Intersection Bodies of Polytopes

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Joint work with Katalin Berlow, Chiara Meroni and Isabelle Shankar

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Women in Algebra and Symbolic Computations II

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https://arxiv.org/abs/2110.05996

https://mathrepo.mis.mpg.de/intersection-bodies

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Definition

Let P be a polytope. Then the *intersection body* IP of P is given by the radial function (restricted to the sphere)

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The degree of the irreducible components of the algebraic boundary of IP is bounded by

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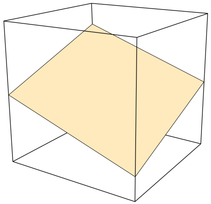
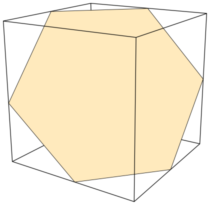
Example: The 3-cube





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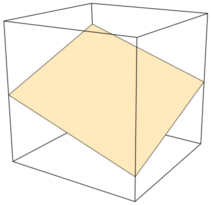

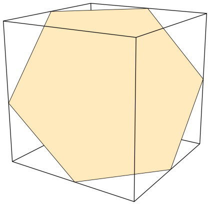

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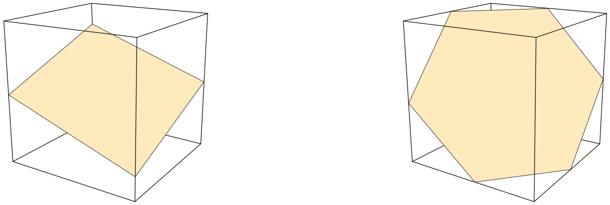

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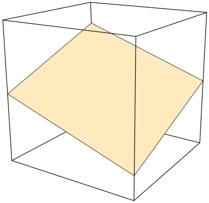 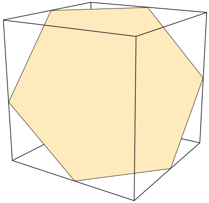

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General idea: Use this to write the volume of $P \cap x^\perp$ in terms of $x \in \mathbb{R}^3$.

Hyperplane Arrangement H

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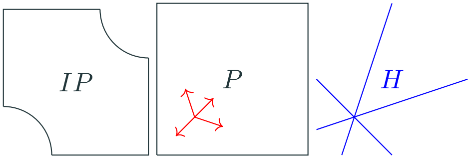
C max chamber of $H \Rightarrow \forall x \in C : x^\perp$ intersects P in fixed set of edges



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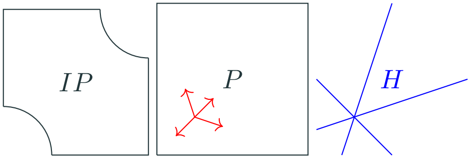

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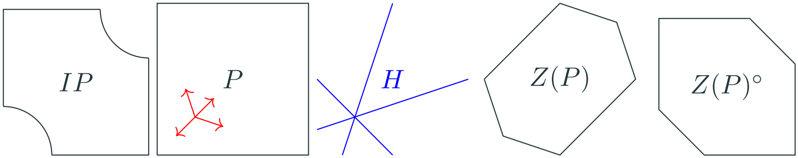
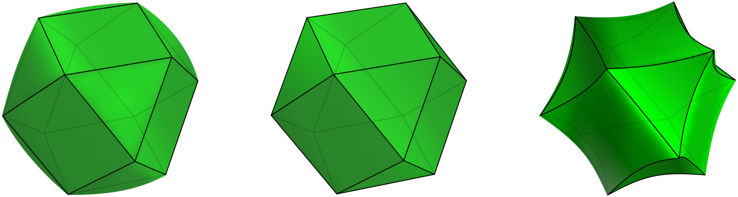


$Z(P)$ can have many Ps!



left: IP_1 for $P_1 = [-1,1]^3$
right: IP_2 for $P_2 = \mathrm{conv}\left(\begin{pmatrix}-1\\-1\\-1\end{pmatrix}, \begin{pmatrix}-1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\-1\end{pmatrix}\right)$
center: $Z(P_1)^\circ = Z(P_2)^\circ$

$\Rightarrow$ The zonotope $Z(P)$ does not determine the polytope P or the intersection body IP!

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Let $C \subseteq H$ be an open chamber. Then there exist polynomials $p(x), q(x) \in \mathbb{R}[x_1, \ldots, x_d]$ such that for all $x \in C$ holds

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What are the degrees of the irreducible components?

Degree Bound

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highest degree of irreducible component $= 3$

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highest degree of irreducible component $= 4$

number of edges of $P_2 - (\dim(P_2) - 1)) = 6 - (3 - 1) = 4$

Corollary

If P is centrally symmetric and centered at the origin, then we can improve the bounds to

$$\deg\left(q(x) - \frac{p(x)}{\|x\|^2}\right) \leq \#\tfrac{1}{2}(\text{ vertices of } P \cap x^{\perp})$$

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CASE STUDY: d-DIMENSIONAL CENTERED CUBE $[-1,1]^d$

Proposition (Berlow, B., Meroni, Shankar)

Let $P = [-1,1]^d$. Then the number of irreducible components of IP of degree 1 is at least $2d$.

Conjecture: This number is exactly $2d$.

$\Rightarrow$ many more non-linear pieces!

dim	# chambers of H	degree bound	deg = 1	2	3	4	5
2	4	1	4				
3	14	5	6		8		
4	104	14	8		32	64	
5	1882	38	10		80	320	1472