Intersection Bodies of Polytopes

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joint work with Katalin Berlow, Chiara Meroni and Isabelle Shankar

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Definition

Let P be a polytope. Then the *intersection body* IP of P is given by the radial function (restricted to the sphere)

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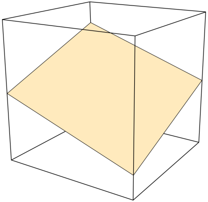
Theorem 2

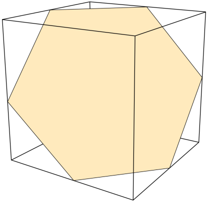
The degree of the algebraic boundary of IP is bounded by

$$\text{number of edges of } P - (\dim(P) - 1)).$$

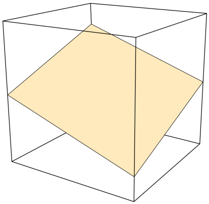
Part 1: IP is semialgebraic*.

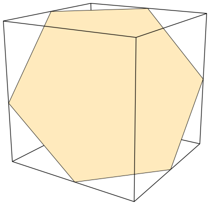
* i.e. a subset of $\mathbb{R}^d$ defined by a boolean combination of polynomial inequalities.





Let $P = [-1,1]^3,\ x \in \mathbb{R}^3$. Then $Q = P \cap x^\perp$ can have different shapes.





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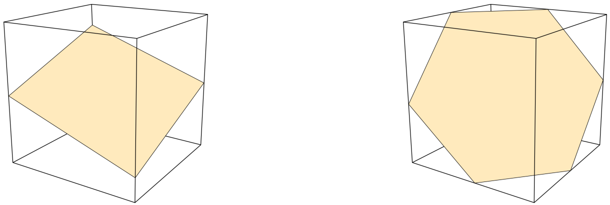
vertices of $Q \longleftrightarrow$ edges of P

Example: The 3-cube



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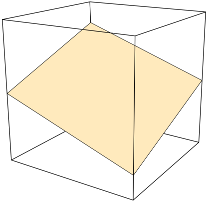

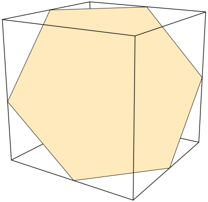

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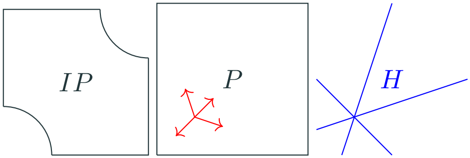
First question:

For which $x \in \mathbb{R}^3$ intersects $x^\perp$ a fixed set of edges of P?

$$H = \{v^{\perp} \mid v \text{ is a vertex of } P \text{ and } v \neq 0\}.$$

C max chamber of $H \Rightarrow \forall x \in C : x^{\perp}$ intersects P in fixed set of edges



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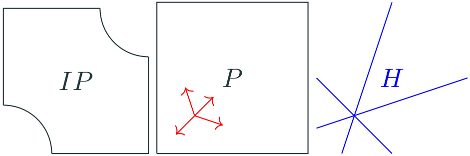
"Pieces" of $IP \longleftrightarrow$ open chambers of H



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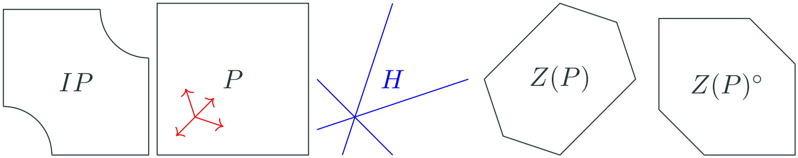
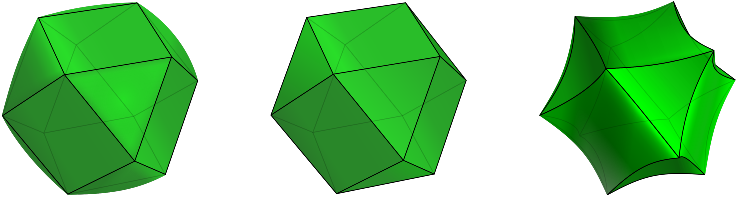


$Z(P)$ CAN HAVE MANY PS!



left: IP_1 for $P_1 = [-1, 1]^3$

right: IP_2 for $P_2 = \operatorname{conv}\left(\begin{pmatrix}-1\\-1\\-1\end{pmatrix}, \begin{pmatrix}-1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\-1\end{pmatrix}\right)$

center: $Z(P_1)^\circ = Z(P_2)^\circ$

$\Rightarrow$ The zonotope $Z(P)$ does not determine the polytope P or the intersection body IP!

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$$Q = \bigcup_{\text{facets of } Q} \operatorname{conv}(\Delta, 0)$$

is a triangulation of Q.

Proof (cont.).

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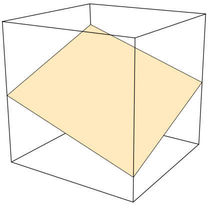
$$IP \cap C = \{x \in C | \rho(x) \geq 1\} = \{x \in C | \frac{p(x)}{\|x\|^2 q(x)} \geq 1\}$$
$$= \{x \in C \mid \|x\|^2 q(x) - p(x) \leq 0\}.$$

$\Rightarrow IP$ is semialgebraic. □

Example: 3-cube | $\rho(x)|_C = \frac{p(x)}{\|x\|^2 q(x)}$



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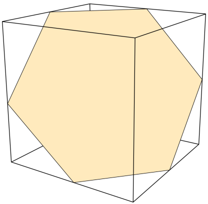

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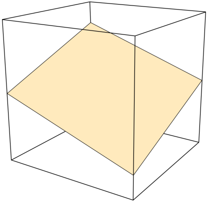

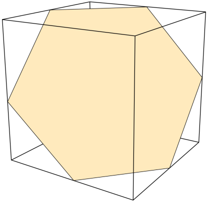

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$$\rho(x,y,z)|_{C_2} = \frac{-\left(x^2-2\,xy+y^2-2\,xz-2\,yz+z^2\right)\left(x^2+y^2+z^2\right)}{(x^2+y^2+z^2)3\,xyz}$$

$$= \frac{-(x^2-2xy+y^2-2xz-2yz+z^2)}{3xyz}$$

Part 2: The algebraic boundary

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What are the degrees of the irreducible components?

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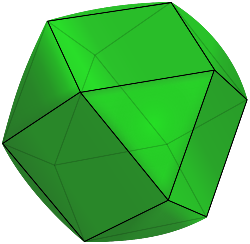

$$q_1 - \frac{p_1}{\|x\|^2} = 3z - \frac{4(x^2+y^2+z^2)}{(x^2+y^2+z^2)} = 3z - 4$$

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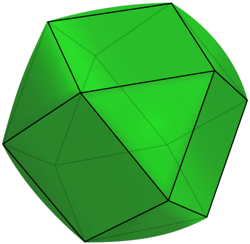

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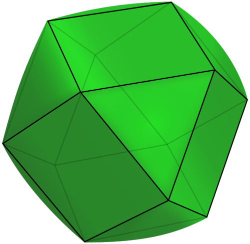

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$\Rightarrow$ If $P = [-1,1]^3$ then the possible degrees are 1 and 3.

What happens if we translate P?

MathRepo

Degree Bound

Theorem 2

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The degree of the algebraic boundary of IP is bounded by

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highest degree of irreducible component $= 4$

number of edges of $P_2 - (\dim(P_2) - 1) = 6 - (3-1) = 4$

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Theorem 2

$$\deg\left(q(x) - \frac{p(x)}{\|x\|^2}\right) \leq \# \text{ vertices of } P \cap x^{\perp}.$$

Corollary

The degree of the algebraic boundary of IP is bounded by

$$\text{number of edges of } P - (\dim(P) - 1).$$

Example: $P_1 = [-1,1]^3$

number of edges of $P_1 - (\dim(P_1) - 1) = 12 - (3-1) = 10 >> 3$

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$\deg\left(q(x) - \frac{p(x)}{\|x\|^2}\right) = \frac{1}{2}$ ($\#$ vertices of a hexagon) $= 3$

Case study: d-dimensional centered cube $[-1,1]^d$

Proposition

Let $P = [-1,1]^d$. Then the number of irreducible components of IP of degree 1 is at least $2d$.

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3	14	5	6		8		
4	104	14	8		32	64	
5	1882	38	10		80	320	1472

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Thank you!