Intersection Bodies of Polytopes

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Overview

Radial functions and star bodies

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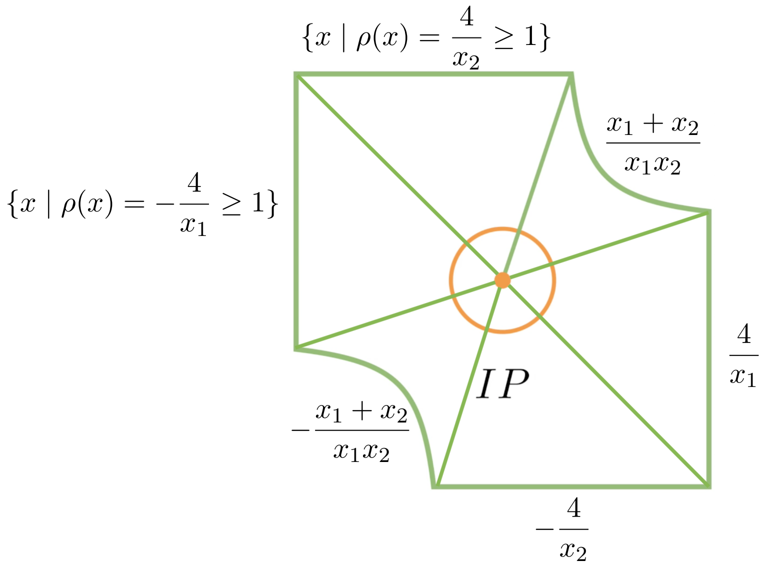
Definition

Let P be a polytope. Then the *intersection body* IP of P is given by the radial function (restricted to the sphere)

$$\rho_{IP}(u) = \mathrm{vol}_{d-1}(P \cap u^{\perp})$$

for $u \in S^{d-1}$.

Radial functions and star bodies


$\{x \mid \rho(x) = \frac{4}{x_2} \geq 1\}$
$\frac{x_1 + x_2}{x_1 x_2}$
$\{x \mid \rho(x) = -\frac{4}{x_1} \geq 1\}$
$\frac{4}{x_1}$
IP
$-\frac{x_1 + x_2}{x_1 x_2}$
$-\frac{4}{x_2}$

Conjecture [Busemann, Petty (1956)]

Let $K, T \subseteq \mathbb{R}^d$ be symmetric convex bodies such that for any hyperplane H through the origin holds

$$\mathrm{vol}_{d-1}(K \cap H) \leq \mathrm{vol}_{d-1}(T \cap H).$$

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Gardner (1994), Koldobsky (1998), Zhang (1999), Gardner-Kodlobsky-Schlumprecht (1999) The conjecture is true if and only if $d \leq 4$.

- $P \subseteq \mathbb{R}^2$ centrally symmetric polygon, centered at the origin
  $\implies IP = 2\varphi_{90}(P)$ ($\varphi_{90} =$ rotation by 90 degrees)
- $K \subseteq \mathbb{R}^d$ is a full-dimensional, convex, centered at the origin
  $\implies IK$ is full-dimensional, convex, centered at the origin
- $K \subseteq \mathbb{R}^d$ star body, $d \geq 3$
  $\implies IK$ is not a polytope

Motivation and Results

Let $P \subseteq \mathbb{R}^d$ be a polytope with intersection body IP.

Goals

- Algorithm to compute the radial function ρ_{IP} explicitly
- Compute the equations of the boundary of IP.

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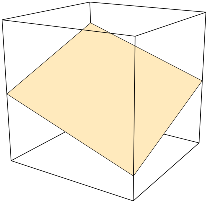
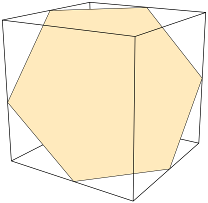
Theorem 2 [Berlow, B., Meroni, Shankar (2021)]

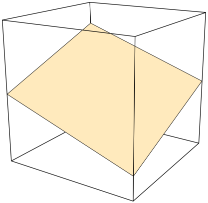
The degree of the irreducible components of the algebraic boundary of IP is bounded by

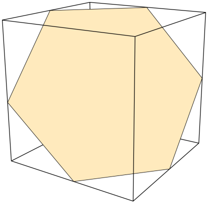
$$\text{number of edges of } P - (\dim(P) - 1)).$$





Let $P = [-1, 1]^3, \ x \in \mathbb{R}^3$. Then $Q = P \cap x^{\perp}$ can have different shapes.





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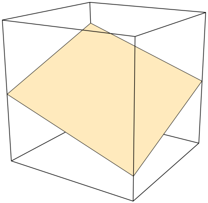
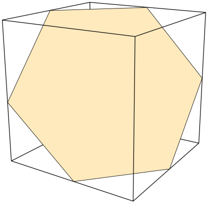
vertices of $Q \longleftrightarrow$ edges of P





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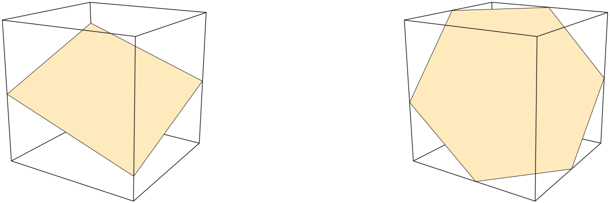
First question:

Which subsets $C \subseteq \mathbb{R}^3$ have the following property:
$\forall x \in C$: $x^\perp$ intersects a fixed set of edges of P.



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General idea: Write the volume of $P \cap x^\perp$ in terms of $x \in \mathbb{R}^3$.

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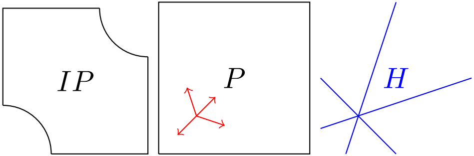

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C max chamber of H
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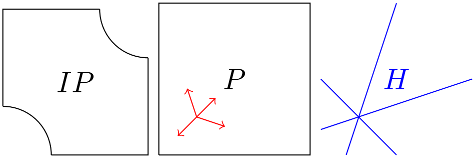

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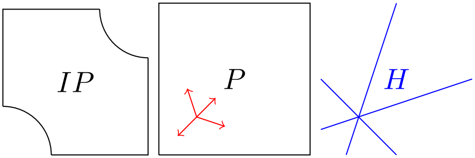
"Pieces" of $\partial IP \longleftrightarrow$ open chambers of H



Computing ρ_{IP} ⟩ Hyperplane Arrangement H

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$$Z(P) = \sum_{v \text{ vertex of } P} [-v, v]$$

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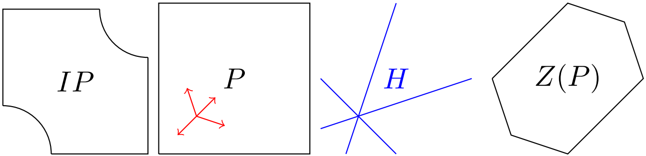

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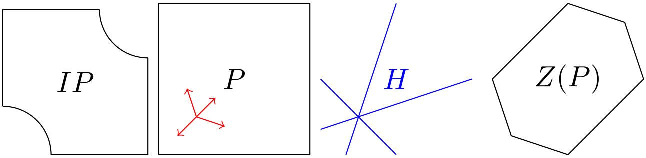

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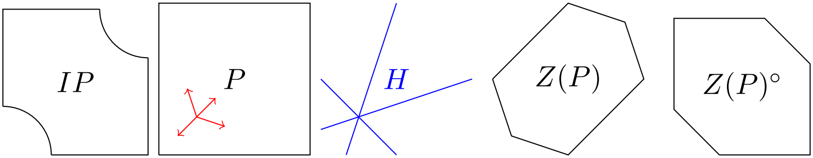
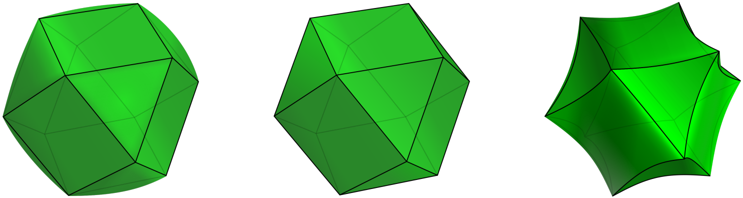
$\longleftrightarrow$ vertices of $Z(P)$
$\longleftrightarrow$ facets of $Z(P)^\circ$



Computing $\rho_{IP} \rangle Z(P)$ can have many Ps!



left: IP_1 for $P_1 = [-1,1]^3$

right: IP_2 for $P_2 = \mathrm{conv}\left(\begin{pmatrix}-1\\-1\\-1\end{pmatrix}, \begin{pmatrix}-1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\-1\end{pmatrix}\right)$

center: $Z(P_1)^\circ = Z(P_2)^\circ$

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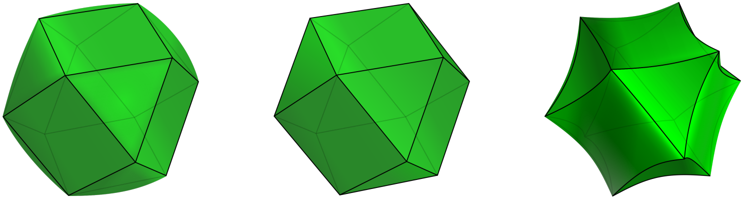

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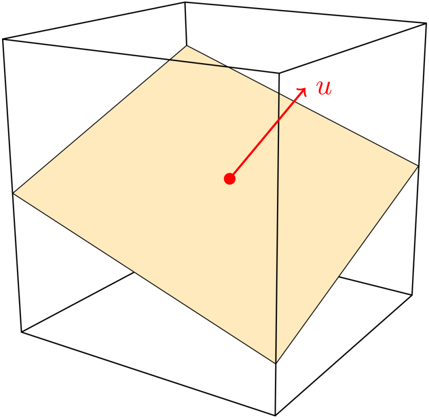
MathRepo

How to compute ρ_{IP}

$C \in H$

$u \in C \cap S^{d-1}$



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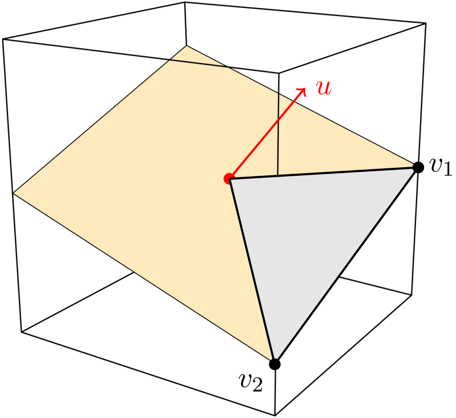

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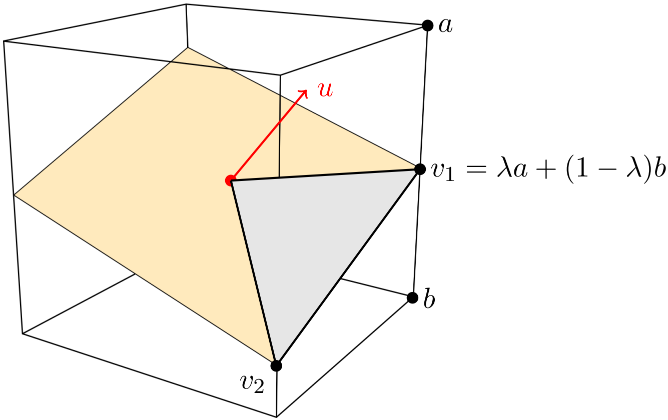

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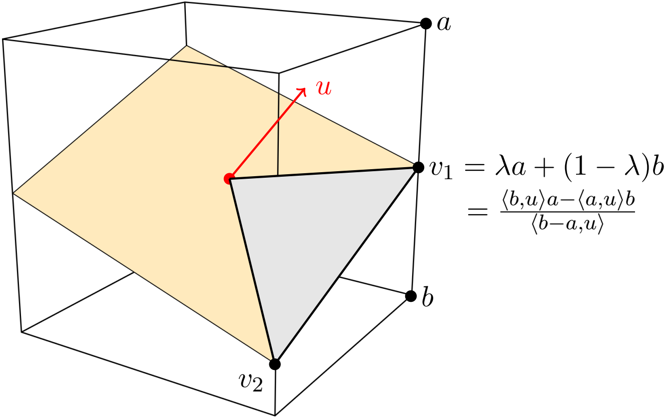

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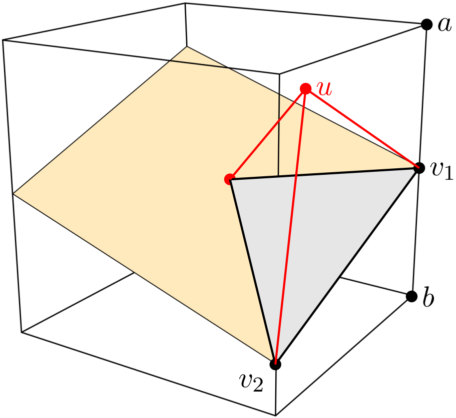

$$v_1 = \lambda a + (1-\lambda) b = \frac{\langle b,u\rangle a - \langle a,u\rangle b}{\langle b-a,u\rangle}$$

$$\mathrm{vol}_2(\mathrm{conv}(0, v_1, v_2)) = 3\,\mathrm{vol}_3(\mathrm{conv}(0, v_1, v_2, u))$$

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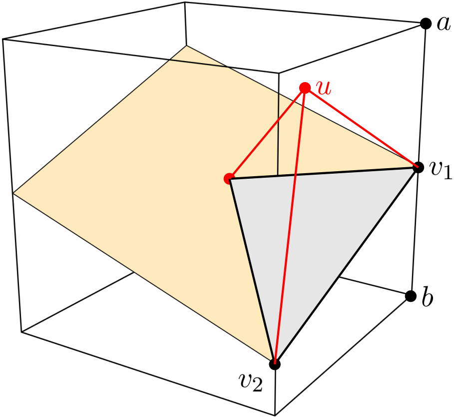

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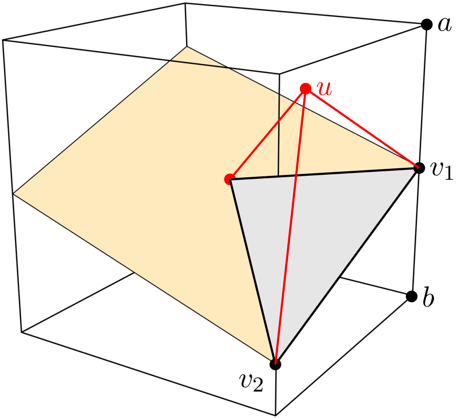

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H has finitely many chambers. Thus,

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How to compute ρ_{IP} 〉 Example

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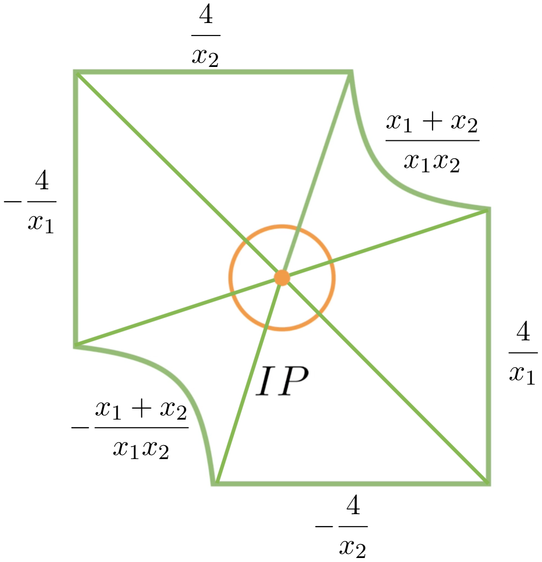

The algebraic boundary

The algebraic boundary $\partial_a IP$ of IP is the Zariski closure of ∂IP,

i.e. the smallest set s.t. $\partial IP \subseteq \partial_a IP$ and there exist polynomials $f_1, \ldots, f_k$ s.t. $\partial_a IP = \{x \in \mathbb{C}^d \mid f_1(x) = \cdots = f_k(x) = 0\}$.

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Let $H = \{C_i \mid i \in I\}$ and $f(x) = \prod_{i \in I} \left(q_i - \frac{p_i}{\|x\|^2} \right)$. Then

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What are the degrees of the irreducible components?

Theorem 2 [Berlow, B., Meroni, Shankar (2021)]

$$\deg\left(q(x) - \frac{p(x)}{\|x\|^2}\right) \leq \# \text{ vertices of } P \cap x^{\perp}.$$

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Corollary

The degrees of the irreducible components of $\partial_a IP$ are bounded by number of edges of $P - (\dim(P) - 1)$.

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$P_2 = [-1, 1]^3$

highest degree of irreducible component $= 3$

number of edges of $P_2 - (\dim(P_2) - 1)) = 12 - (3 - 1) = 10 >> 3$

Corollary

If P is centrally symmetric and centered at the origin, then we can improve these bounds to

$$\deg\left(q(x) - \frac{p(x)}{\|x\|^2}\right) \leq \frac{1}{2}(\#\text{ vertices of } P \cap x^{\perp})$$

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Example: $P_2 = [-1,1]^3$

highest degree of irreducible component $= 3$
$= \frac{1}{2}\left(\#\text{ vertices of a hexagon}\right)$

Case study: $[-1,1]^d$

Proposition [Berlow, B., Meroni, Shankar (2021)]

Let $P = [-1,1]^d$. Then the number of irreducible components of IP of degree 1 is at least $2d$.

dim	# chambers of H	degree bound	$\deg = 1$	2	3	4	5
2	4	1	4				
3	14	5	6		8		
4	104	14	8		32	64	
5	1882	38	10		80	320	1472

Thank you!