Multivariate Volume, Ehrhart and h^*-Polynomials of Polytropes

Marie Brandenburg and Sophia Elia
joint work with Leon Zhang (UC Berkeley)

23 July 2020

Tropical Semiring is $\mathbb{T} = (\mathbb{R} \cup \{\infty\}, \oplus, \odot)$ with addition and multiplication

$$a \oplus b = \min(a, b)$$
$$a \odot b = a + b$$

Tropical Semiring is $\mathbb{T} = (\mathbb{R} \cup \{\infty\}, \oplus, \odot)$ with addition and multiplication

$$a \oplus b = \min(a, b)$$
$$a \odot b = a + b$$

On $\mathbb{T}^n = ((\mathbb{R} \cup \{\infty\})^n, \oplus, \odot)$ addition and multiplication is defined componentwise:

$$\mathbf{v} \oplus \mathbf{w} = \begin{pmatrix} \min(v_1, w_1) \\ \vdots \\ \min(v_n, w_m) \end{pmatrix} \quad \lambda \odot \mathbf{v} = \begin{pmatrix} \lambda + v_1 \\ \vdots \\ \lambda + v_n \end{pmatrix}$$

Tropical Semiring is $\mathbb{T} = (\mathbb{R} \cup \{\infty\}, \oplus, \odot)$ with addition and multiplication

$$a \oplus b = \min(a, b)$$
$$a \odot b = a + b$$

On $\mathbb{T}^n = ((\mathbb{R} \cup \{\infty\})^n, \oplus, \odot)$ addition and multiplication is defined componentwise:

$$\mathbf{v} \oplus \mathbf{w} = \begin{pmatrix} \min(v_1, w_1) \\ \vdots \\ \min(v_n, w_m) \end{pmatrix} \quad \lambda \odot \mathbf{v} = \begin{pmatrix} \lambda + v_1 \\ \vdots \\ \lambda + v_n \end{pmatrix}$$

Example: $$1 \odot \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \oplus \begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} \min(1+0, 5) \\ \min(1+1, 2) \\ \min(1+2, 0) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

Definition

Let $V = \{v_1, \dots, v_r\} \subseteq \mathbb{R}^n$. The tropical convex hull of V is

$$\mathrm{tconv}(V) = \{a_1 \odot \mathbf{v}_1 \oplus \cdots \oplus a_r \odot \mathbf{v}_r \mid a_1, \dots, a_r \in \mathbb{R}\}$$

$$x \in \mathrm{tconv}(V) \implies \lambda \odot x = x + \lambda(1, \dots, 1)^T \in \mathrm{tconv}(V).$$

Identify $\mathrm{tconv}(V)$ with its image in the tropical projective torus $\mathbb{TP}^{n-1} = \mathbb{R}^n / \mathbb{R} \odot (1, \dots, 1)^T$.

Tropical Polytopes

Definition

Let $V = \{v_1, \ldots, v_r\} \subseteq \mathbb{R}^n$. The tropical convex hull of V is

$$\mathrm{tconv}(V) = \{a_1 \odot \mathbf{v}_1 \oplus \cdots \oplus a_r \odot \mathbf{v}_r \mid a_1, \ldots, a_r \in \mathbb{R}\}$$

$$x \in \mathrm{tconv}(V) \implies \lambda \odot x = x + \lambda(1, \ldots, 1)^T \in \mathrm{tconv}(V).$$

Identify $\mathrm{tconv}(V)$ with its image in the tropical projective torus

$$\mathbb{TP}^{n-1} = \mathbb{R}^n / \mathbb{R} \odot (1, \ldots, 1)^T.$$

Example: $\mathrm{tconv}\begin{pmatrix} 0 & 2 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} \min(a_1, a_2+2) \\ \min(a_1, a_2+1) \\ \min(a_1, a_2) \end{pmatrix} \mid a_1, a_2 \in \mathbb{R}, \min(a_1, a_2) = 0 \right\}$

Tropical Polytopes

Definition

Let $V = \{v_1, \ldots, v_r\} \subseteq \mathbb{R}^n$. The tropical convex hull of V is

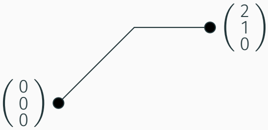
$$\mathrm{tconv}(V) = \{a_1 \odot \mathbf{v}_1 \oplus \cdots \oplus a_r \odot \mathbf{v}_r \mid a_1, \ldots, a_r \in \mathbb{R}\}$$

$$x \in \mathrm{tconv}(V) \implies \lambda \odot x = x + \lambda(1, \ldots, 1)^T \in \mathrm{tconv}(V).$$

Identify $\mathrm{tconv}(V)$ with its image in the tropical projective torus

$$\mathbb{TP}^{n-1} = \mathbb{R}^n / \mathbb{R} \odot (1, \ldots, 1)^T.$$

Example: $$\mathrm{tconv}\begin{pmatrix} 0 & 2 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} \min(a_1, a_2+2) \\ \min(a_1, a_2+1) \\ 0 \end{pmatrix} \;\middle|\; a_1, a_2 \in \mathbb{R}, \min(a_1, a_2) = 0 \right\}$$



Definition

Let $V = \{v_1, \ldots, v_r\} \subseteq \mathbb{R}^n$. The tropical convex hull of V is

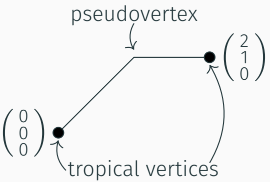
$$\mathrm{tconv}(V) = \{a_1 \odot \mathbf{v}_1 \oplus \cdots \oplus a_r \odot \mathbf{v}_r \mid a_1, \ldots, a_r \in \mathbb{R}\}$$

$$x \in \mathrm{tconv}(V) \implies \lambda \odot x = x + \lambda(1, \ldots, 1)^T \in \mathrm{tconv}(V).$$

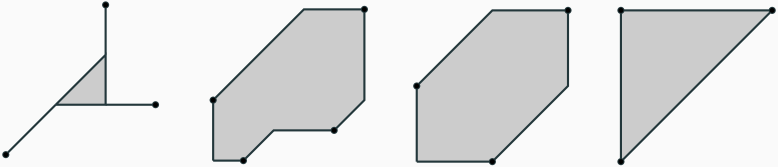
Identify $\mathrm{tconv}(V)$ with its image in the tropical projective torus

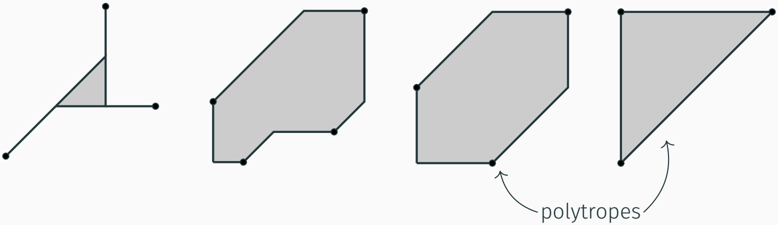
$$\mathbb{TP}^{n-1} = \mathbb{R}^n / \mathbb{R} \odot (1, \ldots, 1)^T.$$

Example: $\mathrm{tconv}\begin{pmatrix} 0 & 2 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} \min(a_1, a_2+2) \\ \min(a_1, a_2+1) \\ 0 \end{pmatrix} \mid a_1, a_2 \in \mathbb{R}, \min(a_1, a_2) = 0 \right\}$



Tropical Polytopes

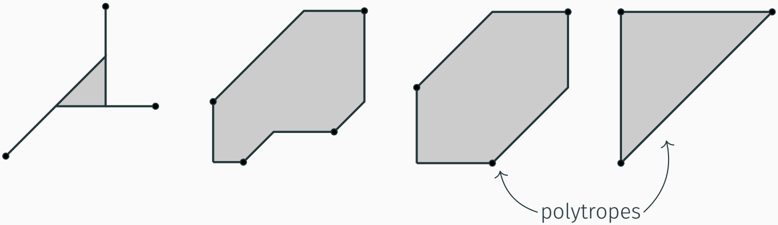




Definition

Polytropes are tropical polytropes that are clasically convex.

Tropical Polytopes



Definition

Polytropes are tropical polytropes that are clasically convex.

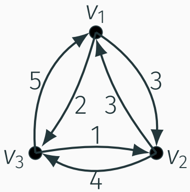
Question

Given the tropical vertices of a tropical polytope P, how can we tell if P is a polytrope?

↳ Kleene stars!

$$c = \begin{pmatrix} 0 & 3 & 2 \\ 3 & 0 & 4 \\ 5 & 1 & 0 \end{pmatrix}$$

$(n \times n)$-matrix with
0's along the diagonal



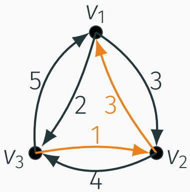
weighted complete digraph K_n

$$c^* = \begin{pmatrix} 0 & 3 & 2 \\ 3 & 0 & 4 \\ 4 & 1 & 0 \end{pmatrix}$$

weight of shortest path from vertex v_i to v_j in K_n

$$c = \begin{pmatrix} 0 & 3 & 2 \\ 3 & 0 & 4 \\ 5 & 1 & 0 \end{pmatrix}$$

$(n \times n)$-matrix with
0's along the diagonal



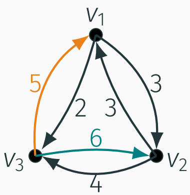
weighted complete digraph K_n

$$c^* = \begin{pmatrix} 0 & 3 & 2 \\ 3 & 0 & 4 \\ 4 & 1 & 0 \end{pmatrix}$$

weight of shortest path from vertex v_i to v_j in K_n

$$c = \begin{pmatrix} 0 & 3 & 2 \\ 3 & 0 & 4 \\ 5 & 6 & 0 \end{pmatrix}$$

$(n \times n)$-matrix with 0's along the diagonal



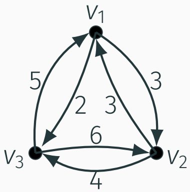
weighted complete digraph K_n

$$c^* = \begin{pmatrix} 0 & 3 & 2 \\ 3 & 0 & 4 \\ 5 & 6 & 0 \end{pmatrix}$$

weight of shortest path from vertex v_i to v_j in K_n

$$c = \begin{pmatrix} 0 & 3 & 2 \\ 3 & 0 & 4 \\ 5 & 6 & 0 \end{pmatrix}$$

$(n \times n)$-matrix with
0's along the diagonal



weighted complete digraph K_n

$$c^* = \begin{pmatrix} 0 & 3 & 2 \\ 3 & 0 & 4 \\ 5 & 6 & 0 \end{pmatrix}$$

weight of shortest path from vertex v_i to v_j in K_n

Definition

A matrix $\mathbf{c} \in \mathbb{R}^{n \times n}$ is a Kleene star if $\mathbf{c} = \mathbf{c}^*$. The polytrope region is the set $\mathcal{P}ol_n = \{\mathbf{c} \in \mathbb{R}^{n \times n} \mid \mathbf{c} \text{ is a Kleene star}\}$.

Proposition (de la Puente '13)

Let $P \subseteq \mathbb{TP}^{n-1}$ be a non-empty set. The following are equivalent:

(1) P is a polytrope.

(2) There is a Kleene star $\mathbf{c} \in \mathcal{P}ol_n$ such that $P = \mathbf{tconv}(\mathbf{c})$.

(3) There is a Kleene star $\mathbf{c} \in \mathcal{P}ol_n$ such that
$P = \{y \in \mathbb{R}^n \mid y_i - y_j \leq c_{ij}, y_n = 0\}$.

Furthermore, the $\mathbf{c}$'s in the last two statements are equal, and are uniquely determined by P.

Definition

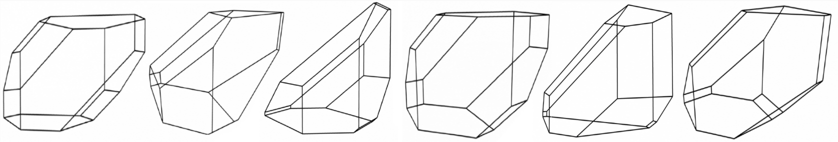
A polytrope is called maximal if it has $\binom{2n-2}{n-1}$ vertices as an ordinary polytope.

Definition

A polytrope is called maximal if it has $\binom{2n-2}{n-1}$ vertices as an ordinary polytope.

dimension	2	3	4	5
# comb. types of max. polytropes	1	6	27248	?

Kulas-Joswig '08, Jiménez-De la Puente '12, Tran '17, Joswig-Schröter '19

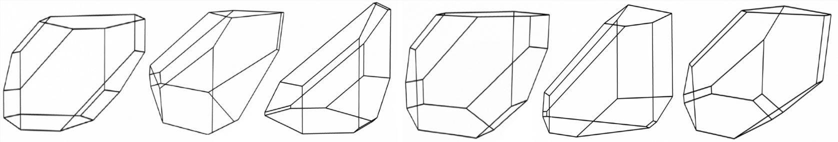


Definition

A polytrope is called maximal if it has $\binom{2n-2}{n-1}$ vertices as an ordinary polytope.

dimension	2	3	4	5
# comb. types of max. polytropes	1	6	27248	?

Kulas-Joswig '08, Jiménez-De la Puente '12, Tran '17, Joswig-Schröter '19



Question

Given two Kleene stars, how can we tell if they define polytropes of the same type?

Kleene Stars and Gröbner Fans

$$\mathcal{P}ol_n = \{\mathbf{c} \in \mathbb{R}^{n \times n} \mid \mathbf{c} \text{ is a Kleene star}\}$$

Gröbner fan $\mathcal{GF}_n$ of the ideal $I = \langle x_{ij}x_{ji} - 1, x_{ij}x_{jk} - x_{ik} \rangle$

$\rightarrow$ Subfan $\mathcal{GF}_n|_{\mathcal{P}ol_n}$

Kleene Stars and Gröbner Fans

$$\mathcal{P}ol_n = \{\mathbf{c} \in \mathbb{R}^{n\times n} \mid \mathbf{c} \text{ is a Kleene star}\}$$

Gröbner fan $\mathcal{GF}_n$ of the ideal $I = \langle x_{ij}x_{ji} - 1, x_{ij}x_{jk} - x_{ik}\rangle$

$\to$ Subfan $\mathcal{GF}_n|_{\mathcal{P}ol_n}$

Theorem (Tran, '17)

$$\text{Cones of } \mathcal{GF}_n|_{\mathcal{P}ol_n} \quad \xleftrightarrow{bij.} \quad \text{types of polytropes in } \mathbb{TP}^{n-1}$$

$$\text{Open max cones of } \mathcal{GF}_n|_{\mathcal{P}ol_n} \quad \xleftrightarrow{bij.} \quad \text{types of max polytropes in } \mathbb{TP}^{n-1}$$

$$\mathcal{P}ol_n = \{\mathbf{c} \in \mathbb{R}^{n\times n} \mid \mathbf{c} \text{ is a Kleene star}\}$$

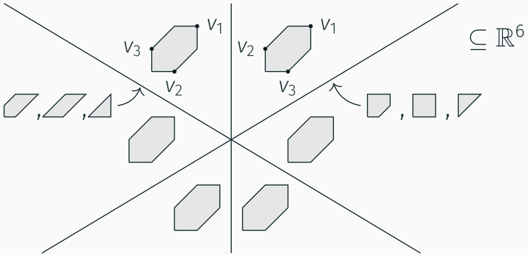
Gröbner fan $\mathcal{GF}_n$ of the ideal $I = \langle x_{ij}x_{ji} - 1, x_{ij}x_{jk} - x_{ik}\rangle$

$\to$ Subfan $\mathcal{GF}_n|_{\mathcal{P}ol_n}$

Theorem (Tran, '17)

Cones of $\mathcal{GF}_n|_{\mathcal{P}ol_n}$ $\xleftrightarrow{bij.}$ types of polytropes in $\mathbb{TP}^{n-1}$

Open max cones of $\mathcal{GF}_n|_{\mathcal{P}ol_n}$ $\xleftrightarrow{bij.}$ types of max polytropes in $\mathbb{TP}^{n-1}$



Goal

We want to compute a polynomial

$$\begin{aligned} p: \quad \text{open max cone of } \mathcal{GF}_n|_{\mathcal{P}ol_n} &\to \mathbb{R} \\ \text{Kleene star } \mathbf{c} &\mapsto \mathrm{Vol}(P_{\mathbf{c}}) \end{aligned}$$

Volume Polynomials

Goal

We want to compute a polynomial

$$\begin{array}{rccc} p: & \text{open max cone of } \mathcal{GF}_n|_{\mathcal{P}ol_n} & \to & \mathbb{R} \\ & \text{Kleene star } \mathbf{c} & \mapsto & \mathrm{Vol}(P_{\mathbf{c}}) \end{array}$$

Theorem

Let X be the smooth toric variety defined by the normal fan Σ of a maximal polytrope $P_{\mathbf{c}}$ and D_{ij} the divisors corresponding to the rays of Σ. Then $[D_{ij}] \in H^2(X, \mathbb{Q})$ and

$$p(a_{12}, a_{13}, \ldots, a_{n(n-1)}) = \int_X \left[\sum a_{ij} D_{ij}\right]^{\dim P}$$

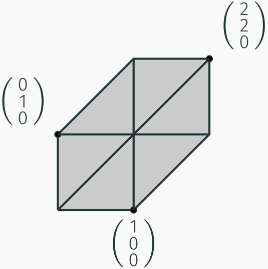
is a polynomial such that $p(\mathbf{c}) = \mathrm{Vol}(P_{\mathbf{c}})$.

There is an algorithm (de Loera-Sturmfels '03) that only depends on the choice of the cone.

Volume Polynomials

$$
\begin{aligned}
Vol(\mathbf{a}) = &- a_{12}^2 - a_{13}^2 - a_{21}^2 - a_{23}^2 - a_{31}^2 - a_{32}^2 + 2a_{12}a_{13} \\
&+ 2a_{13}a_{23} + 2a_{21}a_{23} + 2a_{21}a_{31} + 2a_{12}a_{32} + 2a_{31}a_{32}
\end{aligned}
$$

$$
\mathbf{c} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}
$$



$$
Vol(\mathbf{c}) = 6
$$

Discrete volume $\rightarrow$ lattice point count

$ehr_P(t) = \left| tP \cap \mathbb{Z}^n \right|$

Discrete volume $\rightarrow$ lattice point count

$$ehr_P(t) = \left|tP \cap \mathbb{Z}^n\right|$$
$$= \left| \{x \in \mathbb{Z}^n \mid Ax \leq ta\}\right|,\ A \in \mathbb{R}^{m\times n}, a \in \mathbb{R}^m$$

Discrete volume $\rightarrow$ lattice point count

$$ehr_P(t) = \left|tP \cap \mathbb{Z}^n\right|$$
$$= \left| \{x \in \mathbb{Z}^n \mid Ax \leq ta\}\right|,\ A \in \mathbb{R}^{m\times n}, a \in \mathbb{R}^m$$
$$= \underline{c_d} t^{\dim P} + c_{\dim P-1} t^{d-1} \cdots + c_0$$

$\rightarrow c_{\dim P} = \text{volume}\ (P)$

Discrete volume $\rightarrow$ lattice point count

$$
\begin{aligned}
ehr_P(t) = \left|tP \cap \mathbb{Z}^n\right| \\
&= \left| \{x \in \mathbb{Z}^n \mid Ax \leq ta\}\right|, \; A \in \mathbb{R}^{m\times n}, a \in \mathbb{R}^m \\
&= \underline{c_d} t^{\dim P} + c_{\dim P-1} t^{d-1} \cdots + c_0
\end{aligned}
$$

$\rightarrow c_{\dim P} = \text{volume}\ (P)$

Mulitvariate version- keep $a \in \mathbb{R}^m$ variable

$ehr_P(a_1, \ldots, a_m, t) \rightarrow$ independently move facets.

Discrete volume $\rightarrow$ lattice point count

$$
\begin{aligned}
ehr_P(t) &= \left|tP \cap \mathbb{Z}^n\right| \\
&= \left| \{x \in \mathbb{Z}^n \mid Ax \leq ta\}\right|,\ A \in \mathbb{R}^{m\times n}, a \in \mathbb{R}^m \\
&= \underline{c_d} t^{\dim P} + c_{\dim P-1} t^{d-1} \cdots + c_0
\end{aligned}
$$

$\rightarrow c_{\dim P} = \text{volume}\ (P)$

Mulitvariate version- keep $a \in \mathbb{R}^m$ variable

$ehr_P(a_1, \ldots, a_m, t) \rightarrow$ independently move facets.

Goal

Transform multivariate volume polynomials to multivariate Ehrhart polynomials.

From continuous to discrete volume

Discrete volume $\to$ lattice point count

$$ehr_P(t) = \left|tP \cap \mathbb{Z}^n\right|$$
$$= \left|\ \{x \in \mathbb{Z}^n \mid Ax \leq ta\}\right|,\ A \in \mathbb{R}^{m\times n}, a \in \mathbb{R}^m$$
$$= \underline{c_d} t^{\dim P} + c_{\dim P-1} t^{d-1} \cdots + c_0$$

$\to c_{\dim P} = \text{volume}\ (P)$

Mulitvariate version- keep $a \in \mathbb{R}^m$ variable

$ehr_P(a_1, \ldots, a_m, t) \to$ independently move facets.

Goal

Transform multivariate volume polynomials to multivariate Ehrhart polynomials.

- Use a differential operator - the *Todd Operator*

The *Todd operator* is the differential operator

$$\mathrm{Todd}_h = 1 + \sum_{k \geq 1} (-1)^k \frac{B_k}{k!} \left(\frac{d}{dh} \right)^k .$$

The *Todd operator* is the differential operator

$$\mathrm{Todd}_h = 1 + \sum_{k\geq 1} (-1)^k \frac{B_k}{k!} \left(\frac{d}{dh}\right)^k .$$

Bernoulli numbers: B_k, $k \in \mathbb{Z}_{\geq 0}$:

$$\frac{z}{e^z - 1} = \sum_{k\geq 0} \frac{B_k}{k!} z^k$$

From continuous to discrete volume

The *Todd operator* is the differential operator

$$\mathrm{Todd}_h = 1 + \sum_{k\geq 1} (-1)^k \frac{B_k}{k!} \left(\frac{d}{dh}\right)^k .$$

Bernoulli numbers: B_k, $k \in \mathbb{Z}_{\geq 0}$:

$$\frac{z}{e^z - 1} = \sum_{k\geq 0} \frac{B_k}{k!} z^k$$

Theorem (Khovanskiĭ-Pukhlikov, 1992)

Let $P \subseteq \mathbb{R}^n$ be a unimodular, lattice d-polytope. Then:

$$\#(P \cap \mathbb{Z}^n) = \mathrm{Todd}_{\mathbf{h}} \, \mathrm{vol}(P_{\mathbf{h}})|_{\mathbf{h}=0}.$$

Theorem (Khovanskiĭ-Pukhlikov, 1992)

Let $P \subseteq \mathbb{R}^n$ be a unimodular, lattice d-polytope. Then:

$$\#(P \cap \mathbb{Z}^n) = \mathsf{Todd}_{\mathsf{h}} \operatorname{vol}(P_{\mathsf{h}})|_{\mathsf{h}=0}.$$

unimodular: primitive vertex cone generators form $\mathbb{Z}^d$ basis.

From continuous to discrete volume

Theorem (Khovanskiĭ-Pukhlikov, 1992)

Let $P \subseteq \mathbb{R}^n$ be a unimodular, lattice d-polytope. Then:

$$\#(P \cap \mathbb{Z}^n) = \mathsf{Todd}_{\mathsf{h}} \operatorname{vol}(P_{\mathsf{h}})|_{\mathsf{h}=0}.$$

unimodular: primitive vertex cone generators form $\mathbb{Z}^d$ basis.

For $\mathsf{h} \in \mathbb{R}^m$, the shifted polytope P_{h} is defined as

$$P_{\mathsf{h}} = \{\mathsf{x} \in \mathbb{R}^n : A\mathsf{x} \leq \mathsf{b} + \mathsf{h}\}.$$

Theorem (Khovanskiĭ-Pukhlikov, 1992)

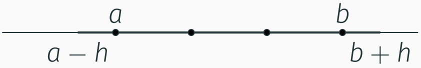
Let $P \subseteq \mathbb{R}^n$ be a unimodular, lattice d-polytope. Then:

$$\#(P \cap \mathbb{Z}^n) = \mathrm{Todd}_{\mathbf{h}} \operatorname{vol}(P_{\mathbf{h}})|_{\mathbf{h}=0}.$$

unimodular: primitive vertex cone generators form $\mathbb{Z}^d$ basis.

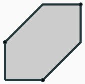
For $\mathbf{h} \in \mathbb{R}^m$, the shifted polytope $P_{\mathbf{h}}$ is defined as

$$P_{\mathbf{h}} = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} \leq \mathbf{b} + \mathbf{h}\}.$$



vol = $b - a + 2h$

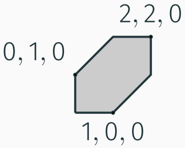
$$\begin{aligned} &\mathrm{Todd}(b - a + 2h)|_{h=0} \\ &= [1 + \tfrac{1}{2}\tfrac{d}{dh}](b - a + 2h)|_{h=0} \\ &= b - a + 2h + 1|_{h=0} \\ &= b - a + 1 \end{aligned}$$



$$\mathrm{vol}(a) = -\frac{1}{2}a_{12}^2 - \frac{1}{2}a_{13}^2 - \frac{1}{2}a_{21}^2 - \frac{1}{2}a_{23}^2 - \frac{1}{2}a_{31}^2 - \frac{1}{2}a_{32}^2$$
$$+a_{12}a_{13} + a_{13}a_{23} + a_{21}a_{23} + a_{21}a_{31} + a_{12}a_{32} + a_{31}a_{32}$$

volume in terms of facet heights

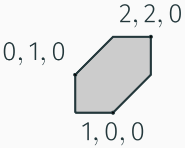
Ehrhart polynomial - Example



$$\mathrm{vol}(a) = -\frac{1}{2}a_{12}^2 - \frac{1}{2}a_{13}^2 - \frac{1}{2}a_{21}^2 - \frac{1}{2}a_{23}^2 - \frac{1}{2}a_{31}^2 - \frac{1}{2}a_{32}^2$$

$$+a_{12}a_{13} + a_{13}a_{23} + a_{21}a_{23} + a_{21}a_{31} + a_{12}a_{32} + a_{31}a_{32}$$

$\rightarrow$volume in terms of facet heights

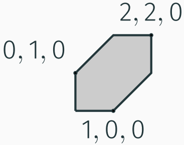


$$\mathrm{vol}(\mathbf{a}) = -\frac{1}{2}a_{12}^2 - \frac{1}{2}a_{13}^2 - \frac{1}{2}a_{21}^2 - \frac{1}{2}a_{23}^2 - \frac{1}{2}a_{31}^2 - \frac{1}{2}a_{32}^2$$
$$+a_{12}a_{13} + a_{13}a_{23} + a_{21}a_{23} + a_{21}a_{31} + a_{12}a_{32} + a_{31}a_{32}$$

$\rightarrow$volume in terms of facet heights

$\rightarrow$ for small $\mathbf{h}$, $\mathrm{vol}(P_{\mathbf{h}}) = \mathrm{vol}(\mathbf{a} + \mathbf{h})$

Ehrhart polynomial - Example



$$\mathrm{vol}(\mathbf{a}) = -\frac{1}{2}a_{12}^2 - \frac{1}{2}a_{13}^2 - \frac{1}{2}a_{21}^2 - \frac{1}{2}a_{23}^2 - \frac{1}{2}a_{31}^2 - \frac{1}{2}a_{32}^2$$

$$+a_{12}a_{13} + a_{13}a_{23} + a_{21}a_{23} + a_{21}a_{31} + a_{12}a_{32} + a_{31}a_{32}$$

$\rightarrow$volume in terms of facet heights

$\rightarrow$ for small $\mathbf{h}$, $\mathrm{vol}(P_{\mathbf{h}}) = \mathrm{vol}(\mathbf{a} + \mathbf{h})$

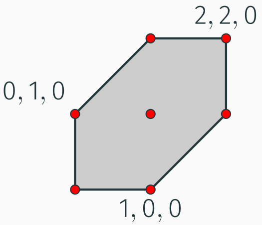
$$\mathrm{Todd}_{\mathbf{h}}\,\mathrm{vol}(\mathbf{a} + \mathbf{h})$$

$$= \left[1 + \frac{1}{2}\frac{\partial}{\partial h_{12}} + \frac{1}{6}\left(\frac{\partial}{\partial h_{12}}\right)^2\right] \cdots \left[1 + \frac{1}{2}\frac{\partial}{\partial h_{32}} + \frac{1}{6}\left(\frac{\partial}{\partial h_{32}}\right)^2\right] \mathrm{vol}(\mathbf{a} + \mathbf{h})\Big|_{\mathbf{h}=0}$$

$$= \left[1 + \frac{1}{2}\frac{\partial}{\partial a_{12}} + \frac{1}{6}\left(\frac{\partial}{\partial a_{12}}\right)^2\right] \cdots \left[1 + \frac{1}{2}\frac{\partial}{\partial a_{32}} + \frac{1}{6}\left(\frac{\partial}{\partial a_{32}}\right)^2\right] \mathrm{vol}(\mathbf{a})$$

$$= vol(\mathbf{a}) + \frac{1}{2}(a_{12} + a_{13} + a_{21} + a_{23} + a_{31} + a_{32}) + 1$$

$$\begin{aligned} ehr(\mathtt{a}) = &-\frac{1}{2}a_{12}^2 - \frac{1}{2}a_{13}^2 - \frac{1}{2}a_{21}^2 - \frac{1}{2}a_{23}^2 - \frac{1}{2}a_{31}^2 - \frac{1}{2}a_{32}^2 \\ &+ a_{12}a_{13} + a_{13}a_{23} + a_{21}a_{23} + a_{21}a_{31} + a_{12}a_{32} + a_{31}a_{32} \\ &+ \frac{1}{2}(a_{12} + a_{13} + a_{21} + a_{23} + a_{31} + a_{32}) + 1 \end{aligned}$$



$$ehr(1, 2, 1, 2, 0, 0) = 7$$

$$ehr(t, 2t, t, 2t, 0, 0) = 3t^2 + 3t + 1$$

From Ehrhart to h^* polynomials

Ehrhart series of a d-dimensional lattice polytope P:

$$Ehr_P(t) = \sum_{k \geq 0} ehr_P(k) t^k = \frac{h^*(t)}{(1-t)^{d+1}}$$

Ehrhart series of a d-dimensional lattice polytope P:

$$Ehr_P(t) = \sum_{k \geq 0} ehr_P(k) t^k = \frac{h^*(t)}{(1-t)^{d+1}}$$

$$\begin{aligned} Ehr_P(t) &= \sum_{k \geq 0} (c_d \cdot k^d + c_{d-1} \cdot k^{d-1} + \cdots + c_0 \cdot k^0) t^k \\ &= \sum_{j=0}^{d} c_j \sum_{k \geq 0} k^j t^k \end{aligned}$$

From Ehrhart to h^* polynomials

Ehrhart series of a d-dimensional lattice polytope P:

$$Ehr_P(t) = \sum_{k \geq 0} ehr_P(k) t^k = \frac{h^*(t)}{(1-t)^{d+1}}$$

$$\begin{aligned} Ehr_P(t) &= \sum_{k \geq 0} (c_d \cdot k^d + c_{d-1} \cdot k^{d-1} + \cdots + c_0 \cdot k^0) t^k \\ &= \sum_{j=0}^{d} c_j \sum_{k \geq 0} k^j t^k \end{aligned}$$

$\longrightarrow$ recognize the Eulerian polynomials:

$$\sum_{k \geq 0} k^j t^k = \frac{A_j(t)}{(1-t)^{j+1}}$$

From Ehrhart to h^* polynomials

Ehrhart series of a d-dimensional lattice polytope P:

$$Ehr_P(t) = \sum_{k \geq 0} ehr_P(k) t^k = \frac{h^*(t)}{(1-t)^{d+1}}$$

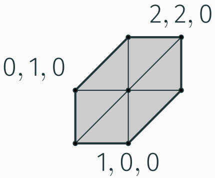
$$\begin{aligned} Ehr_P(t) &= \sum_{k \geq 0} (c_d \cdot k^d + c_{d-1} \cdot k^{d-1} + \cdots + c_0 \cdot k^0) t^k \\ &= \sum_{j=0}^{d} c_j \sum_{k \geq 0} k^j t^k \end{aligned}$$

$\longrightarrow$ recognize the Eulerian polynomials:

$$\sum_{k \geq 0} k^j t^k = \frac{A_j(t)}{(1-t)^{j+1}}$$

$$\Rightarrow \quad h_P^*(t) = \sum_{j=0}^{d} c_j A_j(t) (1-t)^{d-j}$$

$h^*(t) = \sum_{j=0}^{d} c_j A_j(t)(1-t)^{d-j}$



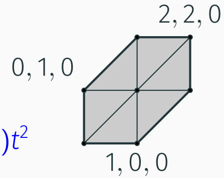
$h^*(t) = \sum_{j=0}^{d} c_j A_j(t)(1-t)^{d-j}$

$ehr(\mathbf{a}, t) =$

$$-\frac{1}{2}(a_{12}^2 + a_{13}^2 + a_{21}^2 + a_{23}^2 + a_{31}^2 + a_{32}^2)t^2$$

$$+ (a_{12}a_{13} + a_{13}a_{23} + a_{21}a_{23} + a_{21}a_{31} + a_{12}a_{32} + a_{31}a_{32})t^2$$

$$+ \frac{1}{2}(a_{12} + a_{13} + a_{21} + a_{23} + a_{31} + a_{32})t + 1$$



$\rightarrow$ collect terms of like degree

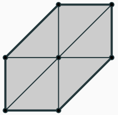
$h^*(t) = \sum_{j=0}^{d} c_j A_j(t)(1-t)^{d-j}$

$ehr(\mathbf{a}, t) =$

$-\frac{1}{2}(a_{12}^2 + a_{13}^2 + a_{21}^2 + a_{23}^2 + a_{31}^2 + a_{32}^2)t^2$

$+ (a_{12}a_{13} + a_{13}a_{23} + a_{21}a_{23} + a_{21}a_{31} + a_{12}a_{32} + a_{31}a_{32})t^2$

$+ \frac{1}{2}(a_{12} + a_{13} + a_{21} + a_{23} + a_{31} + a_{32})t + 1$



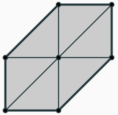
$\rightarrow$ collect terms of like degree

$\rightarrow$ apply formula to find h^*

$$h^*(\mathbf{a}, t) = \left(\sum_{i \neq j \in [3]} -\frac{1}{2}[a_{ij}^2 + a_{ij}] + \sum_{i \neq j \neq k \in [3]} [a_{ij}a_{ik} + a_{ji}a_{ki}] + 1 \right) t^2$$

$$+ \left(\sum_{i \neq j \in [3]} \frac{1}{2}[a_{ij} - a_{ij}^2] + \sum_{i \neq j \neq k \in [3]} [a_{ij}a_{ik} + a_{ji}a_{ki}] - 2 \right) t + 1$$

$h^*(t) = \sum_{j=0}^{d} c_j A_j(t)(1-t)^{d-j}$



$$ehr(\mathbf{a}, t) =$$

$$-\frac{1}{2}(a_{12}^2 + a_{13}^2 + a_{21}^2 + a_{23}^2 + a_{31}^2 + a_{32}^2)t^2$$

$$+ (a_{12}a_{13} + a_{13}a_{23} + a_{21}a_{23} + a_{21}a_{31} + a_{12}a_{32} + a_{31}a_{32})t^2$$

$$+ \frac{1}{2}(a_{12} + a_{13} + a_{21} + a_{23} + a_{31} + a_{32})t + 1$$

$\rightarrow$ collect terms of like degree

$\rightarrow$ apply formula to find h^*

$$h^*(\mathbf{a}, t) = \left(\sum_{i \neq j \in [3]} -\frac{1}{2}[a_{ij}^2 + a_{ij}] + \sum_{i \neq j \neq k \in [3]} [a_{ij}a_{ik} + a_{ji}a_{ki}] + 1 \right) t^2$$

$$+ \left(\sum_{i \neq j \in [3]} \frac{1}{2}[a_{ij} - a_{ij}^2] + \sum_{i \neq j \neq k \in [3]} [a_{ij}a_{ik} + a_{ji}a_{ki}] - 2 \right) t + 1$$

$h^*(1, 2, 1, 2, 0, 0, t) = t^2 + 4t + 1$

Result:

Multivariate volume, Ehrhart, and h^* polynomials for all polytropes up to dimension 4.

$\longrightarrow$ fast computation time.

Result:

Multivariate volume, Ehrhart, and h^* polynomials for all polytropes up to dimension 4.

$\longrightarrow$ fast computation time. Volume polynomial of a 3-polytrope:

$$\begin{aligned}
&2a_{12}^3 - 3a_{12}^2a_{13} + a_{13}^3 - 3a_{12}^2a_{14} + 6a_{12}a_{13}a_{14} - 3a_{13}^2a_{14} + a_{21}^3 - 3a_{13}^2a_{23} \\
&+ 6a_{13}a_{14}a_{23} - 3a_{14}^2a_{23} - 3a_{14}a_{23}^2 - 3a_{21}a_{23}^2 + a_{23}^3 - 3a_{21}^2a_{24} + 6a_{14}a_{23}a_{24} \\
&+ 6a_{21}a_{23}a_{24} - 3a_{14}a_{24}^2 - 3a_{23}a_{24}^2 + a_{24}^3 - 3a_{21}^2a_{31} + 6a_{21}a_{24}a_{31} - 3a_{24}^2a_{31} \\
&- 3a_{24}a_{31}^2 + a_{31}^3 - 3a_{12}^2a_{32} + 6a_{12}a_{14}a_{32} - 3a_{14}^2a_{32} - 3a_{31}^2a_{32} - 3a_{14}a_{32}^2 \\
&+ 6a_{14}a_{24}a_{34} + 6a_{24}a_{31}a_{34} + 6a_{14}a_{32}a_{34} + 6a_{31}a_{32}a_{34} - 3a_{14}a_{34}^2 - 3a_{24}a_{34}^2 \\
&- 3a_{31}a_{34}^2 - 3a_{32}a_{34}^2 + 2a_{34}^3 + 6a_{21}a_{31}a_{41} - 3a_{31}^2a_{41} + 6a_{31}a_{32}a_{41} - 3a_{32}^2a_{41} \\
&- 3a_{21}a_{41}^2 - 3a_{32}a_{41}^2 + a_{41}^3 - 3a_{12}^2a_{42} + 6a_{12}a_{13}a_{42} - 3a_{13}^2a_{42} + 6a_{12}a_{32}a_{42} \\
&+ 6a_{32}a_{41}a_{42} - 3a_{13}a_{42}^2 - 3a_{32}a_{42}^2 - 3a_{41}a_{42}^2 + a_{42}^3 - 3a_{21}^2a_{43} + 6a_{13}a_{23}a_{43} \\
&+ 6a_{21}a_{23}a_{43} - 3a_{23}^2a_{43} + 6a_{21}a_{41}a_{43} - 3a_{41}^2a_{43} + 6a_{13}a_{42}a_{43} + 6a_{41}a_{42}a_{43} \\
&- 3a_{13}a_{43}^2 - 3a_{21}a_{43}^2 - 3a_{42}a_{43}^2 + a_{43}^3
\end{aligned}$$

dimension	2	3	4	5
# comb. types of max. polytropes	1	6	27248	?

Kulas-Joswig ’08, Jiménez-De la Puente ’12, Tran ’17, Joswig-Schröter ’19

dimension	2	3	4	5
# comb. types of max. polytropes	1	6	27248	?

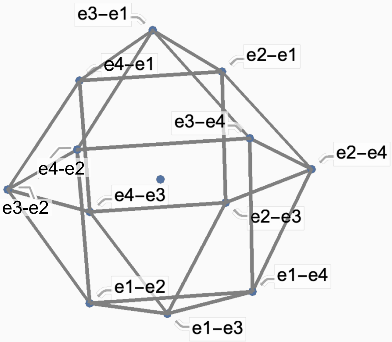
Kulas-Joswig ’08, Jiménez-De la Puente ’12, Tran ’17, Joswig-Schröter ’19

Theorem: Joswig-Schröter ’19

Maximal n-polytropes $\Leftrightarrow$ regular central triangulations of the fundamental polytope FP_{n+1}.

fundamental polytope:

$$FP_n = \mathrm{conv}\{\mathbf{e}_i - \mathbf{e}_j \mid i \neq j \in [n]\}.$$



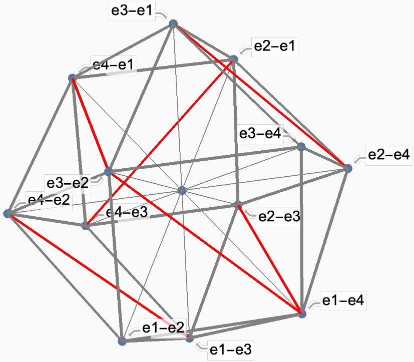
A regular central triangulation of FP_4 is determined by a triangulation of each of the six square facets.

Result

The coefficients of the six 3-dimensional volume polynomials correspond to the six triangulations of the fundamental polytope.

Result

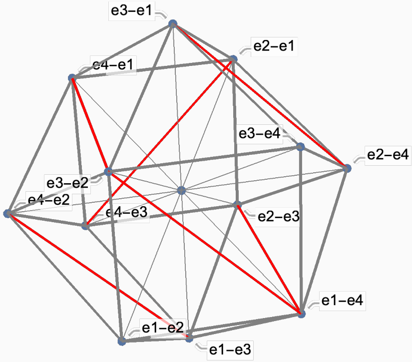
The coefficients of the six 3-dimensional volume polynomials correspond to the six triangulations of the fundamental polytope.



$$\begin{pmatrix} 0 & 11 & 20 & 29 \\ 21 & 0 & 19 & 20 \\ 20 & 29 & 0 & 11 \\ 19 & 20 & 21 & 0 \end{pmatrix}$$

Result

The coefficients of the six 3-dimensional volume polynomials correspond to the six triangulations of the fundamental polytope.



$$\begin{pmatrix} 0 & 11 & 20 & 29 \\ 21 & 0 & 19 & 20 \\ 20 & 29 & 0 & 11 \\ 19 & 20 & 21 & 0 \end{pmatrix}$$

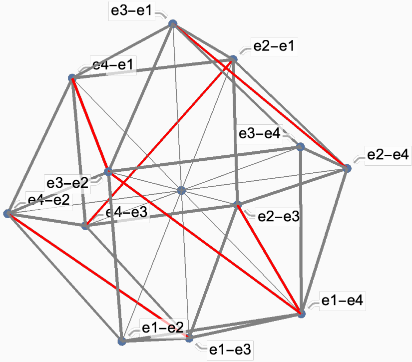
$\{e_1 - e_2, e_3 - e_2, e_1 - e_4, 0\}$ form a simplex

$\Rightarrow$ coefficient of $a_{12}a_{32}a_{14}$ is 6.

Understanding the Coeffiecients: dimension 3

Result

The coefficients of the six 3-dimensional volume polynomials correspond to the six triangulations of the fundamental polytope.



$$\begin{pmatrix} 0 & 11 & 20 & 29 \\ 21 & 0 & 19 & 20 \\ 20 & 29 & 0 & 11 \\ 19 & 20 & 21 & 0 \end{pmatrix}$$

$\{e_1 - e_2, e_3 - e_2, e_1 - e_4, 0\}$ form a simplex
$\Rightarrow$ coefficient of $a_{12}a_{32}a_{14}$ is 6.

$e_1 - e_2$ neighbors $e_3 - e_2$ and $e_3 - e_2$ is adjacent to a triangulating edge in the square
$\Rightarrow$ coefficient of $a_{12}^2 a_{32}$ is -3.

$$\begin{aligned}
&2a_{12}^3 - 3a_{12}^2a_{13} + a_{13}^3 - 3a_{12}^2a_{14} + 6a_{12}a_{13}a_{14} - 3a_{13}^2a_{14} + a_{21}^3 - 3a_{13}^2a_{23} \\
&+ 6a_{13}a_{14}a_{23} - 3a_{14}^2a_{23} - 3a_{14}a_{23}^2 - 3a_{21}a_{23}^2 + a_{23}^3 - 3a_{21}^2a_{24} + 6a_{14}a_{23}a_{24} \\
&+ 6a_{21}a_{23}a_{24} - 3a_{14}a_{24}^2 - 3a_{23}a_{24}^2 + a_{24}^3 - 3a_{21}^2a_{31} + 6a_{21}a_{24}a_{31} - 3a_{24}^2a_{31} \\
&- 3a_{24}a_{31}^2 + a_{31}^3 {\color{blue}- 3a_{12}^2a_{32}} + {\color{red}6a_{12}a_{14}a_{32}} - 3a_{14}^2a_{32} - 3a_{31}^2a_{32} - 3a_{14}a_{32}^2 \\
&+ 6a_{14}a_{24}a_{34} + 6a_{24}a_{31}a_{34} + 6a_{14}a_{32}a_{34} + 6a_{31}a_{32}a_{34} - 3a_{14}a_{34}^2 - 3a_{24}a_{34}^2 \\
&- 3a_{31}a_{34}^2 - 3a_{32}a_{34}^2 + 2a_{34}^3 + 6a_{21}a_{31}a_{41} - 3a_{31}^2a_{41} + 6a_{31}a_{32}a_{41} - 3a_{32}^2a_{41} \\
&- 3a_{21}a_{41}^2 - 3a_{32}a_{41}^2 + a_{41}^3 - 3a_{12}^2a_{42} + 6a_{12}a_{13}a_{42} - 3a_{13}^2a_{42} + 6a_{12}a_{32}a_{42} \\
&+ 6a_{32}a_{41}a_{42} - 3a_{13}a_{42}^2 - 3a_{32}a_{42}^2 - 3a_{41}a_{42}^2 + a_{42}^3 - 3a_{21}^2a_{43} + 6a_{13}a_{23}a_{43} \\
&+ 6a_{21}a_{23}a_{43} - 3a_{23}^2a_{43} + 6a_{21}a_{41}a_{43} - 3a_{41}^2a_{43} + 6a_{13}a_{42}a_{43} + 6a_{41}a_{42}a_{43} \\
&- 3a_{13}a_{43}^2 - 3a_{21}a_{43}^2 - 3a_{42}a_{43}^2 + a_{43}^3
\end{aligned}$$

Understanding the Coeffiecients: dimension 4

$\rightarrow$ less straight forward than dim 3

$\rightarrow$ less straight forward than dim 3

$\rightarrow$ Embed the 27,248 normalized volume polynomials in the vector space of homogeneous polynomials of degree 4, having dimension $\binom{23}{4} = 8855$. The affine span has dimension 70.

$\rightarrow$ less straight forward than dim 3

$\rightarrow$ Embed the 27,248 normalized volume polynomials in the vector space of homogeneous polynomials of degree 4, having dimension $\binom{23}{4} = 8855$. The affine span has dimension 70.

Partition	Example monomial	Possible coefficients	Coefficient sum
4	a_{12}^4	$-6, -3, -2, -1, 0, 1, 2, 3$	-20
3 + 1	$a_{12}^3 a_{13}$	$-4, 0, 4, 8$	320
2 + 2	$a_{12}^2 a_{13}^2$	$0, 6$	300
2+1+1	$a_{12} a_{13} a_{14}^2$	$-12, 0, 12$	-2160
1+1+1+1	$a_{12} a_{13} a_{14} a_{15}$	$0, 24$	1680

- Used tools from algebraic geometry and Ehrhart theory to quickly produce multivariate volume, Ehrhart, and h^*-polynomials for all polytropes up to dimension 4

- Used tools from algebraic geometry and Ehrhart theory to quickly produce multivariate volume, Ehrhart, and h^*-polynomials for all polytropes up to dimension 4

- Analyzed the coefficients of these polynomials

- Used tools from algebraic geometry and Ehrhart theory to quickly produce multivariate volume, Ehrhart, and h^*-polynomials for all polytropes up to dimension 4

- Analyzed the coefficients of these polynomials

- Question: How do the coefficients of the volume polynomials of maximal $(n-1)$-dimensional polytropes reflect the combinatorics of the corresponding regular central subdivision of FP_n?

- Used tools from algebraic geometry and Ehrhart theory to quickly produce multivariate volume, Ehrhart, and h^*-polynomials for all polytropes up to dimension 4

- Analyzed the coefficients of these polynomials

- Question: How do the coefficients of the volume polynomials of maximal $(n-1)$-dimensional polytropes reflect the combinatorics of the corresponding regular central subdivision of FP_n?

- Question: Why is the dimension of the affine span so low in dimension 4? What is the convex hull?

- Used tools from algebraic geometry and Ehrhart theory to quickly produce multivariate volume, Ehrhart, and h^*-polynomials for all polytropes up to dimension 4

- Analyzed the coefficients of these polynomials

- Question: How do the coefficients of the volume polynomials of maximal $(n-1)$-dimensional polytropes reflect the combinatorics of the corresponding regular central subdivision of FP_n?

- Question: Why is the dimension of the affine span so low in dimension 4? What is the convex hull?

The Mahler Conjecture (1938)

For $K \subseteq \mathbb{R}^n$ compact, centrally symmetric, full-dimensional, convex:

$$\mathrm{vol}(K)\,\mathrm{vol}(K^\circ) \geq \frac{4^n}{n!}$$

The Mahler Conjecture (1938)

For $K \subseteq \mathbb{R}^n$ compact, centrally symmetric, full-dimensional, convex:

$$\mathrm{vol}(K)\,\mathrm{vol}(K^\circ) \geq \frac{4^n}{n!}$$

- Holds for three dimensional centrally symmetric polytropes, de la Puente and Clavería (2018)

The Mahler Conjecture (1938)

For $K \subseteq \mathbb{R}^n$ compact, centrally symmetric, full-dimensional, convex:

$$\mathrm{vol}(K)\,\mathrm{vol}(K^\circ) \geq \frac{4^n}{n!}$$

- Holds for three dimensional centrally symmetric polytropes, de la Puente and Clavería (2018)

- Question: Can our volume polynomials be used to prove the Mahler conjecture for 4-dimensional centrally symmetric polytropes?

Thank You!