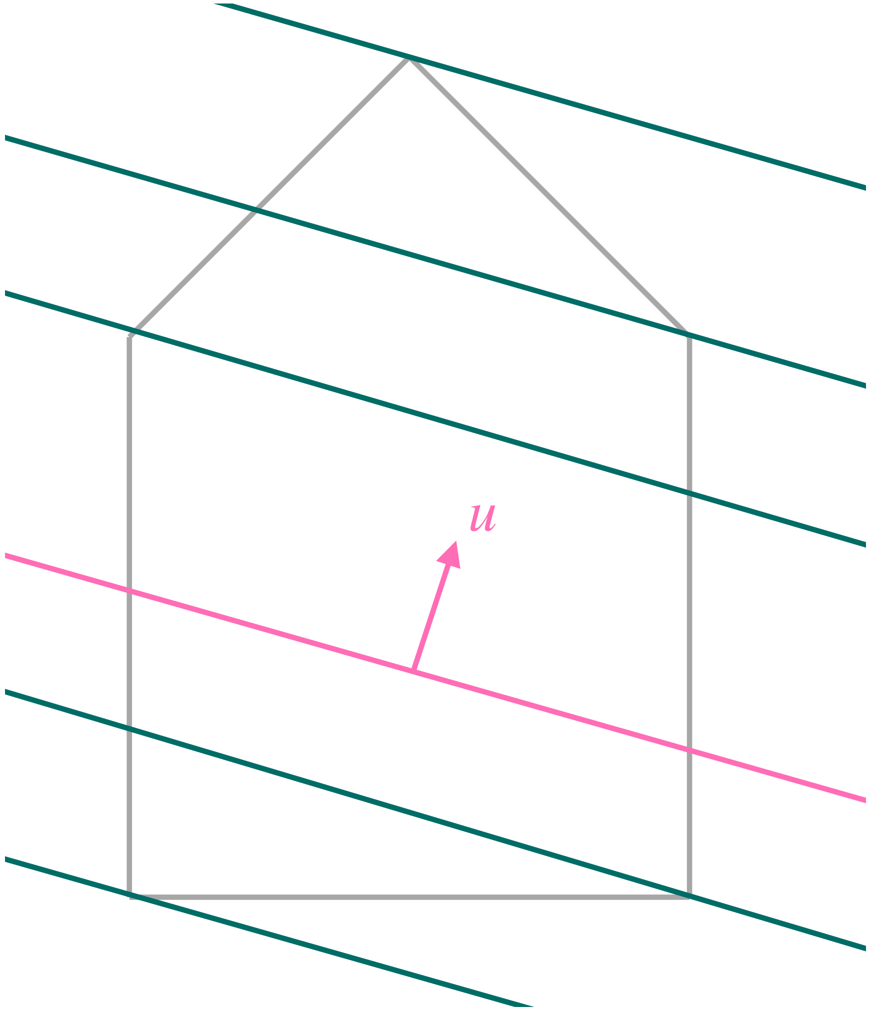
$$\mathscr{R}_{\uparrow}(P) = \{(v_i - v_j)^{\perp} \mid v_i, v_j \text{ are vertices of } P\}$$

$\longrightarrow$ with each region of $\mathscr{R}_{\uparrow}(P)$ the induced ordering given by $\mathscr{C}^u_{\uparrow}(P)$ is fixed

Translational Approach

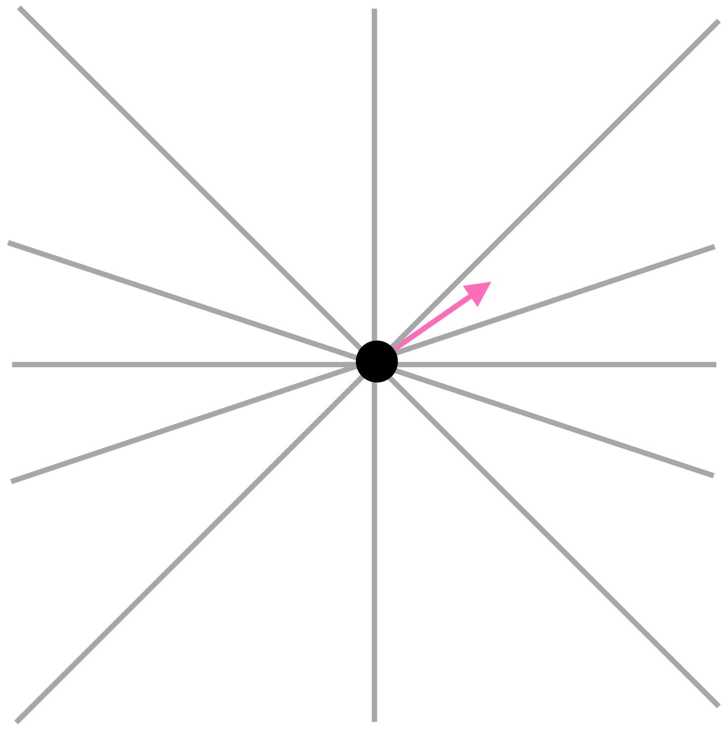


$\mathscr{R}_{\uparrow}(P) = \{(v_i - v_j)^{\perp} \mid v_i, v_j \text{ are vertices of } P\}$

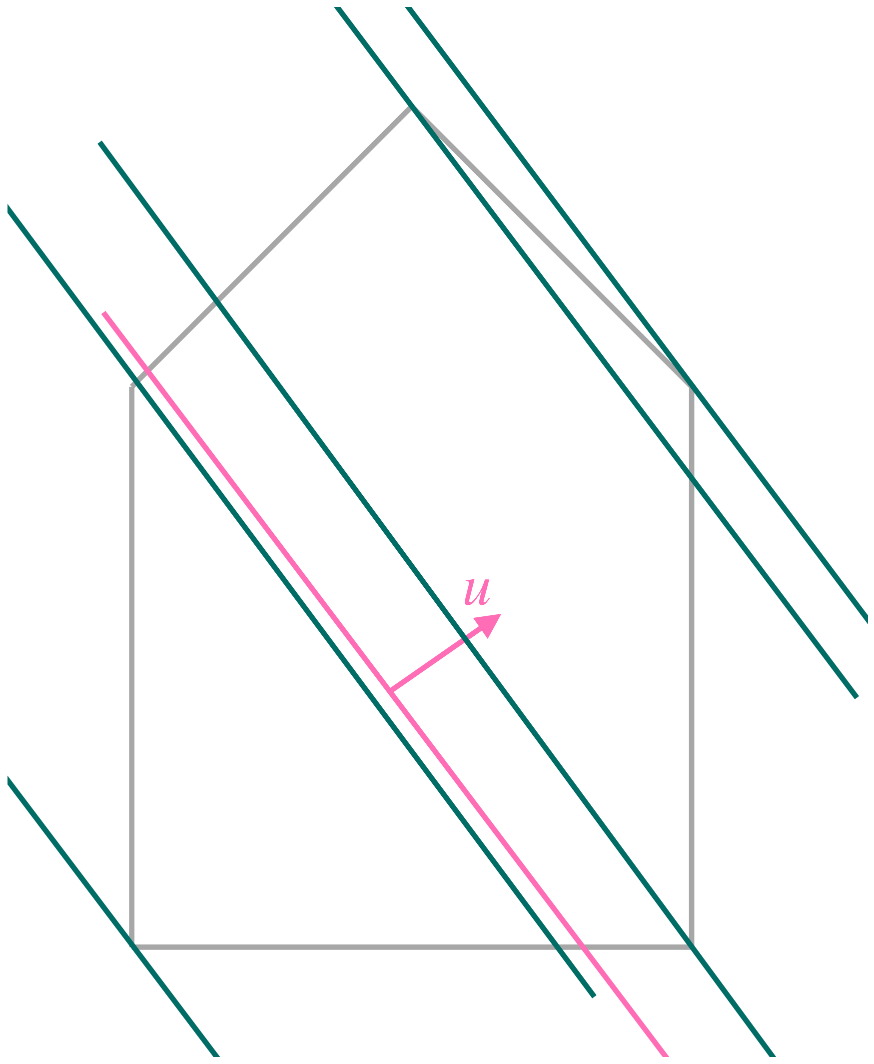


$\mathscr{C}^{u}_{\uparrow}(P) = \{H(\langle v, u\rangle \mid v \text{ is a vertex of } P\}$

Translational Approach



$\mathscr{R}_\uparrow(P) = \{(v_i - v_j)^\perp \mid v_i, v_j \text{ are vertices of } P\}$



$\mathscr{C}^u_\uparrow(P) = \{H(\langle v, u\rangle \mid v \text{ is a vertex of } P\}$

Translational Approach

Theorem (B.-Meroni-De Loera '23):

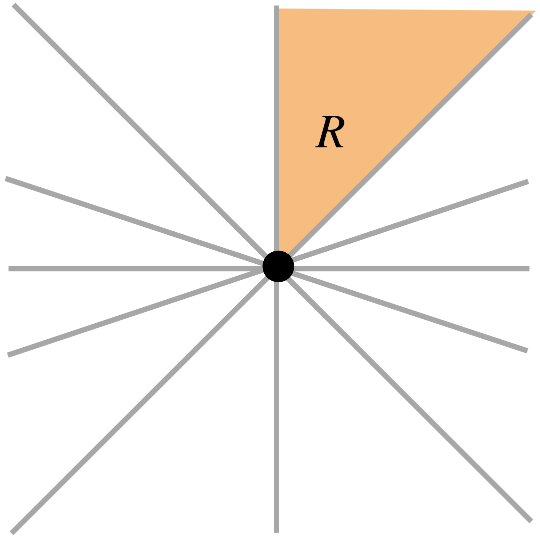
Let $P \subseteq \mathbb{R}^d$ be a polytope and $f(x) = \sum_\alpha c_\alpha x^\alpha$ be a polynomial in variables $x_1, \ldots, x_d$. Fix a region $R \in \mathcal{R}_\uparrow(P)$ of the sweep arrangement, a unit direction $u \in R \cap S^{d-1}$ and a chamber $C(u) \in \mathcal{C}^u_\uparrow(P)$ of the parallel arrangement.

Restricted to $u \in R \cap S^{d-1}$ and $H(\beta) \in C(u)$, the integral

$$\int_{P \cap H(\beta)} f(x) \, \mathrm{d}x$$

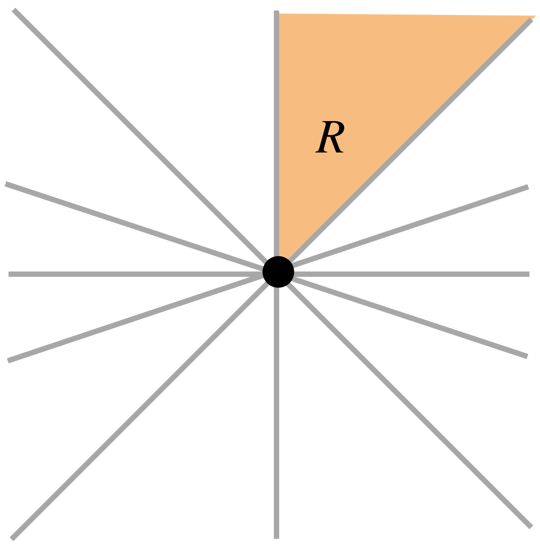
is a rational function in variables $u_1, \ldots, u_d, \beta$.

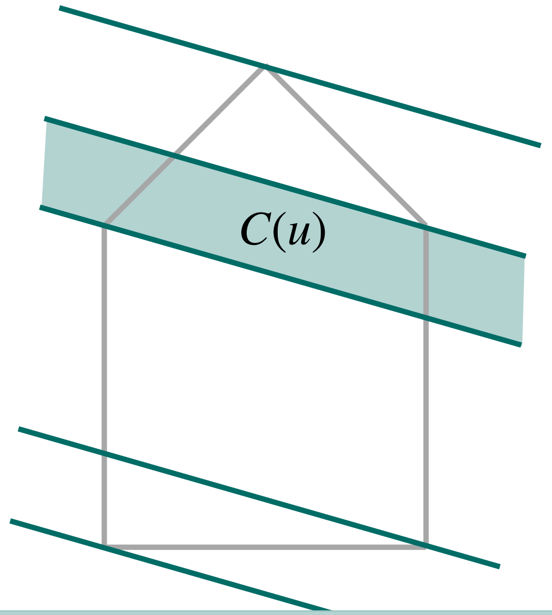
Translational Approach



$$(u_1, u_2) \in R \iff \begin{aligned} u_1 &\geq 0 \\ u_1 - u_2 &\leq 0 \end{aligned}$$

Translational Approach

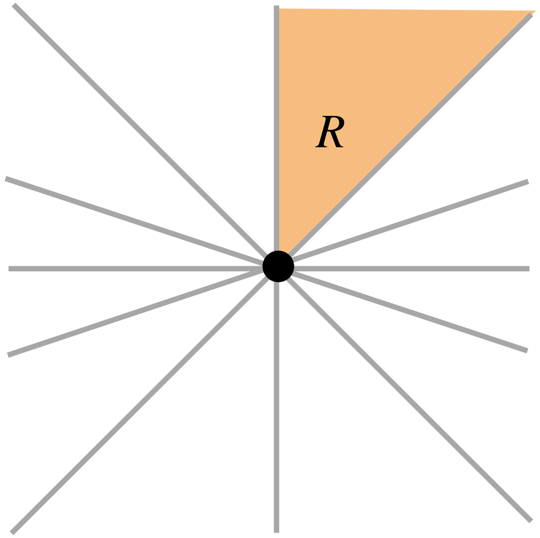


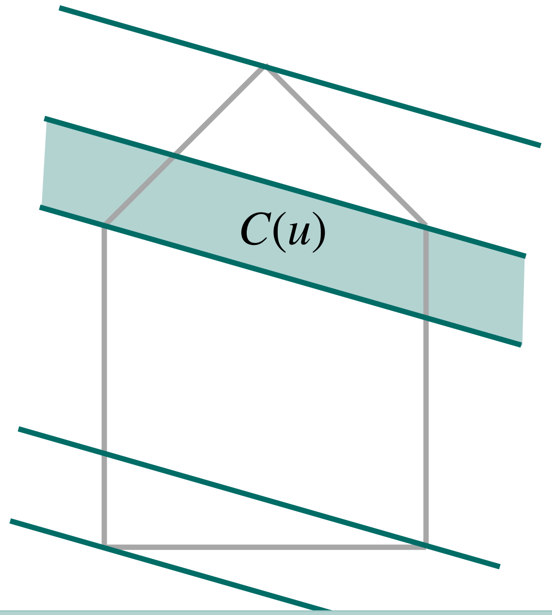


$$(u_1, u_2) \in R \iff \begin{aligned} u_1 &\geq 0 \\ u_1 - u_2 &\leq 0 \end{aligned}$$

If $(u_1, u_2) \in R \cap S^{d-1}$ then $\beta \in C(u) \iff$
$u_1 - u_2 \leq \beta \leq -u_1 + u_2$

Translational Approach



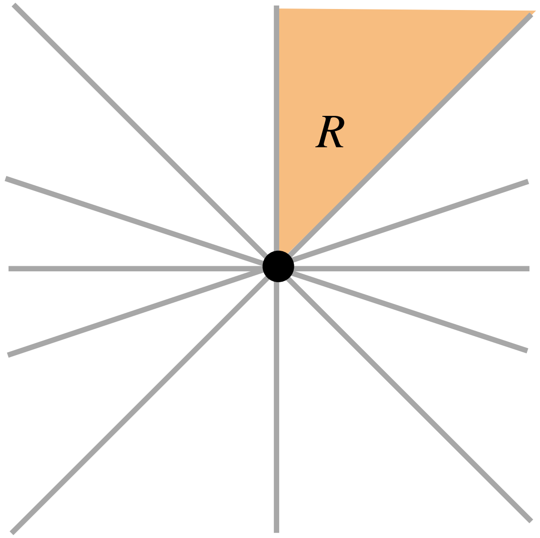


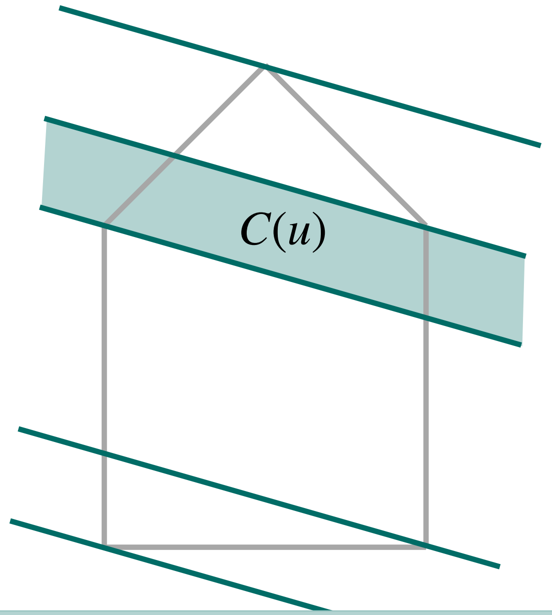
$$(u_1, u_2) \in R \iff \begin{array}{r} u_1 \geq 0 \\ u_1 - u_2 \leq 0 \end{array}$$

If $(u_1, u_2) \in R \cap S^{d-1}$ then $\beta \in C(u) \iff$
$$u_1 - u_2 \leq \beta \leq -u_1 + u_2$$

If $u \in R \cap S^{d-1}$ and $H(\beta) \in C(u)$ then
$$\text{vol}((P + t) \cap u^{\perp}) = \frac{-(\beta - u_1 - 3u_2)}{u_2(u_1 + u_2)}$$

Translational Approach





$(u_1, u_2) \in R \iff u_1 \geq 0$, $u_1 - u_2 \leq 0$

If $(u_1, u_2) \in R \cap S^{d-1}$ then $\beta \in C(u) \iff u_1 - u_2 \leq \beta \leq -u_1 + u_2$

If $u \in R \cap S^{d-1}$ and $H(\beta) \in C(u)$ then

$$\mathrm{vol}((P+t) \cap u^{\perp}) = \frac{-(\beta - u_1 - 3u_2)}{u_2(u_1 + u_2)}$$

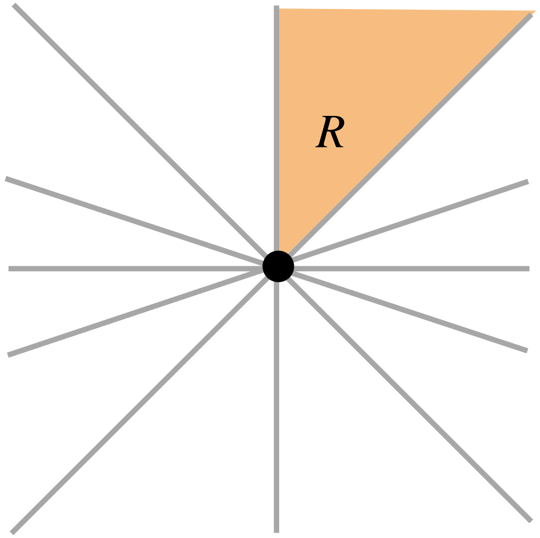
Let the computer find the biggest slice:

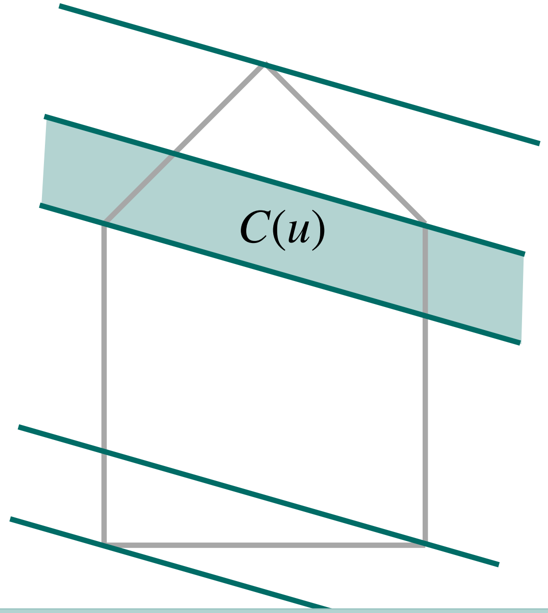
maximize $\dfrac{-(\beta - u_1 - 3u_2)}{u_2(u_1 + u_2)}$

s.t $(u_1, u_2) \in R \cap S^{d-1}$

$H(\beta) \in C(u)$

Translational Approach



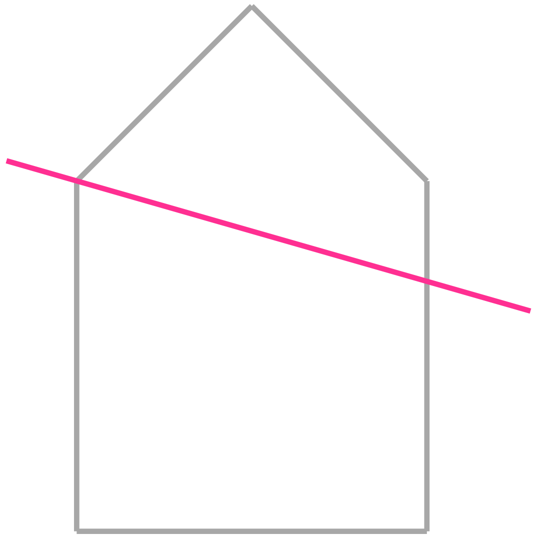


Let the computer find the biggest slice:

maximize	$\dfrac{-(\beta - u_1 - 3u_2)}{u_2(u_1 + u_2)}$
s.t	$(u_1, u_2) \in R \cap S^{d-1}$
	$H(\beta) \in C(u)$

$(u_1, u_2) \in R \iff \begin{array}{r} u_1 \geq 0 \\ u_1 - u_2 \leq 0 \end{array}$

If $(u_1, u_2) \in R \cap S^{d-1}$ then $\beta \in C(u) \iff$
$u_1 - u_2 \leq \beta \leq -u_1 + u_2$



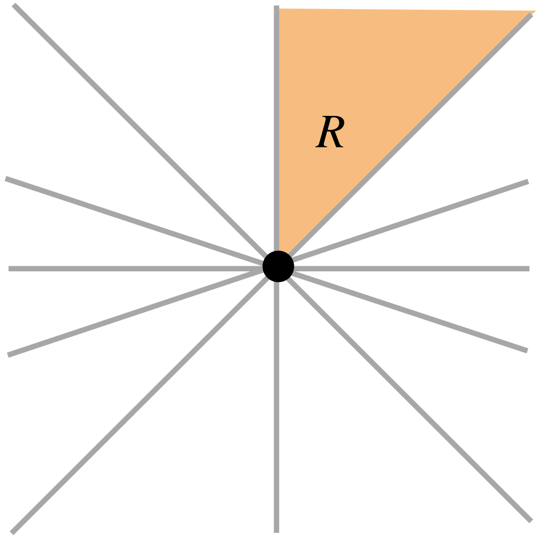
If $u \in R \cap S^{d-1}$ and $H(\beta) \in C(u)$ then

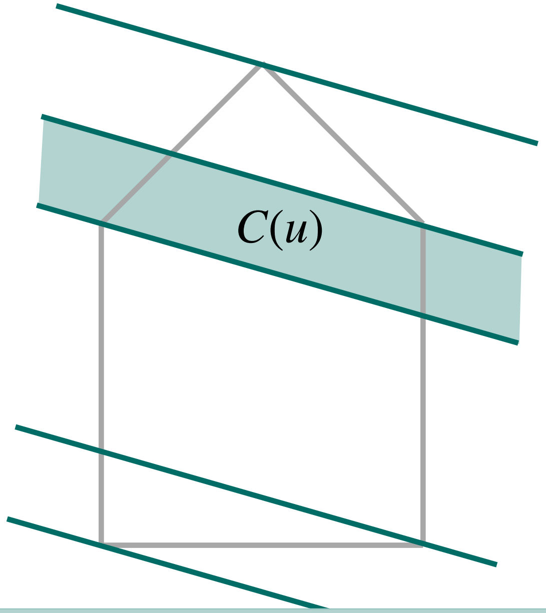
$$\text{vol}((P + t) \cap u^{\perp}) = \frac{-(\beta - u_1 - 3u_2)}{u_2(u_1 + u_2)}$$

Translational Approach

Let the computer find the biggest slice:

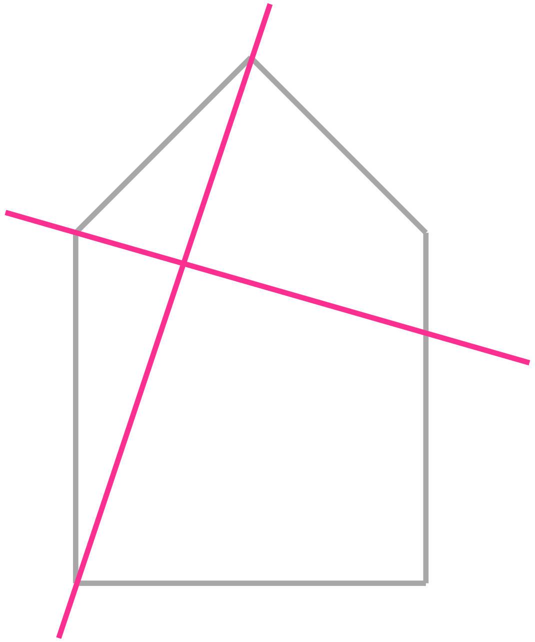
maximize	$\dfrac{-(\beta - u_1 - 3u_2)}{u_2(u_1 + u_2)}$
s.t	$(u_1, u_2) \in R \cap S^{d-1}$
	$H(\beta) \in C(u)$





$(u_1, u_2) \in R \iff \begin{array}{r} u_1 \geq 0 \\ u_1 - u_2 \leq 0 \end{array}$

If $(u_1, u_2) \in R \cap S^{d-1}$ then $\beta \in C(u) \iff$
$u_1 - u_2 \leq \beta \leq -u_1 + u_2$



If $u \in R \cap S^{d-1}$ and $H(\beta) \in C(u)$ then

$$\mathrm{vol}((P + t) \cap u^{\perp}) = \frac{-(\beta - u_1 - 3u_2)}{u_2(u_1 + u_2)}$$

ROTATION VS TRANSLATION

COMPARISON OF THE APPROACHES

Comparison

Running time of the algorithm $\longleftrightarrow$ number of chambers in the arrangements

$n =$ # vertices of P

Comparison

Running time of the algorithm $\longleftrightarrow$ number of chambers in the arrangements

n = # vertices of P

Rotational Approach

Arrangement	#Chambers (incl. lower-dim. cells)
$\mathscr{C}_{\circlearrowleft}(P)$	$O(n^d 2^d)$
$\mathscr{R}_{\circlearrowleft}(P)$	$O(n^{d^2} 2^d)$
Total	$O(n^{d^2+d} 2^d)$

Comparison

Running time of the algorithm $\longleftrightarrow$ number of chambers in the arrangements

$n =$ # vertices of P

Rotational Approach

Arrangement	#Chambers (incl. lower-dim. cells)
$\mathcal{C}_{\circlearrowleft}(P)$	$O(n^{d}2^{d})$
$\mathcal{R}_{\circlearrowleft}(P)$	$O(n^{d^2}2^{d})$
Total	$O(n^{d^2+d}2^{d})$

Translational Approach

Arrangement	#Chambers (incl. lower-dim.)
$\mathcal{C}_{\uparrow}(P)$	$O(n)$
$\mathcal{R}_{\uparrow}(P)$	$O(n^{2d}2^{d})$
Total	$O(n^{2d+1}2^{d})$

Comparison

Running time of the algorithm $\longleftrightarrow$ number of chambers in the arrangements

n = # vertices of P

Rotational Approach

Arrangement	#Chambers (incl. lower-dim. cells)
$\mathscr{C}_{\circlearrowleft}(P)$	$O(n^d 2^d)$
$\mathscr{R}_{\circlearrowleft}(P)$	$O(n^{d^2} 2^d)$
Total	$O(n^{d^2+d} 2^d)$

Translational Approach

Arrangement	#Chambers (incl. lower-dim.)
$\mathscr{C}_{\uparrow}(P)$	$O(n)$
$\mathscr{R}_{\uparrow}(P)$	$O(n^{2d} 2^d)$
Total	$O(n^{2d+1} 2^d)$

If $d \in \mathbb{N}$ is fixed then all of these are polynomials in n

$\longrightarrow$ both approaches yield algorithms in polynomial running time

Comparison

Running time of the algorithm $\longleftrightarrow$ number of chambers in the arrangements

n = # vertices of P

Rotational Approach

Arrangement	#Chambers (incl. lower-dim. cells)
$\mathscr{C}_{\circlearrowleft}(P)$	$O(n^d 2^d)$
$\mathscr{R}_{\circlearrowleft}(P)$	$O(n^{d^2} 2^d)$
Total	$O(n^{d^2+d} 2^d)$

Translational Approach

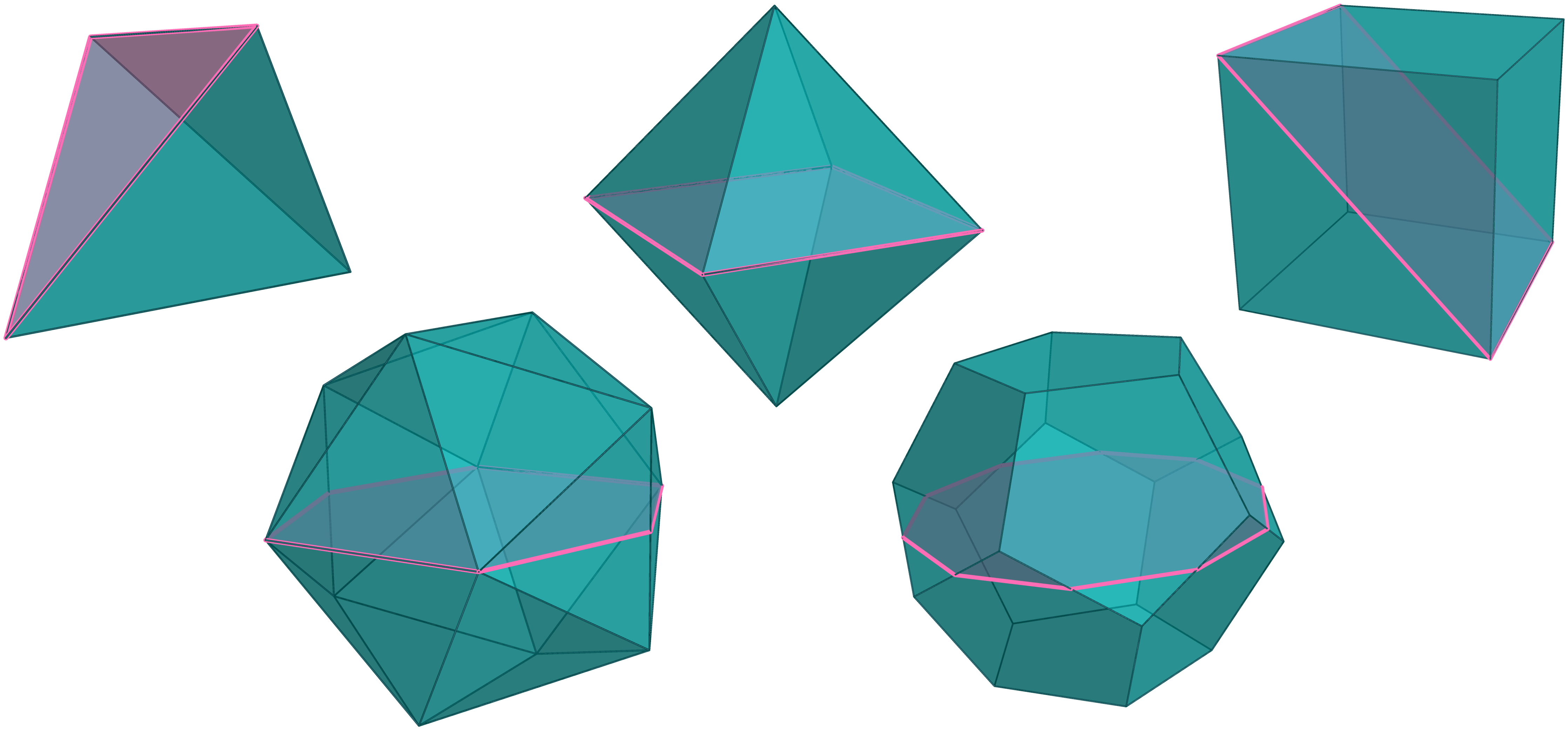
Arrangement	#Chambers (incl. lower-dim.)
$\mathscr{C}_{\uparrow}(P)$	$O(n)$
$\mathscr{R}_{\uparrow}(P)$	$O(n^{2d} 2^d)$
Total	$O(n^{2d+1} 2^d)$

If $d \in \mathbb{N}$ is fixed then all of these are polynomials in n

$\longrightarrow$ both approaches yield algorithms in polynomial running time

$\longrightarrow$ Translational approach runs much faster

Maximum Volume Slices of Platonic solids



VARIATIONS

What Else?

With the same methods we can understand...

- Intersections with half-spaces
- Projections onto hyperplanes
- Combinatorial types

What Else?

With the same methods we can understand...

- Intersections with half-spaces
- Projections onto hyperplanes
- Combinatorial types

We can optimize for...

- volume
- Integral of a polynomial
- Number of k-dimensional faces

What Else?

With the same methods we can understand…

- Intersections with half-spaces
- Projections onto hyperplanes
- Combinatorial types

We can optimize for…

- volume
- Integral of a polynomial
- Number of k-dimensional faces

We can compute all of this in polynomial time in fixed dimension

(! Most of these problems are known to be (at least) NP-HARD in non-fixed dimension !)

WHAT ELSE?

WITH THE SAME METHODS WE CAN UNDERSTAND…

- Intersections with half-spaces
- Projections onto hyperplanes
- Combinatorial types

WE CAN OPTIMIZE FOR…

- volume
- Integral of a polynomial
- Number of k-dimensional faces

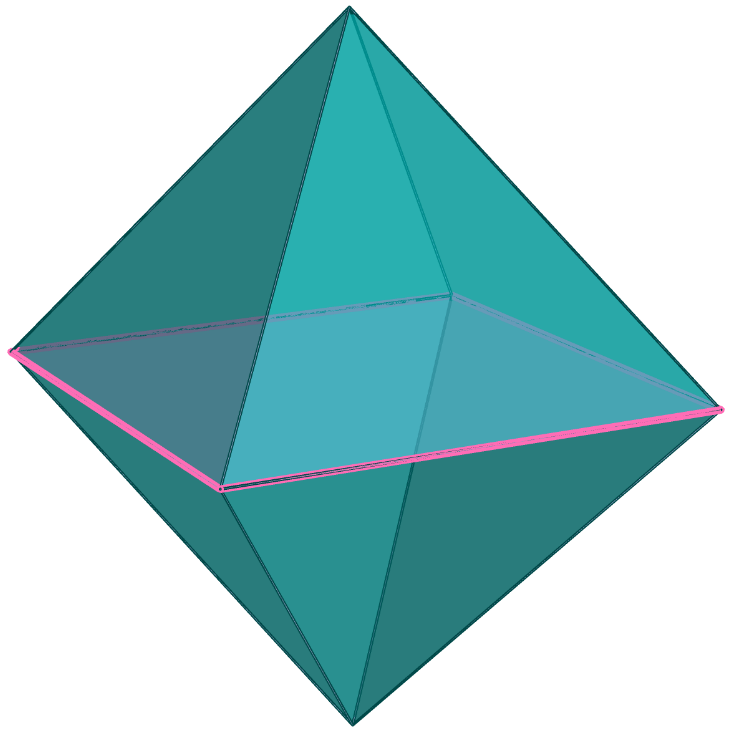
WE CAN COMPUTE ALL OF THIS IN POLYNOMIAL TIME IN FIXED DIMENSION

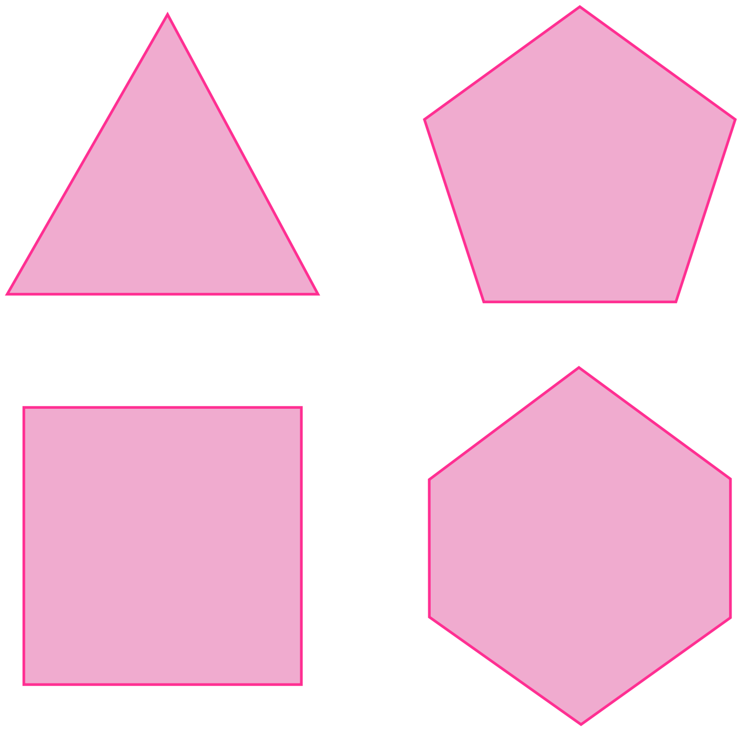
(! MOST OF THESE PROBLEMS ARE KNOWN TO BE (AT LEAST) NP-HARD IN NON-FIXED DIMENSION !)

Combinatorial Types of Sections of the Cross-Polytope

$P = \mathrm{conv}(\pm e_i \mid i \in [d])$

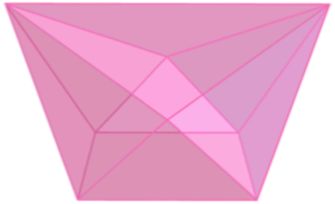
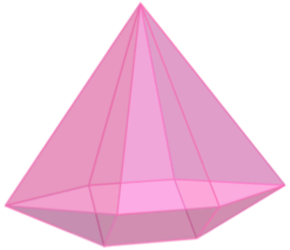
$d = 3$





Combinatorial Types of Sections of the Cross-Polytope

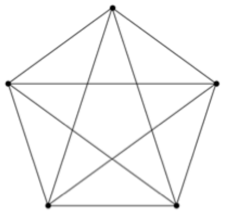
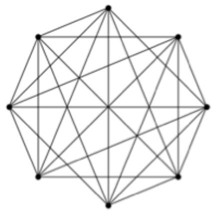
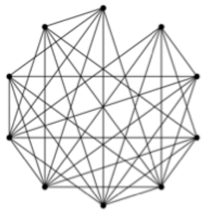
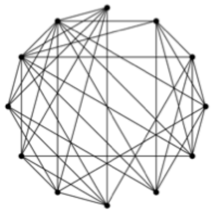
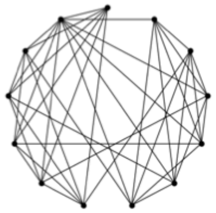
$P = \mathrm{conv}(\pm e_i \mid i \in [d])$ $\qquad$ $d = 4$

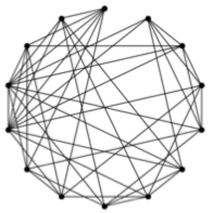
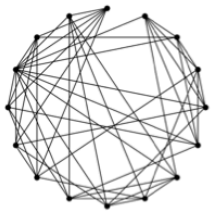
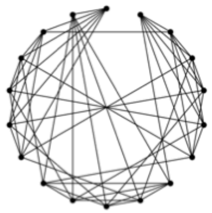
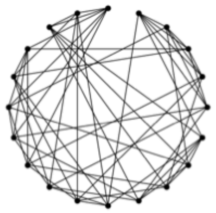
$P \cap H$					
f-vector	$(4, 6, 4)$	$(6, 12, 8)$	$(8, 18, 12)$	$(8, 17, 11)$	$(9, 19, 12)$
H	$x_1 + x_2 + x_3 + x_4 = 1$	$2x_1 = 1$	$x_1 + x_2 + x_3 = 0$	$2x_1 + 2x_2 + x_3 + x_4 = 1$	$2x_1 + 2x_2 + x_3 = 1$

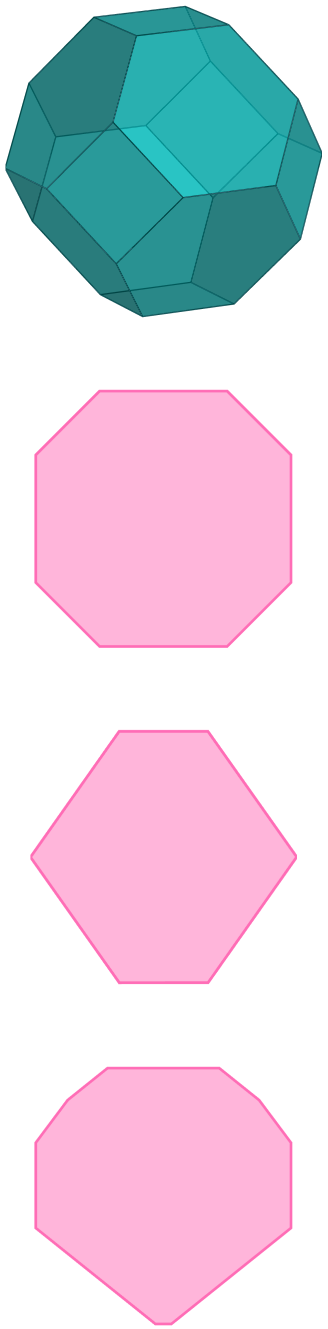
$P \cap H$				
f-vector	$(8, 18, 12)$	$(10, 21, 13)$	$(12, 24, 14)$	$(12, 24, 14)$
H	$x_1 + x_2 + x_3 = 0$	$+ 2x_2 + 2x_3 + x_4 = 1$	$x_1 + x_2 + x_3 + x_4 = 0$	$2x_1 + 2x_2 + 2x_3 = 1$

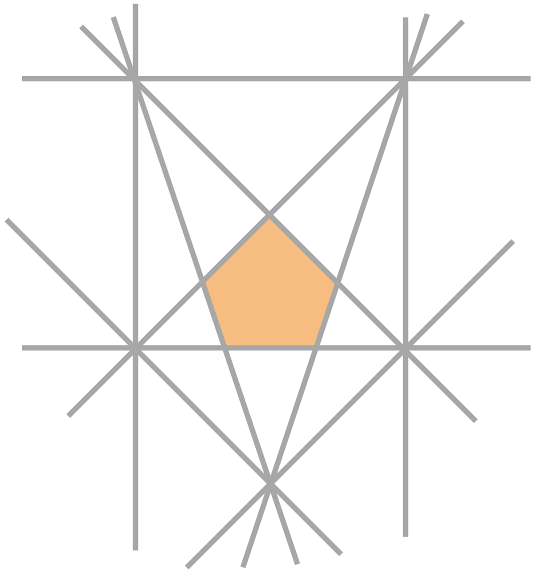
Combinatorial Types of Sections of the Cross-Polytope

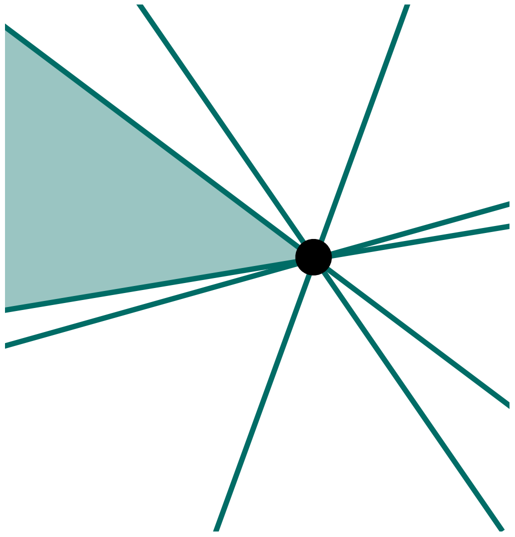
$P = \mathrm{conv}(\pm e_i \mid i \in [d])$ $\qquad\qquad$ $d = 5$

$P \cap H$								
f-vector	$(5, 10, 10, 5)$	$(8, 24, 32, 16)$	$(10, 34, 48, 24)$	$(11, 36, 48, 23)$	$(12, 39, 51, 24)$	$(13, 41, 52, 24)$	$(14, 42, 52, 24)$	$(14, 48, 62, 28)$
H	$x_1 + x_2 + x_3 + x_4 + x_5 = 1$	$2x_1 = 1$	$x_1 + x_2 + x_3 = 0$	$2x_1 + 2x_2 + x_3 + x_4 + x_5 = 1$	$2x_1 + 2x_2 + x_3 + x_4 = 1$	$2x_1 + 2x_2 + x_3 = 1$	$2x_1 + 2x_2 = 1$	$x_1 + x_2 + x_3 + x_4 = 0$

$P \cap H$						
f-vector	$(14, 46, 59, 27)$	$(16, 51, 63, 28)$	$(17, 54, 66, 29)$	$(18, 54, 64, 28)$	$(20, 60, 70, 30)$	$(20, 60, 70, 30)$
H	$2x_1 + 2x_2 + 2x_3 + x_4 + x_5 = 1$	$2x_1 + 2x_2 + 2x_3 + x_4 = 1$	$2x_1 + 2x_2 + 2x_3 + 2x_4 + x_5 = 1$	$2x_1 + 2x_2 + 2x_3 = 1$	$2x_1 + 2x_2 + 2x_3 + 2x_4 = 1$	$x_1 + x_2 + x_3 + x_4 + x_5 = 0$







THANK YOU!