Intersection Bodies of Polytopes

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Combinatorial Coworkspace

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Overview

1. Definition
2. History
3. Computing Intersection Bodies
4. The algebraic boundary

Radial functions and star bodies

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Definition 〉 Radial functions and star bodies

A bounded set K is a star body if for every $s \in K$ holds $[0, s] \subseteq K$. The radial function of K is

$$\rho_K : \mathbb{R}^d \to \mathbb{R}$$
$$x \mapsto \max(\lambda \in \mathbb{R} \mid \lambda x \in K).$$

Note that $\rho_K(\lambda x) = \frac{1}{\lambda} x$ for $\lambda > 0$.

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Definition

Let P be a polytope. Then the *intersection body* IP of P is given by the radial function (restricted to the sphere)

$$\rho_{IP}(u) = \mathrm{vol}_{d-1}(P \cap u^{\perp})$$

for $u \in S^{d-1}$.

Conjecture [Busemann, Petty (1956)]

Let $K, T \subseteq \mathbb{R}^d$ be symmetric convex bodies such that for any hyperplane H through the origin holds

$$\mathrm{vol}_{d-1}(K \cap H) \leq \mathrm{vol}_{d-1}(T \cap H).$$

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Gardner (1994), Koldobsky (1998), Zhang (1999), Gardner-Kodlobsky-Schlumprecht (1999) The conjecture is true if and only if $d \leq 4$.

- For any intersection body holds $IK = -IK$
- $P \subseteq \mathbb{R}^2$ polygon and $P = -P$
  $\implies IP = 2\varphi_{90}(P)$ (φ_{90} = rotation by 90 degrees)
- $K \subseteq \mathbb{R}^d$ is a full-dimensional, convex body and $K = -K$
  $\implies IK$ is a full-dimensional convex body (and $IK = -IK$)
- $K \subseteq \mathbb{R}^d$ star body, $d \geq 3$
  $\implies IK$ is not a polytope (Campi '99, Zhang '99)

Motivation and Results

Let $P \subseteq \mathbb{R}^d$ be a polytope with intersection body IP.

Goals

- Compute the radial function ρ_{IP} explicitly
- Understand the boundary of IP & its equations.

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IP is semialgebraic, i.e. a subset of $\mathbb{R}^d$ defined by finite unions and intersections of polynomial inequalities.

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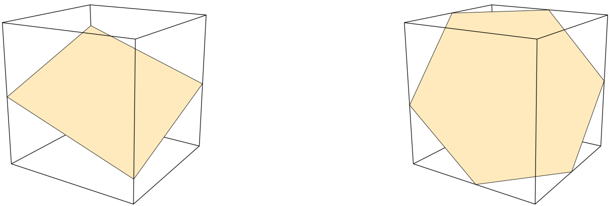
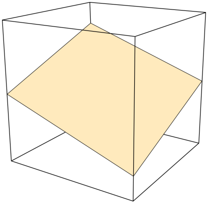
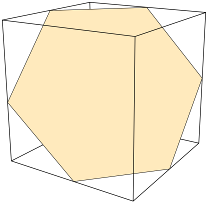
Theorem 2

The degree of the irreducible components of the algebraic boundary of IP is bounded by

$$\text{number of edges of } P - (\dim(P) - 1)).$$



Let $P = [-1,1]^3,\ x \in \mathbb{R}^3$. Then $Q = P \cap x^{\perp}$ can have different shapes.

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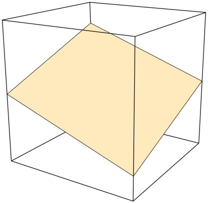

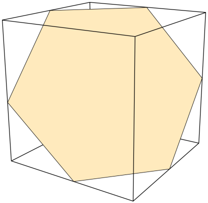

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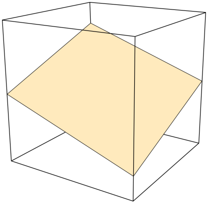
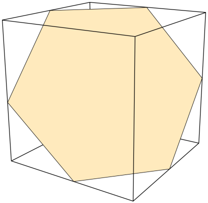
Which subsets $C \subseteq \mathbb{R}^3$ have the following property:
$\forall x \in C$: $x^{\perp}$ intersects a fixed set of edges of P.





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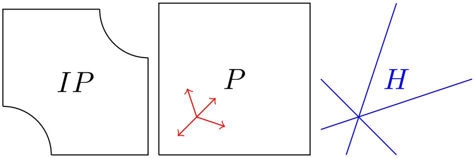
General idea: Write the volume of $P \cap x^\perp$ in terms of $x \in \mathbb{R}^3$.

$H = \{v^\perp \mid v$ is a vertex of P and $v \neq 0\}$ hyperplane arrangement.



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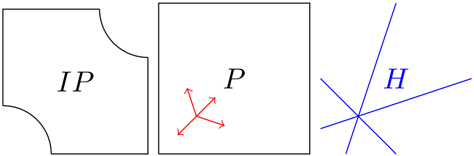

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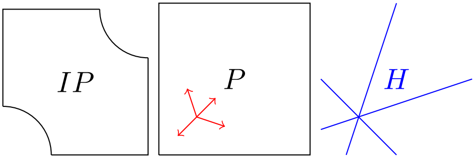
"Pieces" of $\partial IP \longleftrightarrow$ open chambers of H



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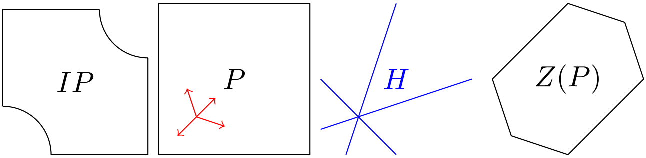

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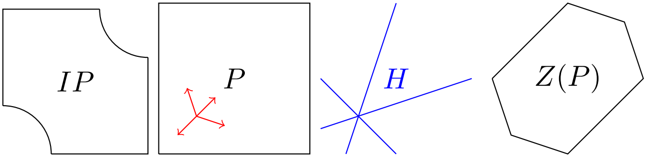

Computing ρ_{IP} ⟩ Hyperplane Arrangement H

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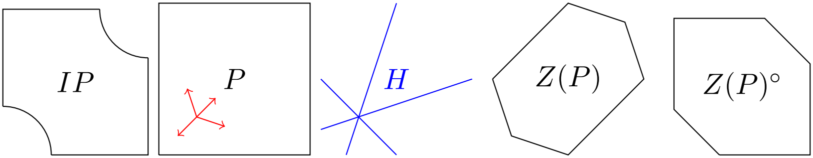
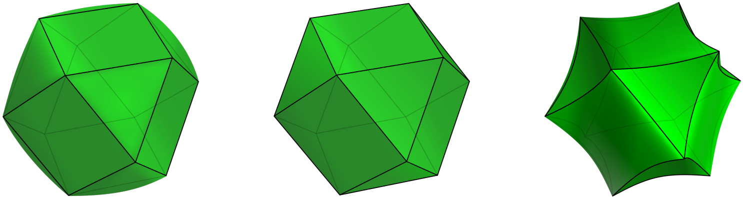
$\longleftrightarrow$ vertices of $Z(P)$
$\longleftrightarrow$ facets of $Z(P)^\circ$



Computing $\rho_{IP} \rangle Z(P)$ can have many Ps!



left: IP_1 for $P_1 = [-1,1]^3$

right: IP_2 for $P_2 = \mathrm{conv}\left(\begin{pmatrix}-1\\-1\\-1\end{pmatrix}, \begin{pmatrix}-1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\-1\end{pmatrix}\right)$

center: $Z(P_1)^\circ = Z(P_2)^\circ$

$\Rightarrow$ The zonotope $Z(P)$ does not determine the polytope P or the intersection body IP!

Computing $\rho_{IP} \rangle IP$ is semialgebraic

Lemma

Let $C \subseteq H$ be an open chamber. Then there exist polynomials $p(x), q(x) \in \mathbb{R}[x_1, \ldots, x_d]$ such that for all $u \in C \cap S^{d-1}$ holds

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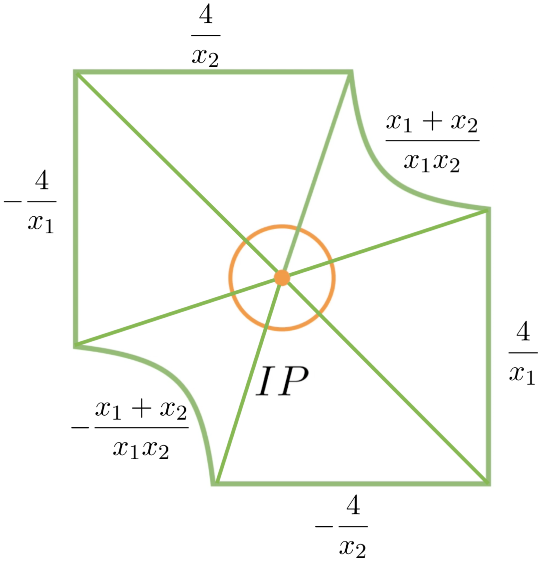
Computing ρ_{IP} 〉 Example

$$\rho(x)|_C = \frac{p(x)}{\|x\|^2 q(x)}, \quad IP \cap C = \{x \in C \mid \|x\|^2 q(x) - p(x) \leq 0\}$$



The algebraic boundary

The algebraic boundary $\partial_a IP$ of IP is the Zariski closure of ∂IP,

i.e. the smallest set s.t. $\partial IP \subseteq \partial_a IP$ and there exist polynomials $f_1, \ldots, f_k$ s.t. $\partial_a IP = \{x \in \mathbb{C}^d \mid f_1(x) = \cdots = f_k(x) = 0\}$.

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Proposition.

Let $H = \{C_i \mid i \in I\}$. Then

$$\partial_a IP = \bigcup_{i \in I} \underbrace{\mathcal{V}\left(q_i - \frac{p_i}{\|x\|^2}\right)}_{\text{irreducible components}}$$

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What are the degrees of the irreducible components?

Theorem 2 (Berlow-B.-Meroni-Shankar, '21)

$$\deg\left(q(x) - \frac{p(x)}{\|x\|^2}\right) \leq \# \text{ vertices of } P \cap x^{\perp}.$$

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Example:

$P_1 = \operatorname{conv}\left(\begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\right)$

highest degree of irreducible component $= 4$

number of edges of $P_1 - (\dim(P_1) - 1)) = 6 - (3 - 1) = 4$

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$P_2 = [-1, 1]^3$

highest degree of irreducible component $= 3$

number of edges of $P_2 - (\dim(P_2) - 1)) = 12 - (3 - 1) = 10 >> 3$

Corollary

If $P = -P$ then we can improve these bounds to

$$\deg\left(q(x) - \frac{p(x)}{\|x\|^2}\right) \leq \frac{1}{2}(\#\text{ vertices of } P \cap x^\perp)$$

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Example: $P_2 = [-1, 1]^3$

highest degree of irreducible component $= 3$

$= \frac{1}{2}\,(\#$ vertices of a hexagon$)$

Case study: $[-1,1]^d$

Proposition

Let $P = [-1,1]^d$. Then the number of irreducible components of IP of degree 1 is at least $2d$.

dim	# chambers of H	degree bound	$\deg = 1$	2	3	4	5
2	4	1	4				
3	14	5	6		8		
4	104	14	8		32	64	
5	1882	38	10		80	320	1472

Thank you!