Competitive Equilibrium and Lattice Polytopes

arXiv:2107.08813

MOR Seminar | University of Twente

05 October 2022

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based on joint work with Christian Haase and Ngoc Mai Tran

Max-Planck-Institut für Mathematik in den Naturwissenschaften

MAX-PLANCK-GESELLSCHAFT

Overview

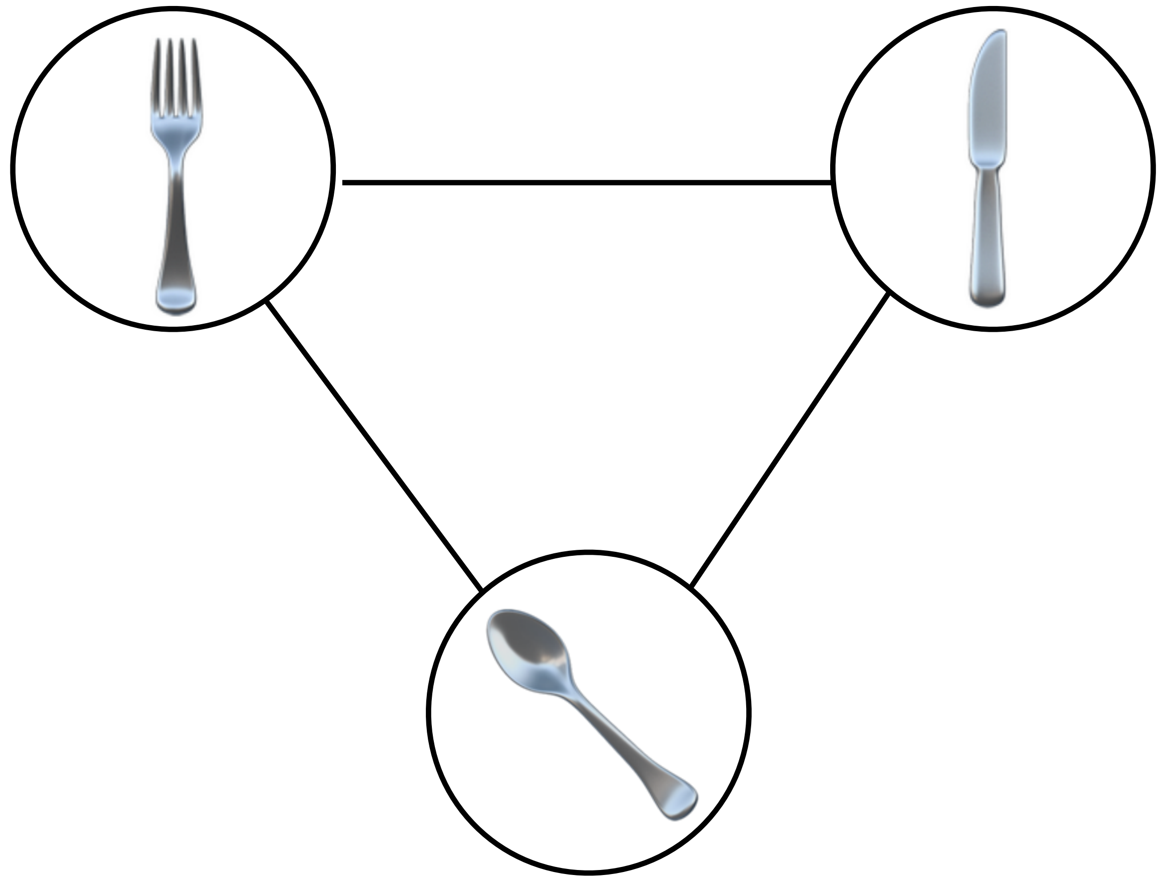
1. **First Example**

2. **Mathematical Model | Connections to Polytopes**

3. **Can we guarantee the existence of a competitive equilibrium?**
   (Answer: yes, if $G = K_n$)

First Example

The cutlery auction at dinner time



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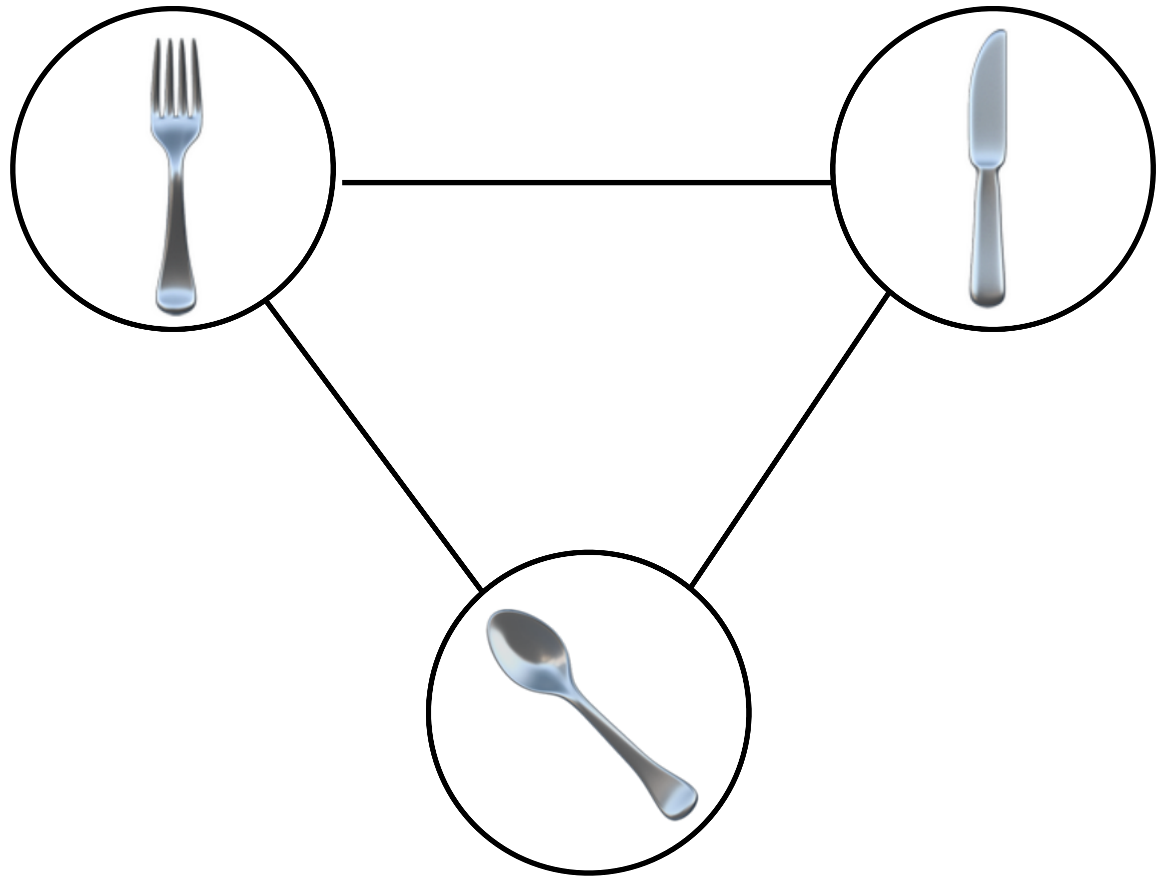

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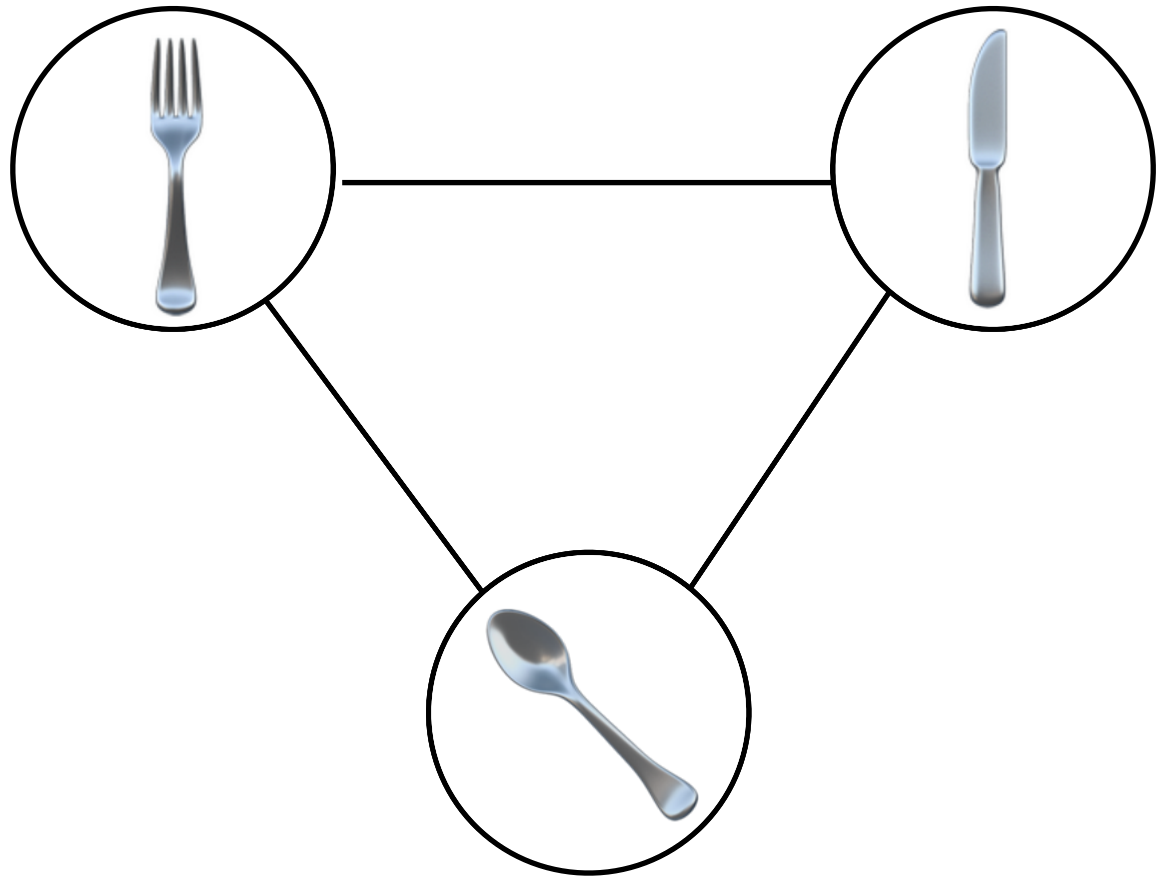

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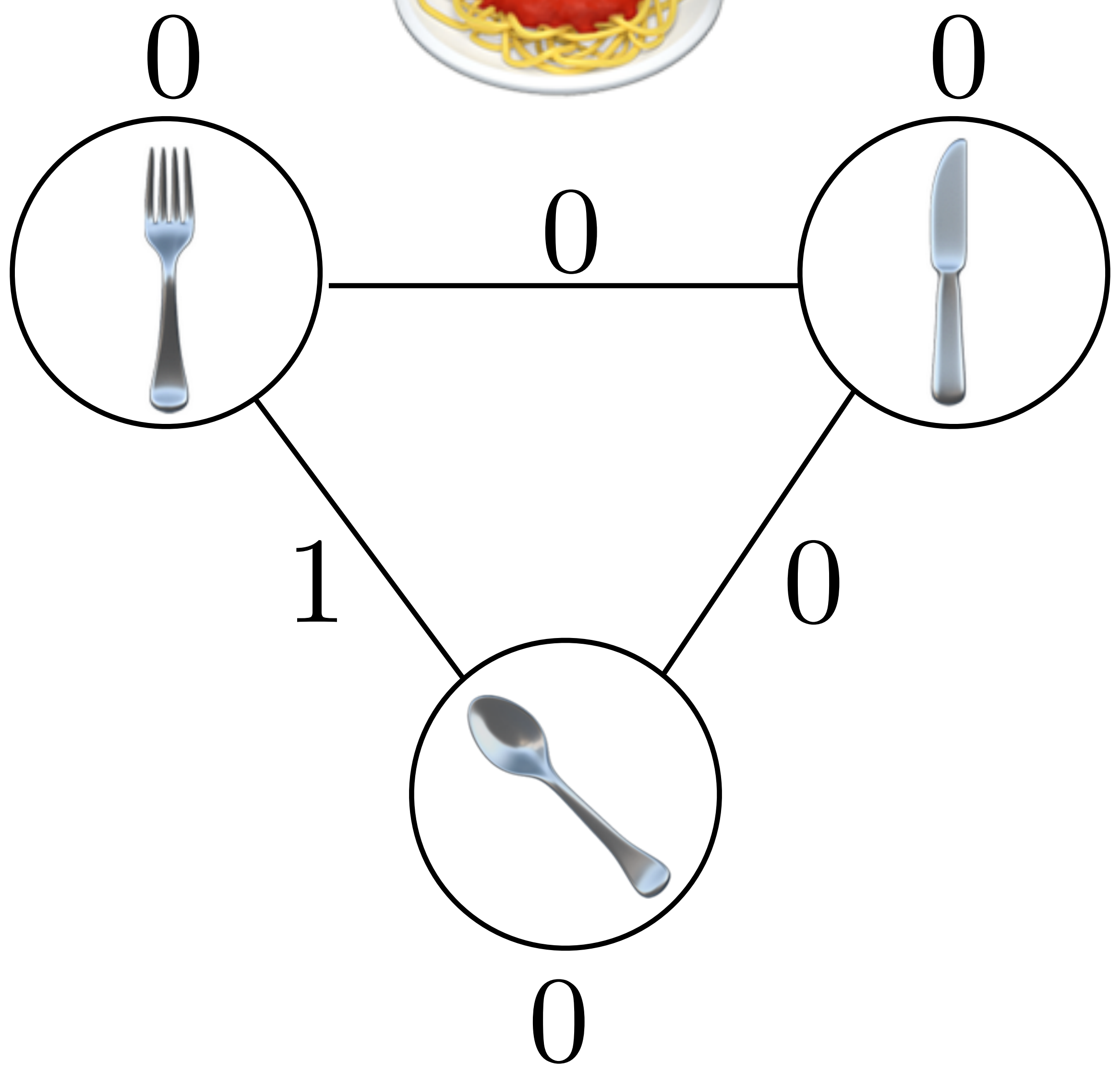

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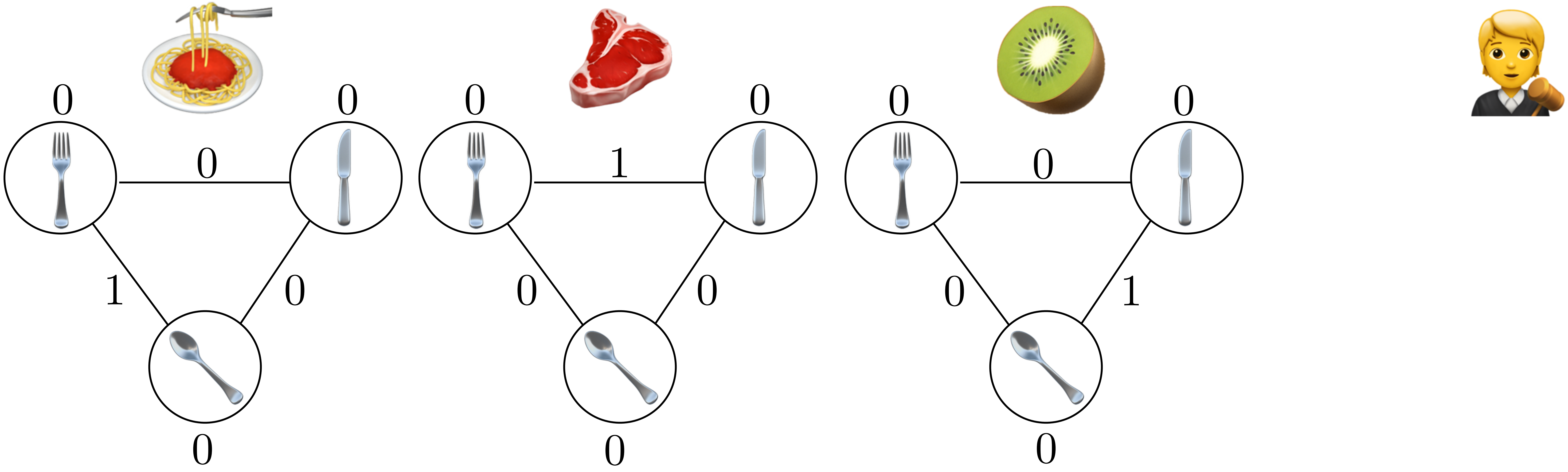

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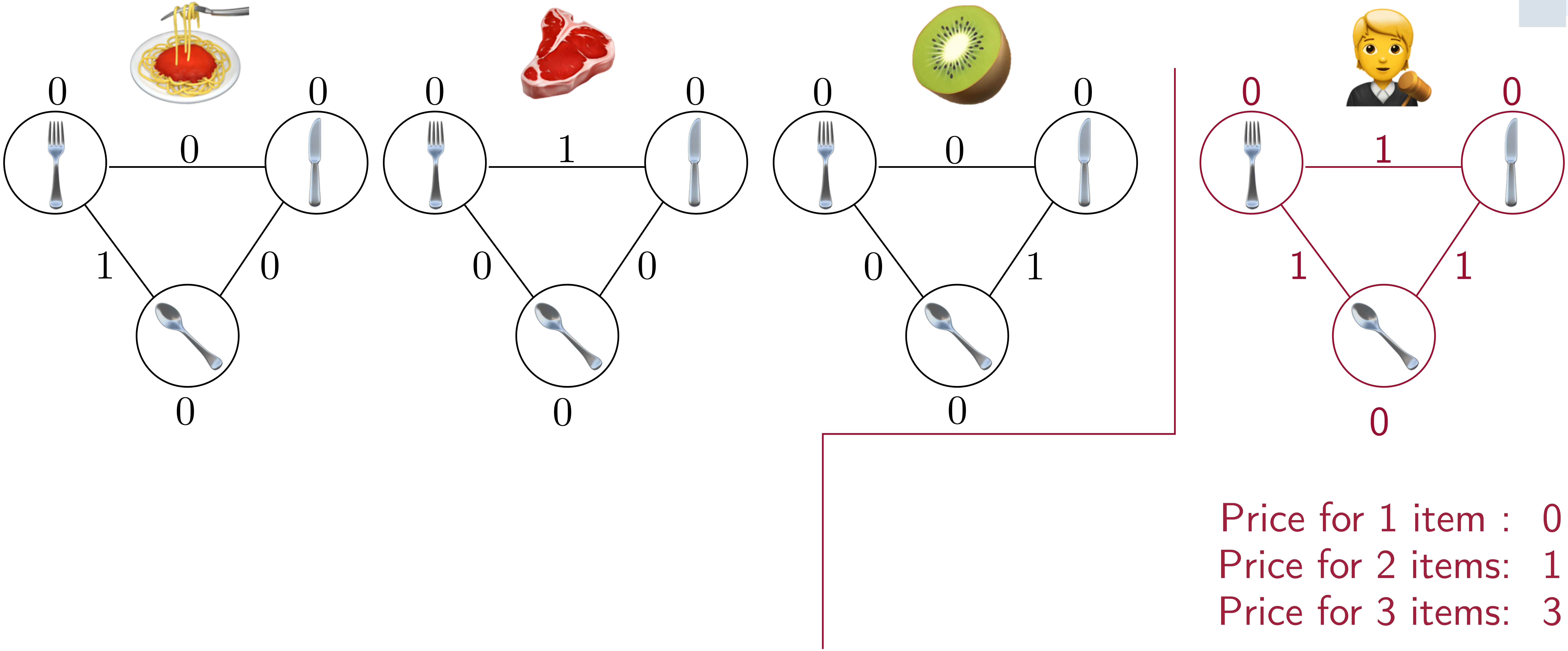

Price for 1 item : 0
Price for 2 items: 1
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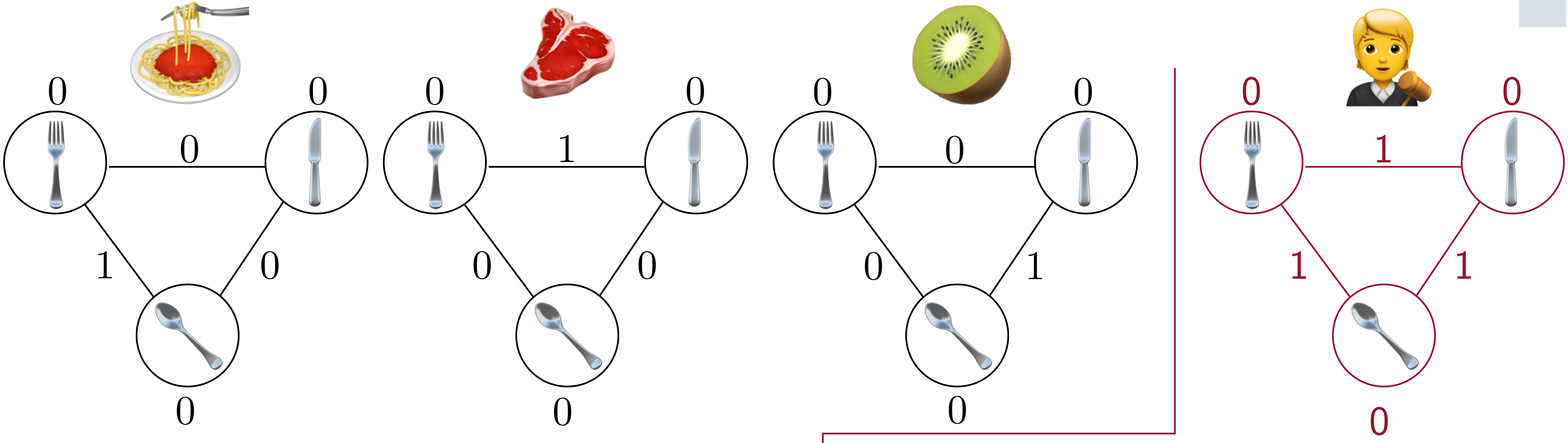

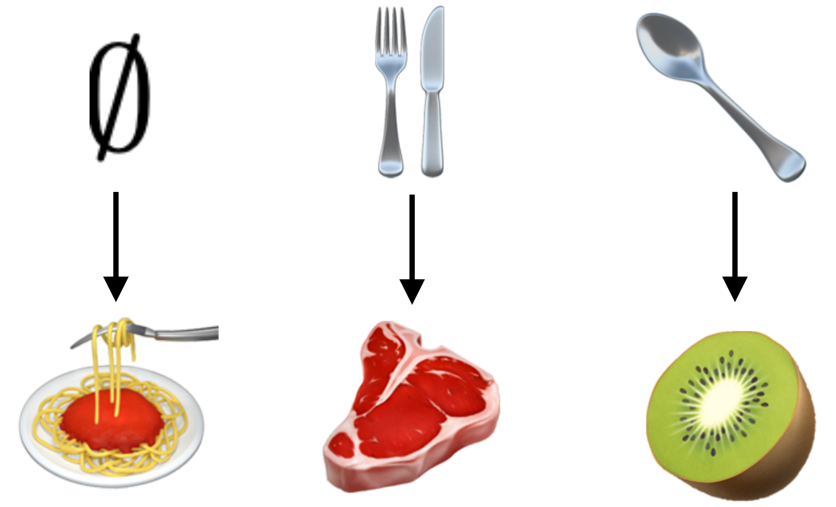

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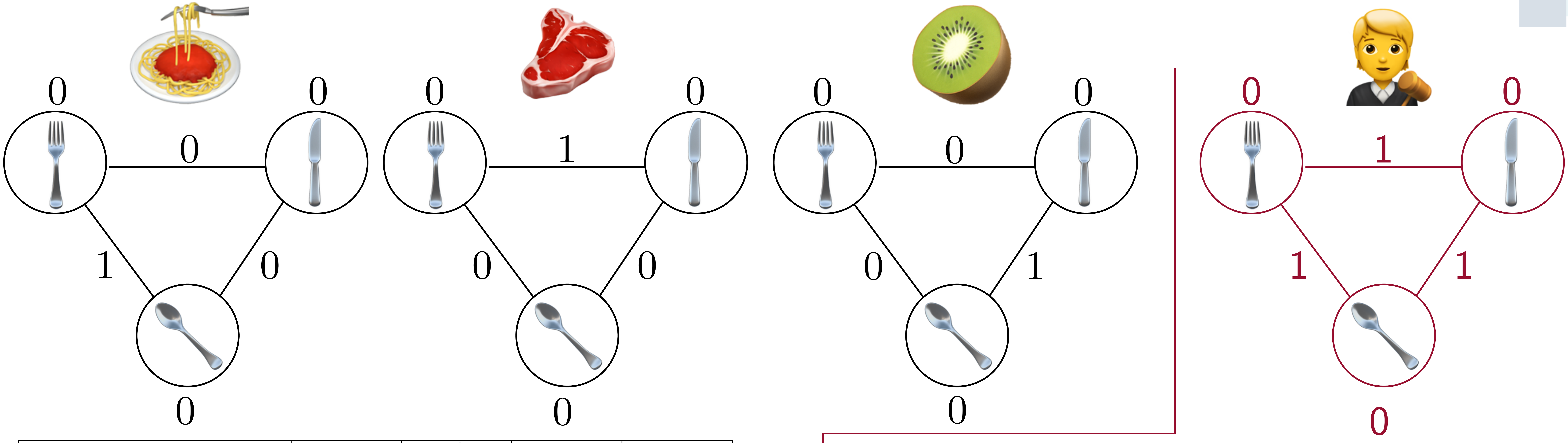

Opinion of 🥩	∅	🔪	🍴🔪	🍴🔪🥄
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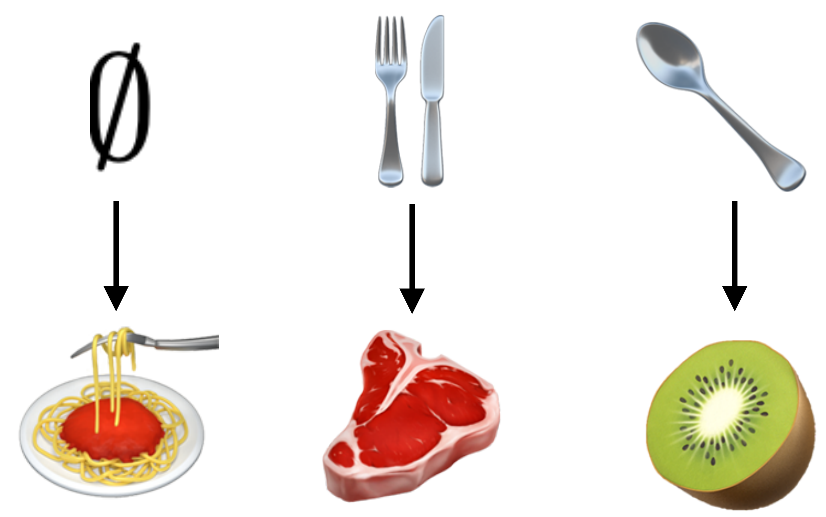

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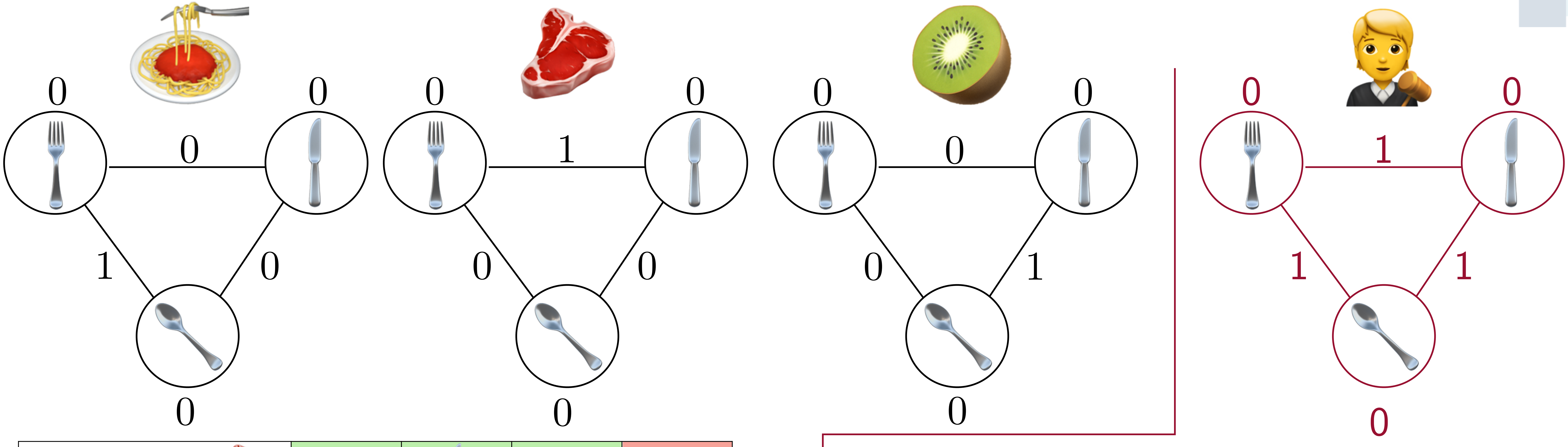

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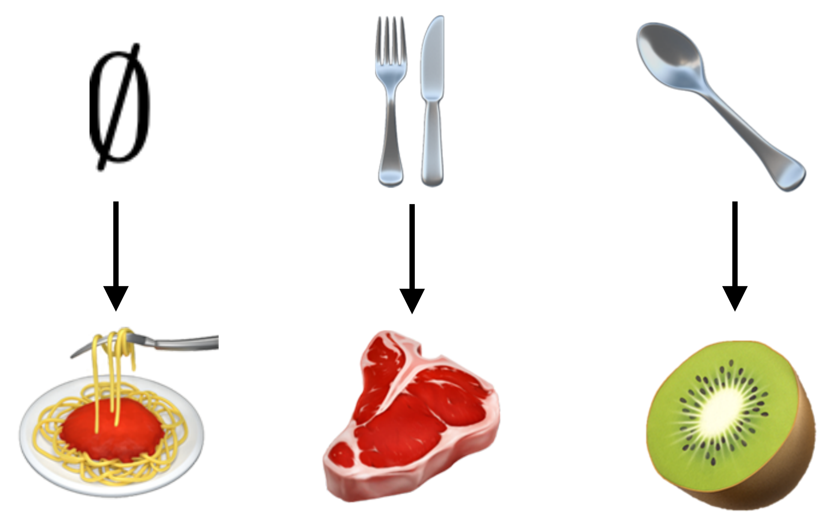

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Polytopes

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A *lattice polytope* is the convex hull of finitely many points

$$P = \mathrm{conv}(v_1, \ldots, v_n) \text{ for } v_1, \ldots, v_n \in \mathbb{Z}^d$$

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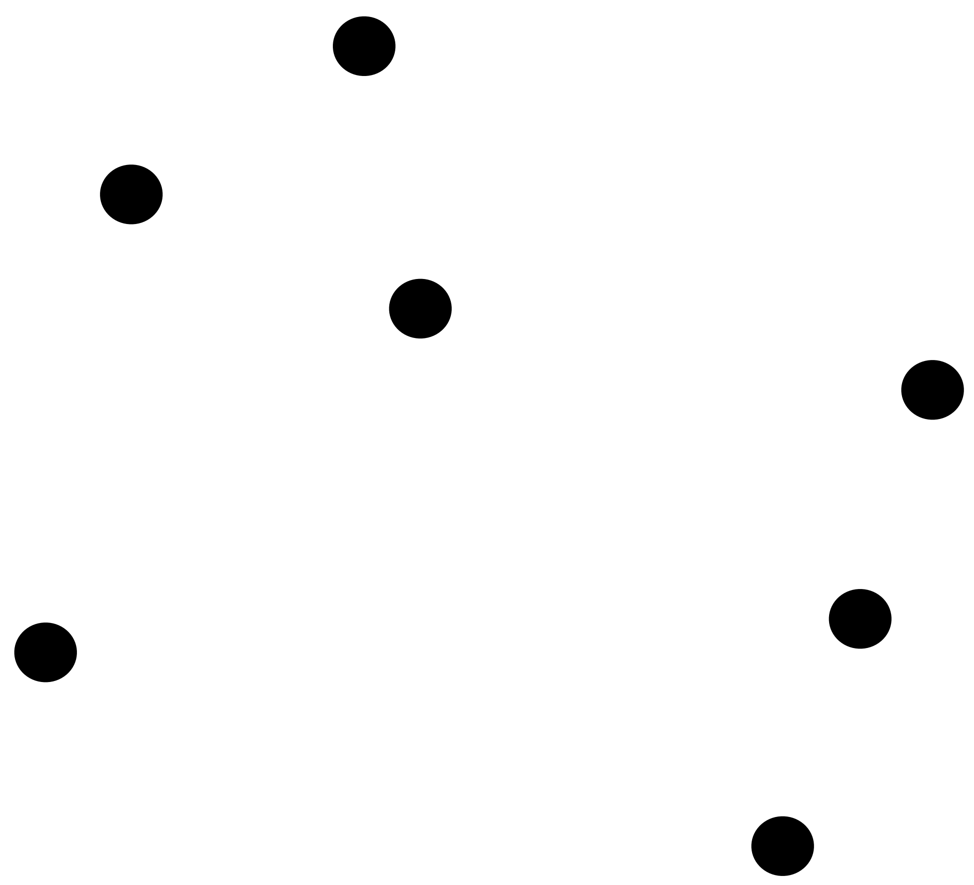

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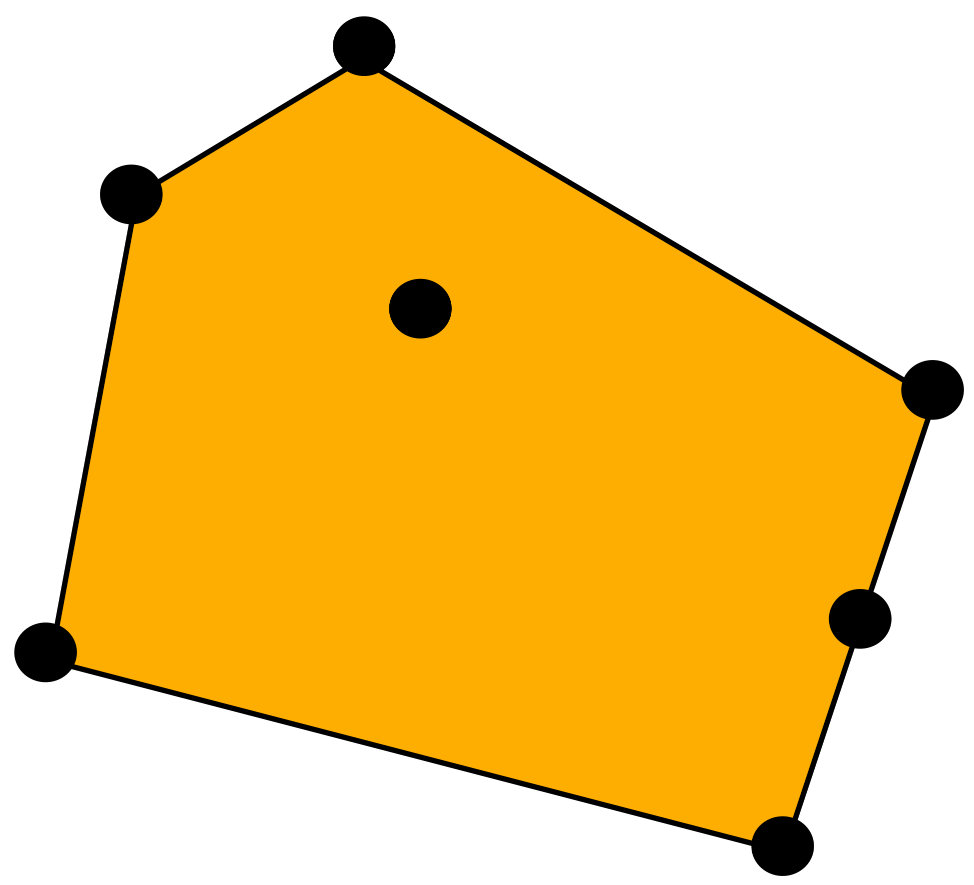

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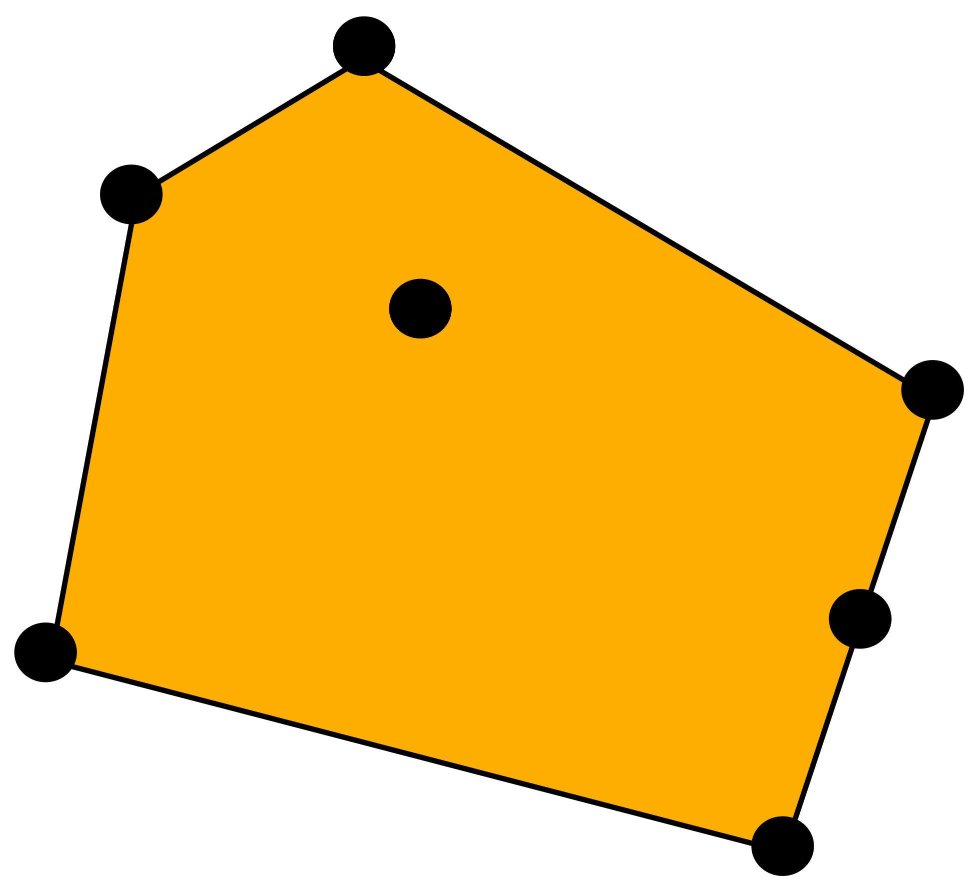

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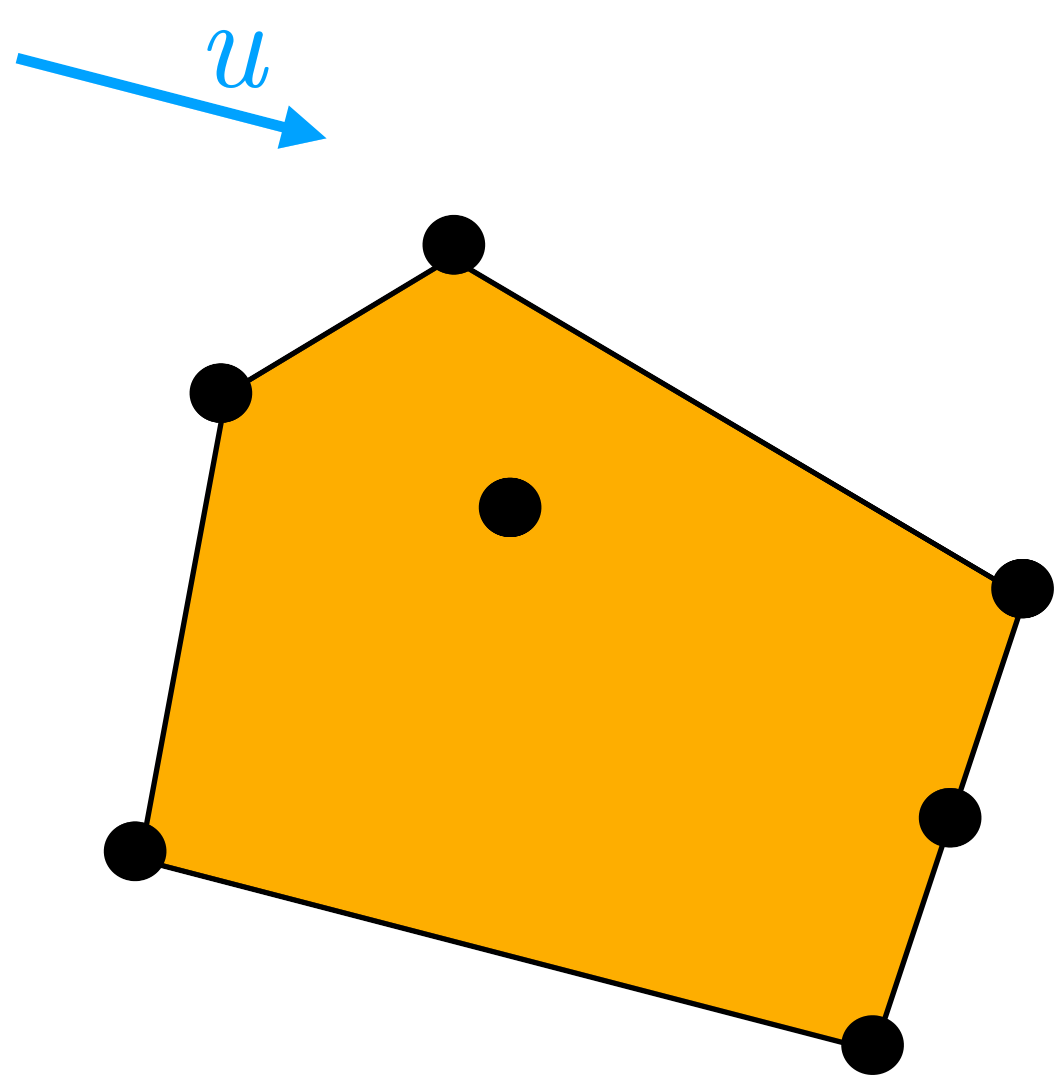

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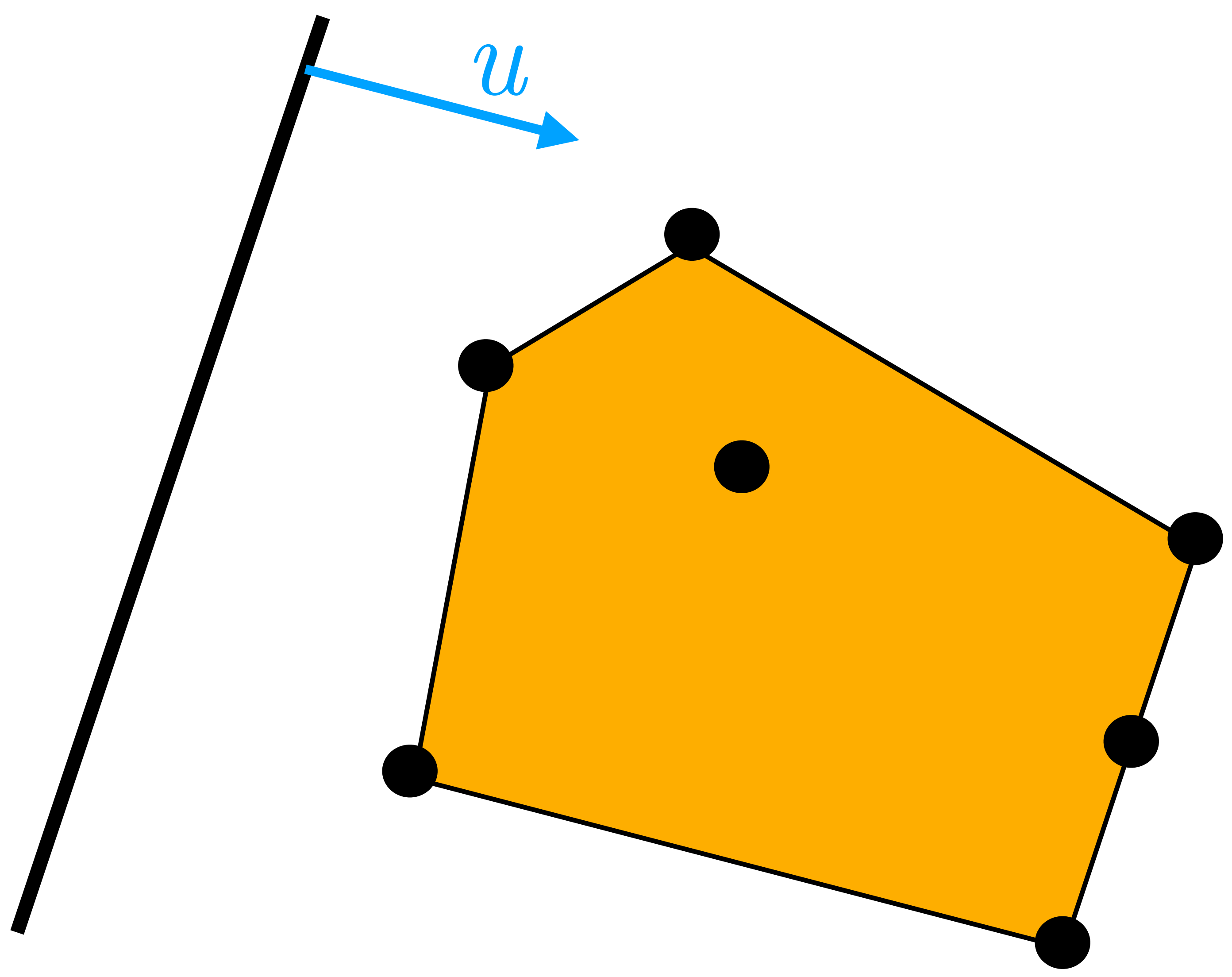

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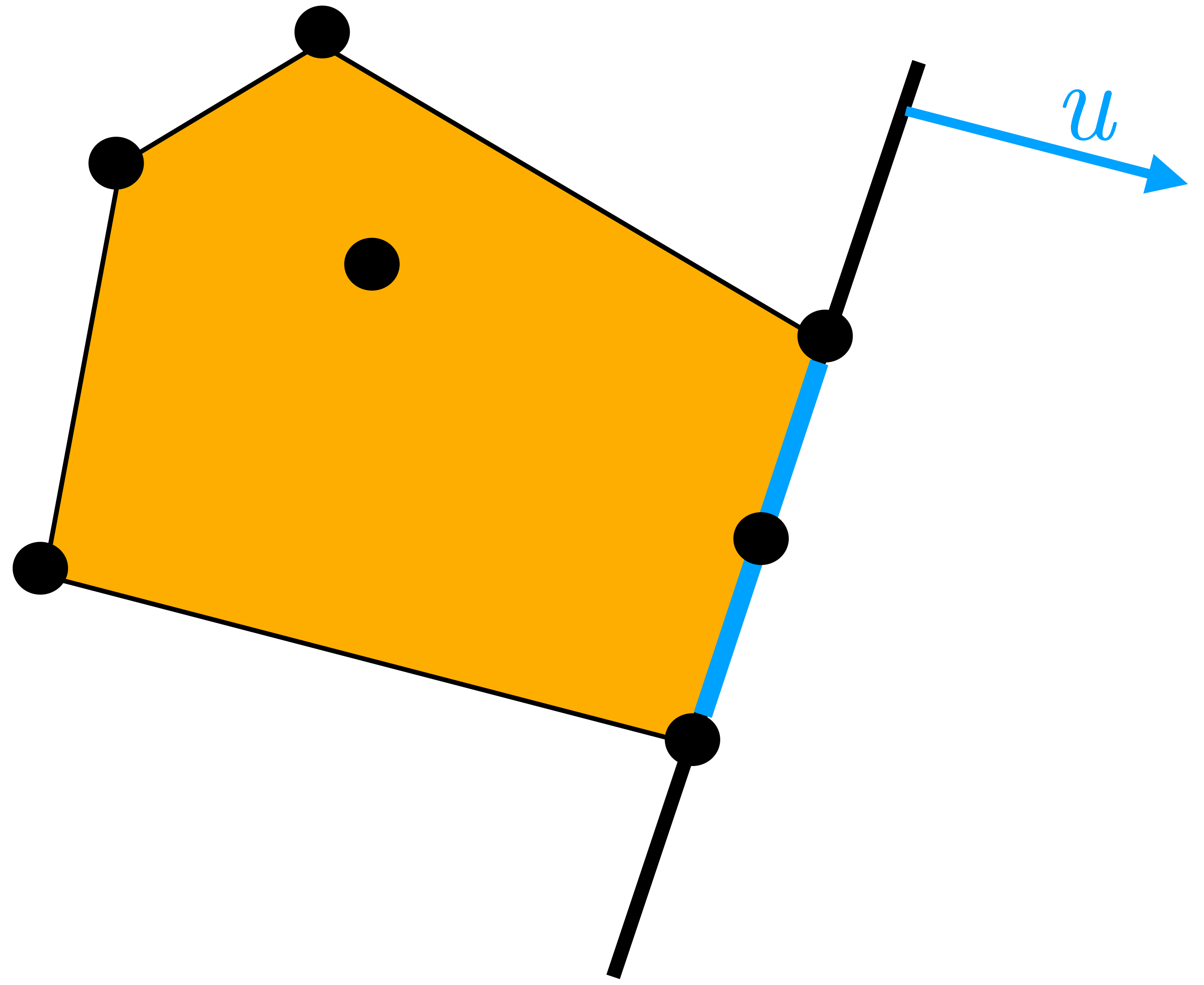

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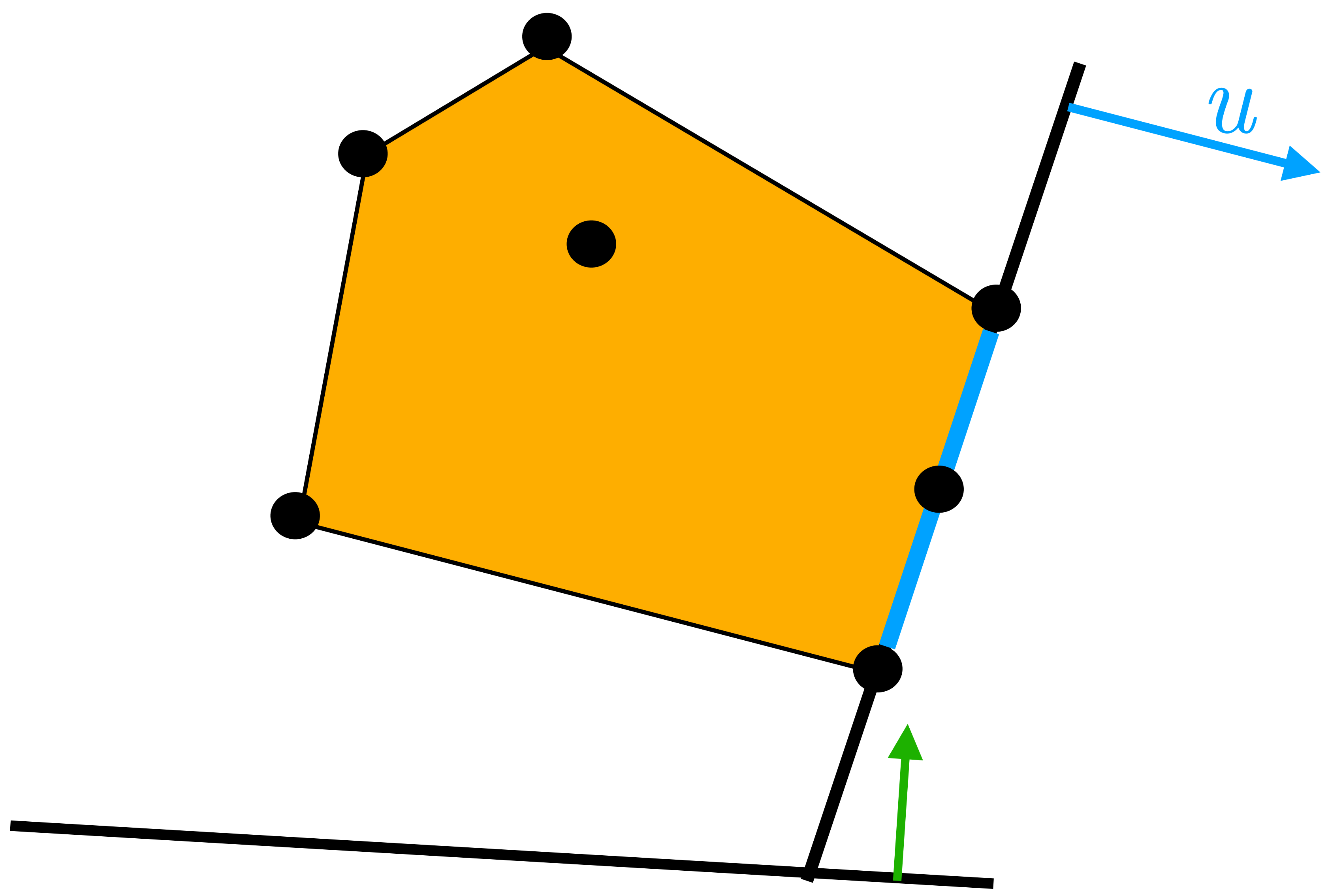

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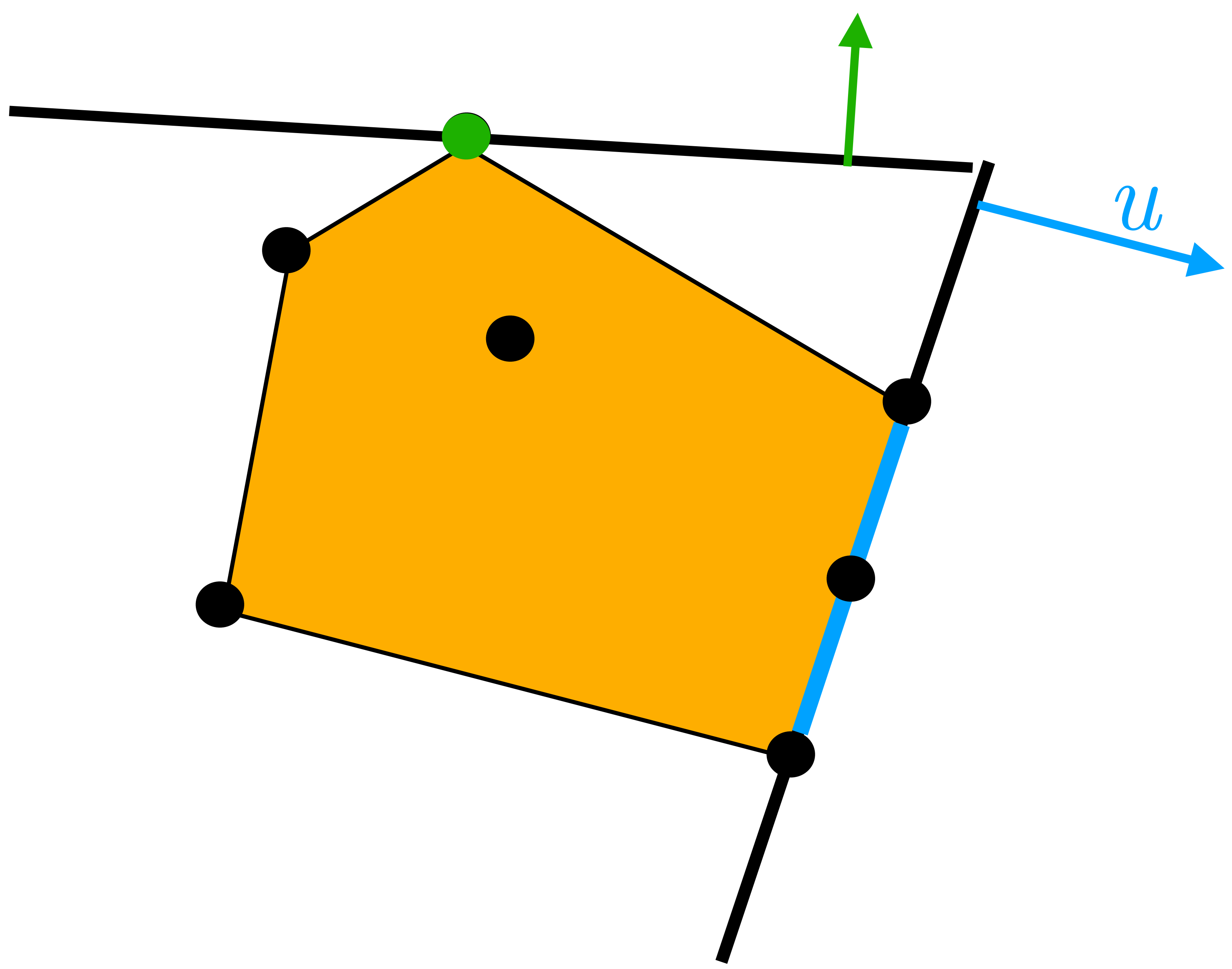

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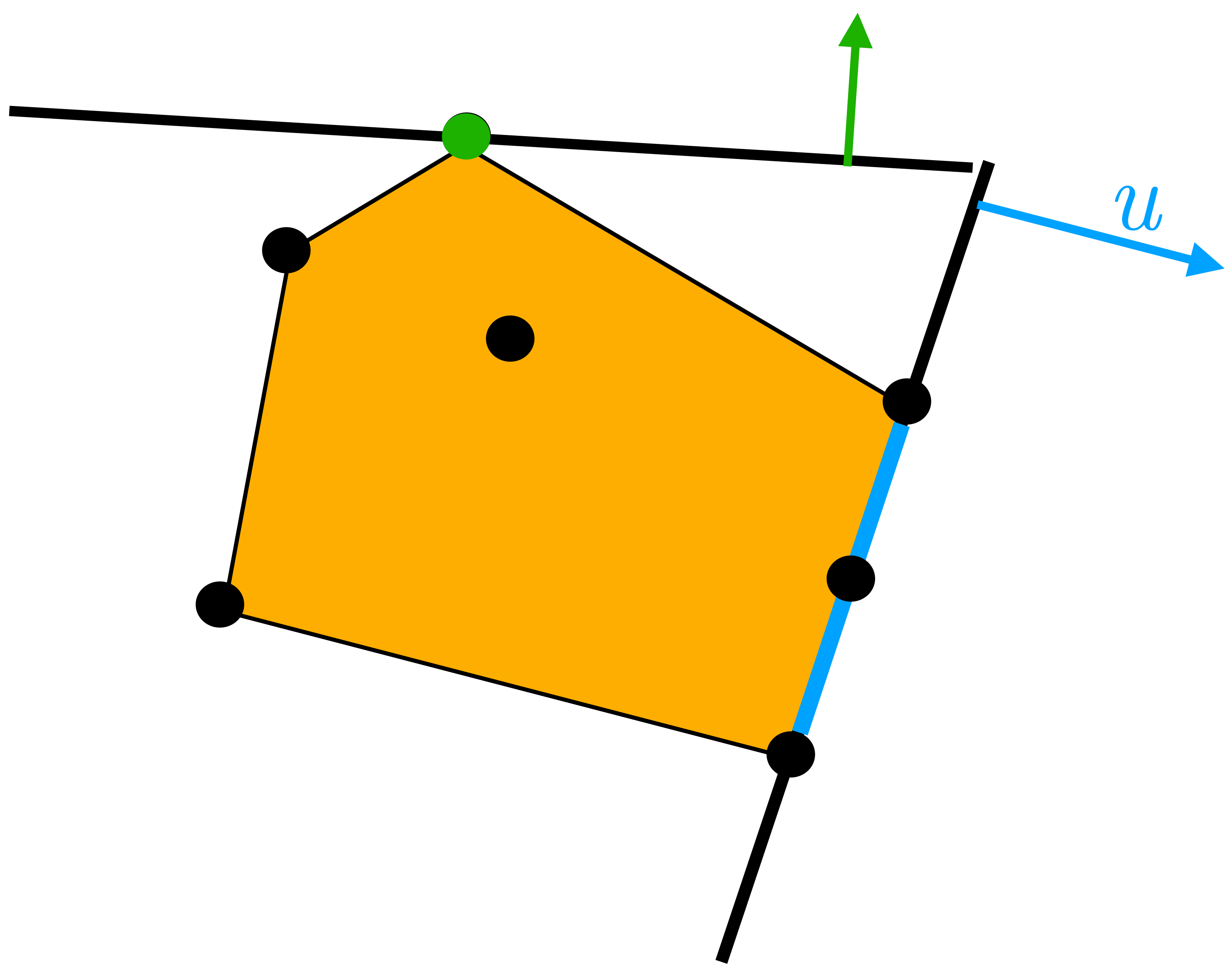

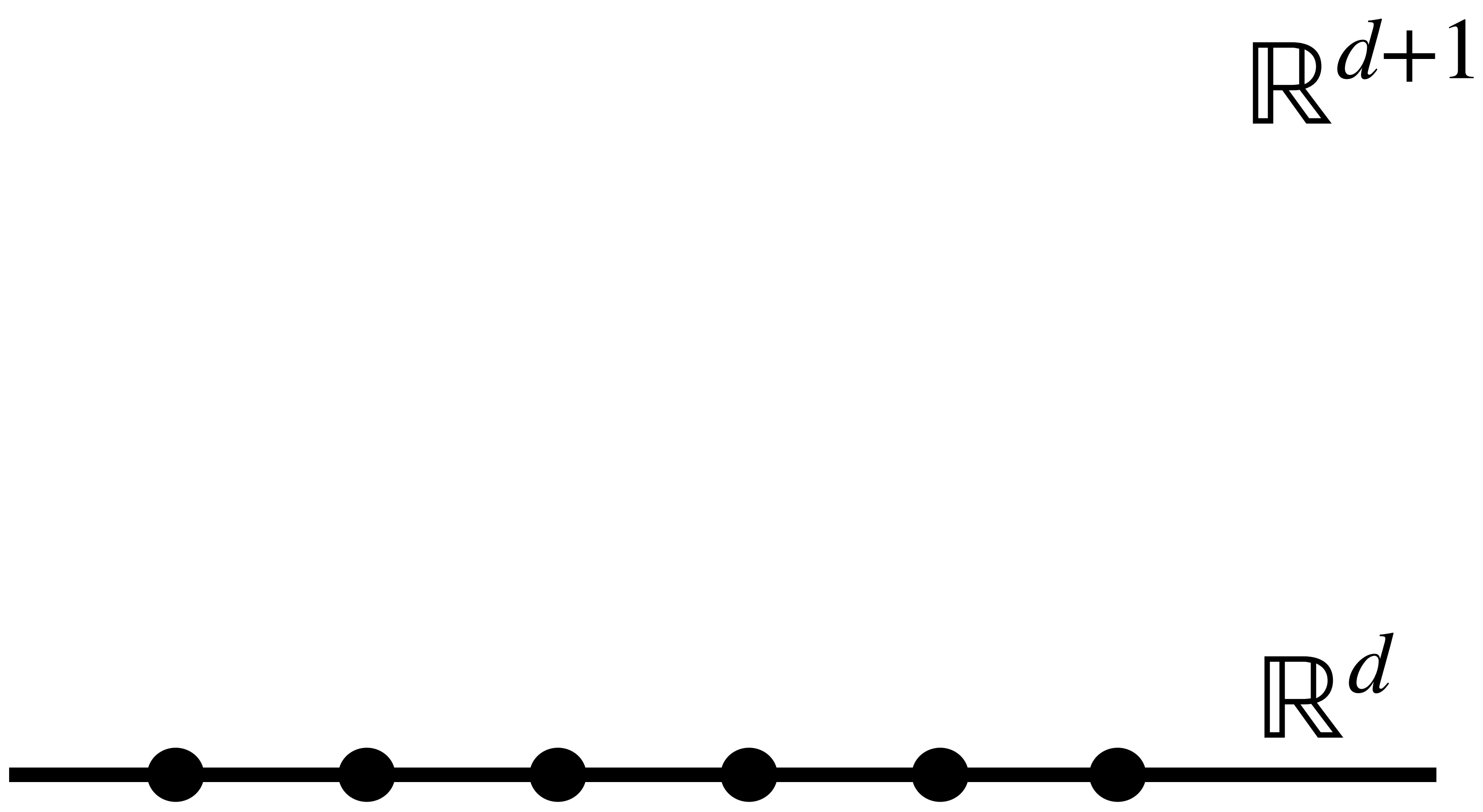

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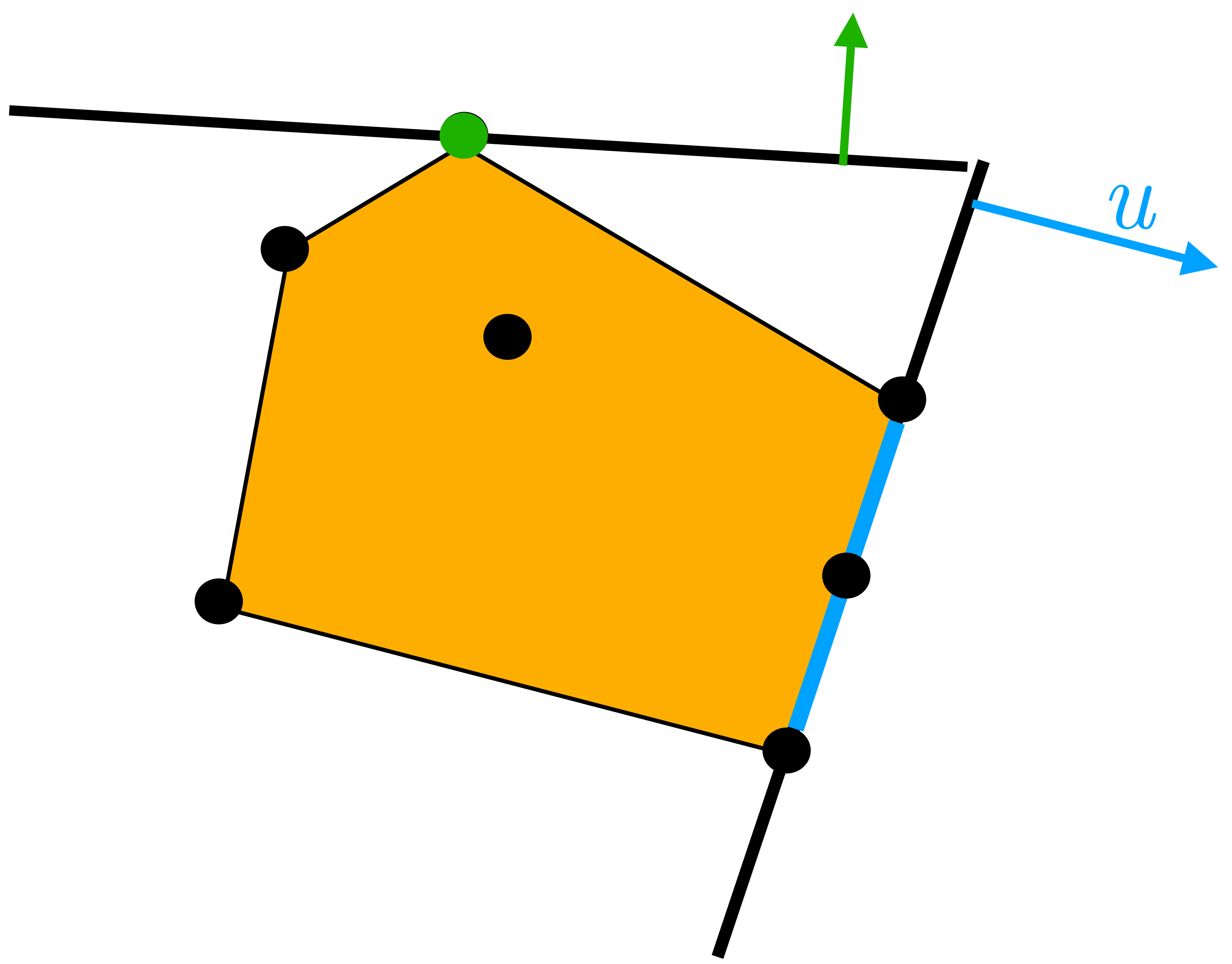

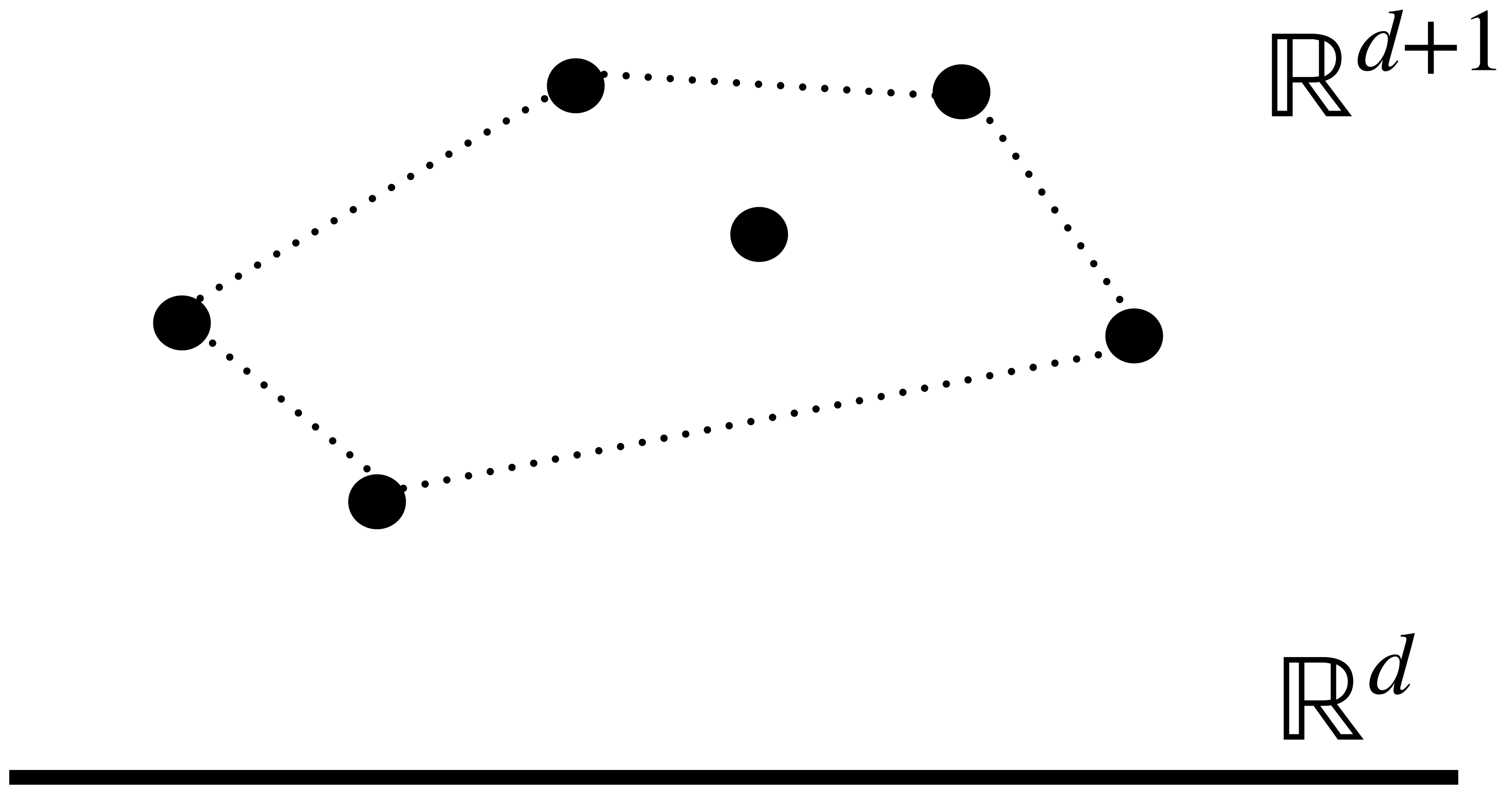

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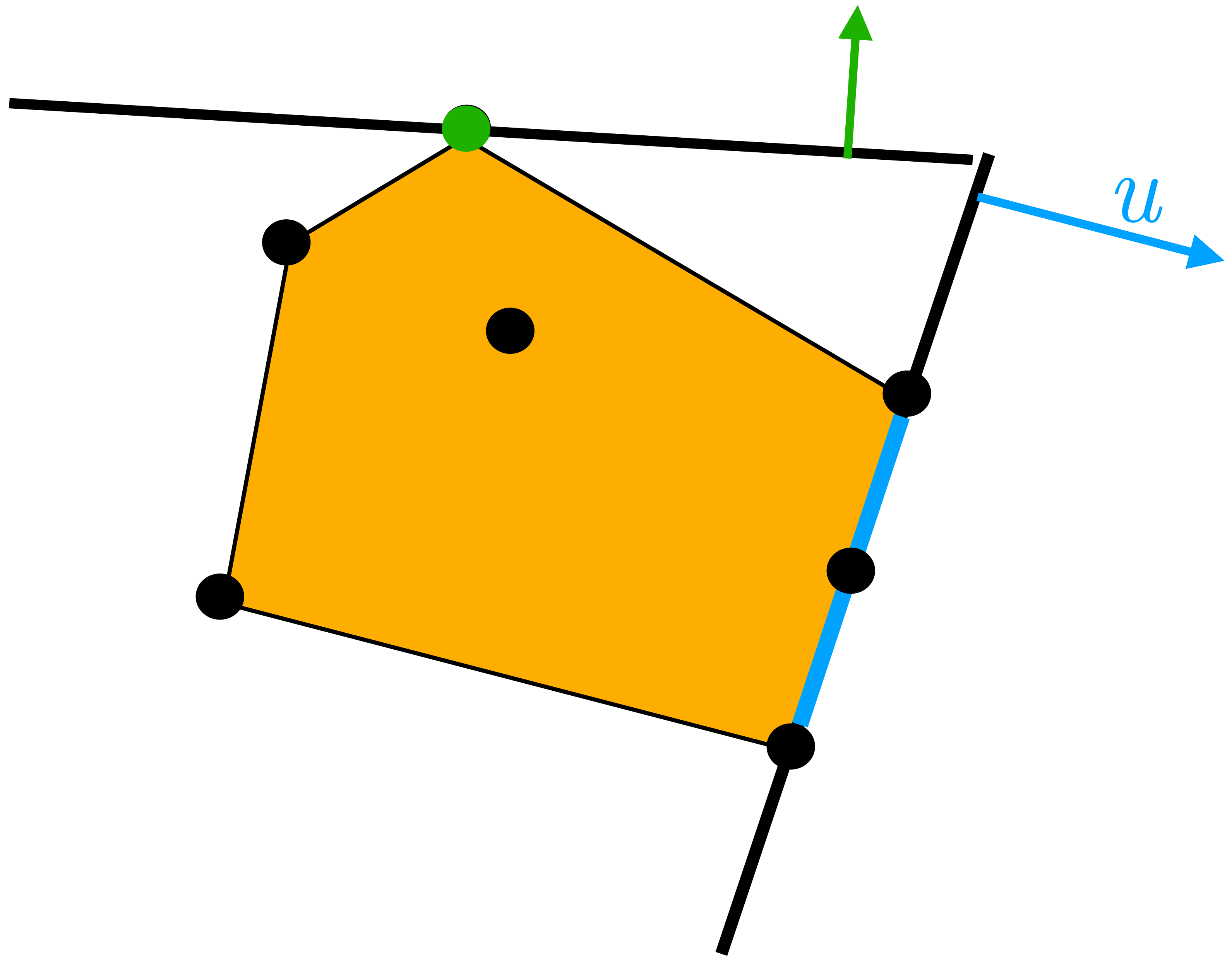

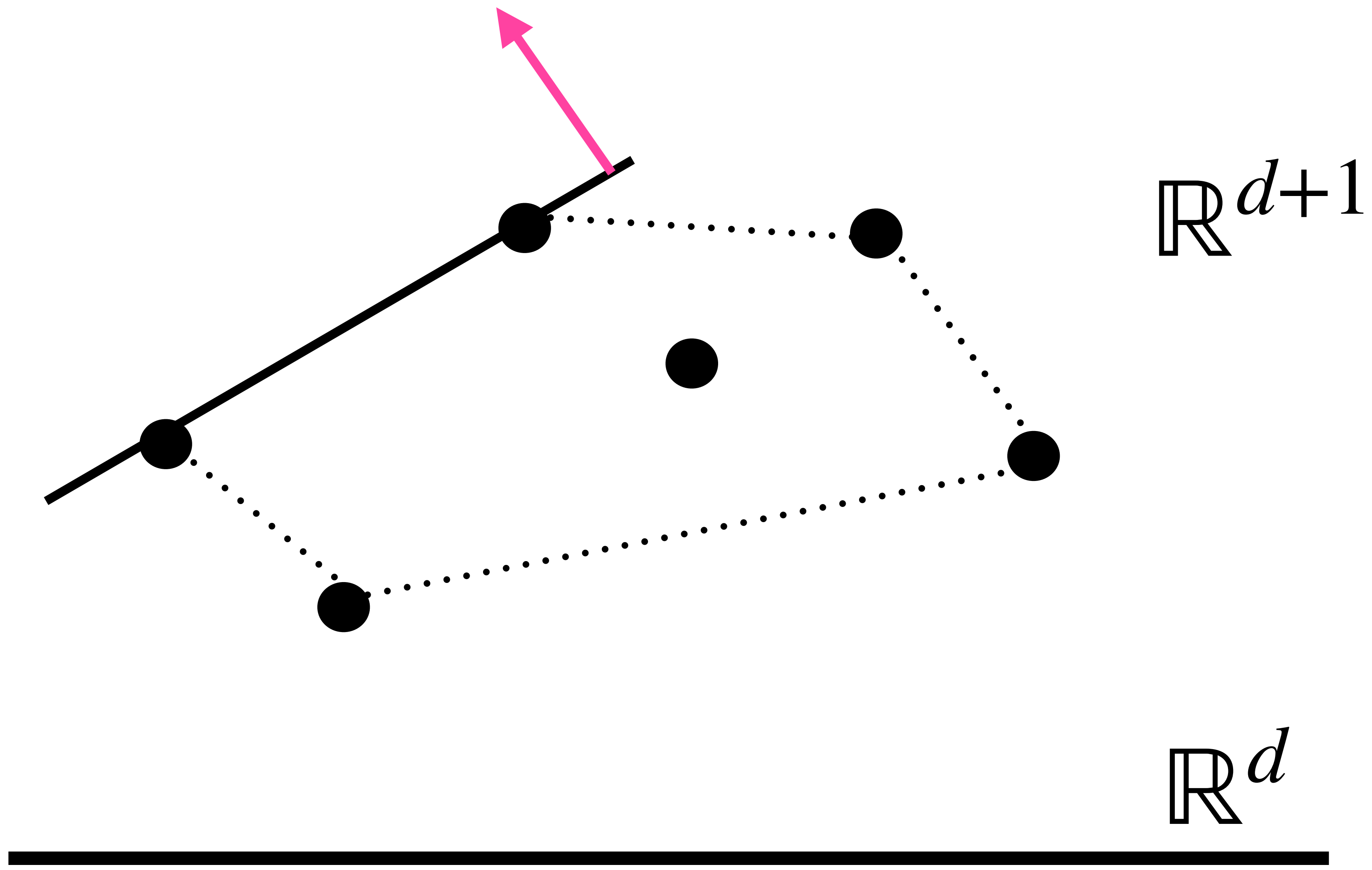

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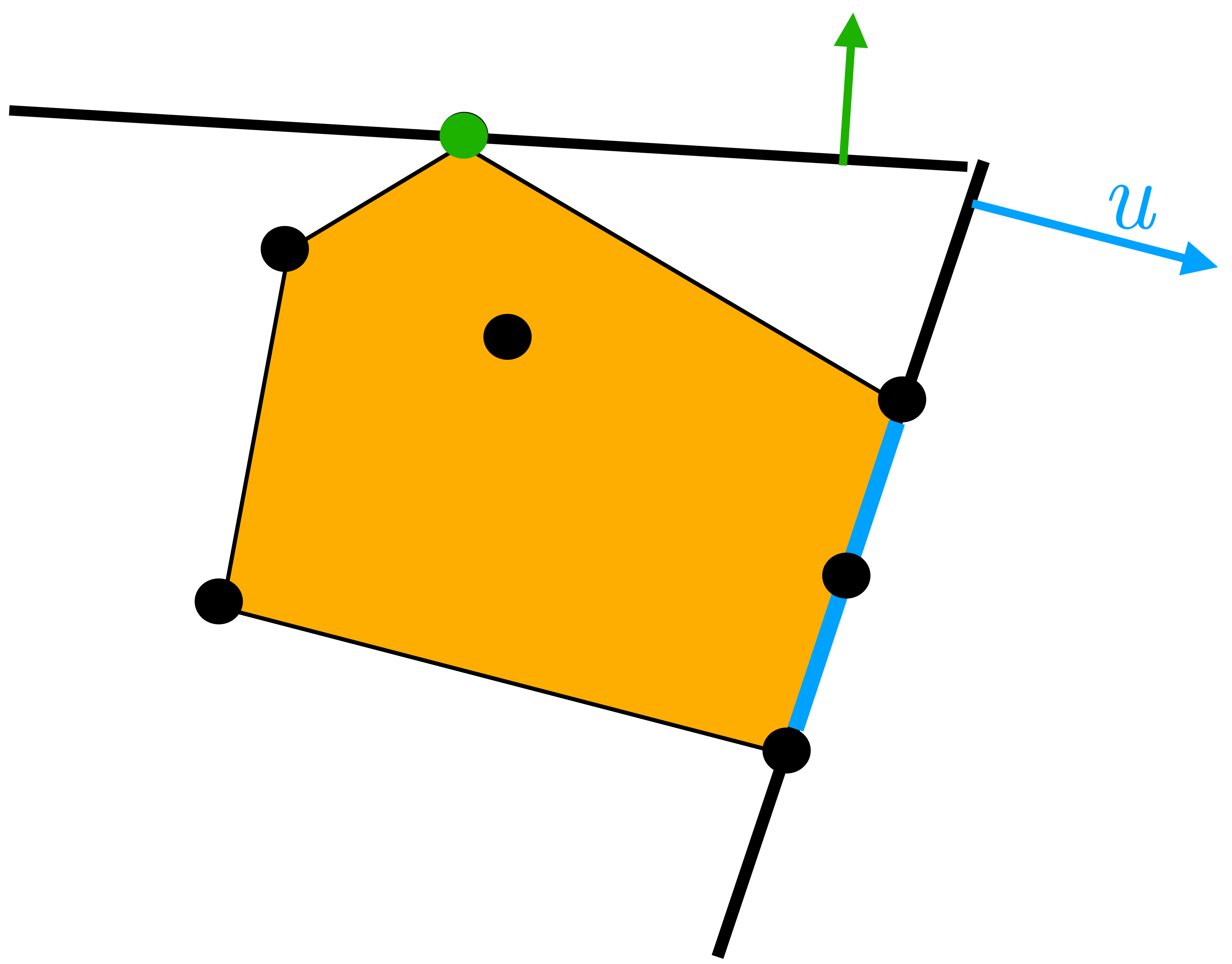

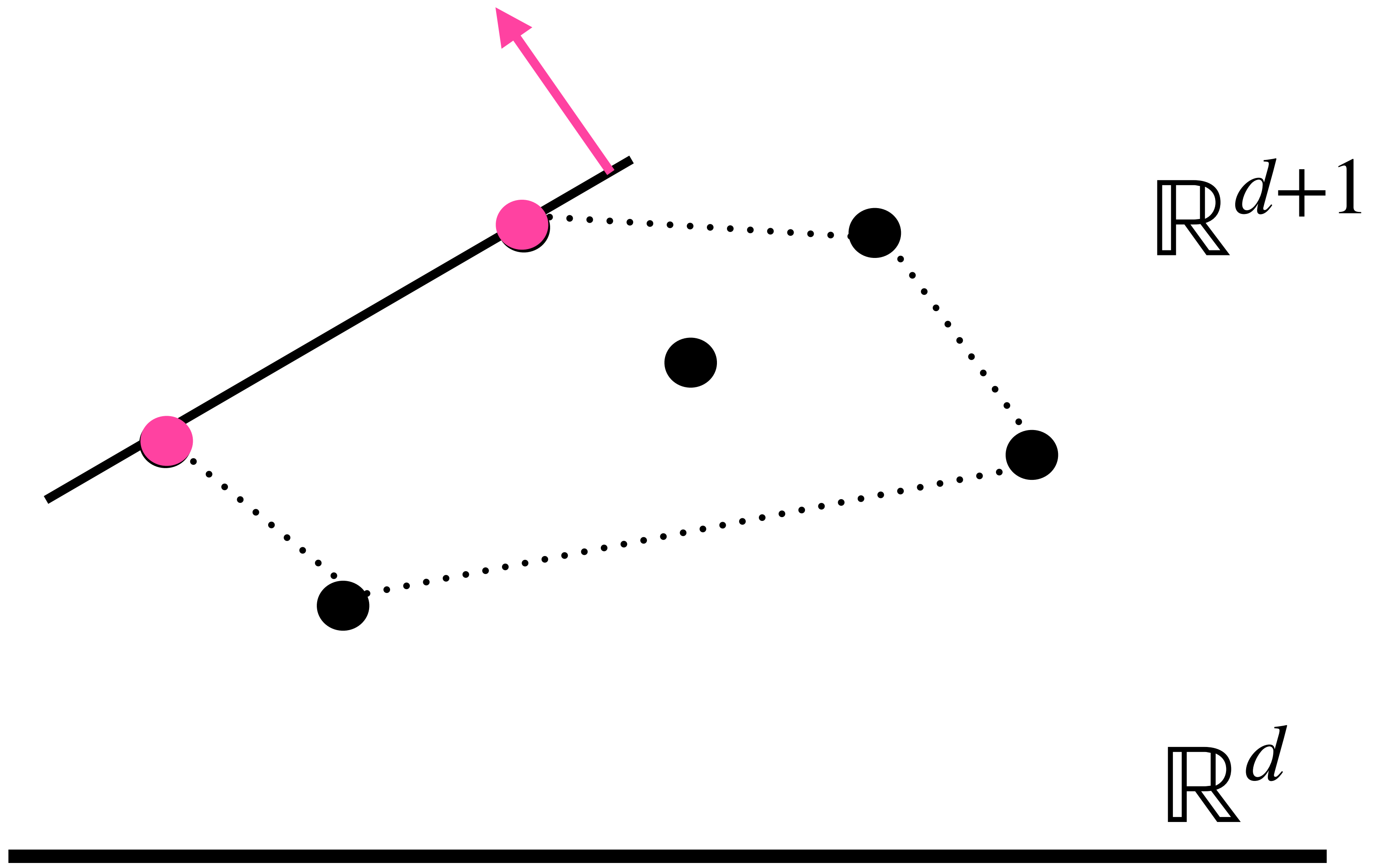

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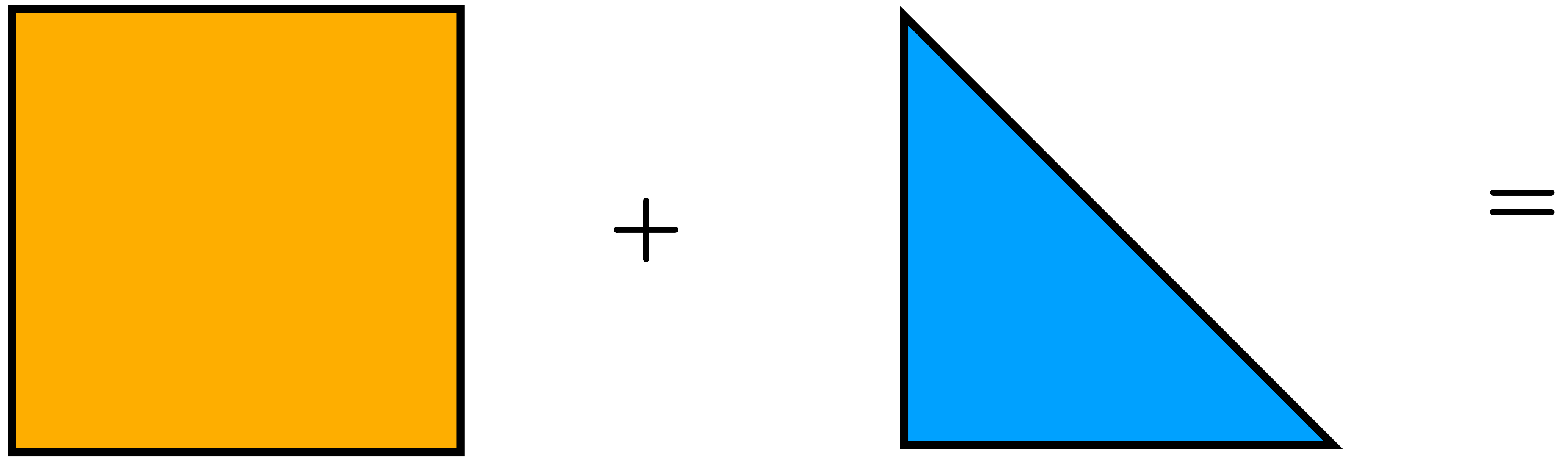

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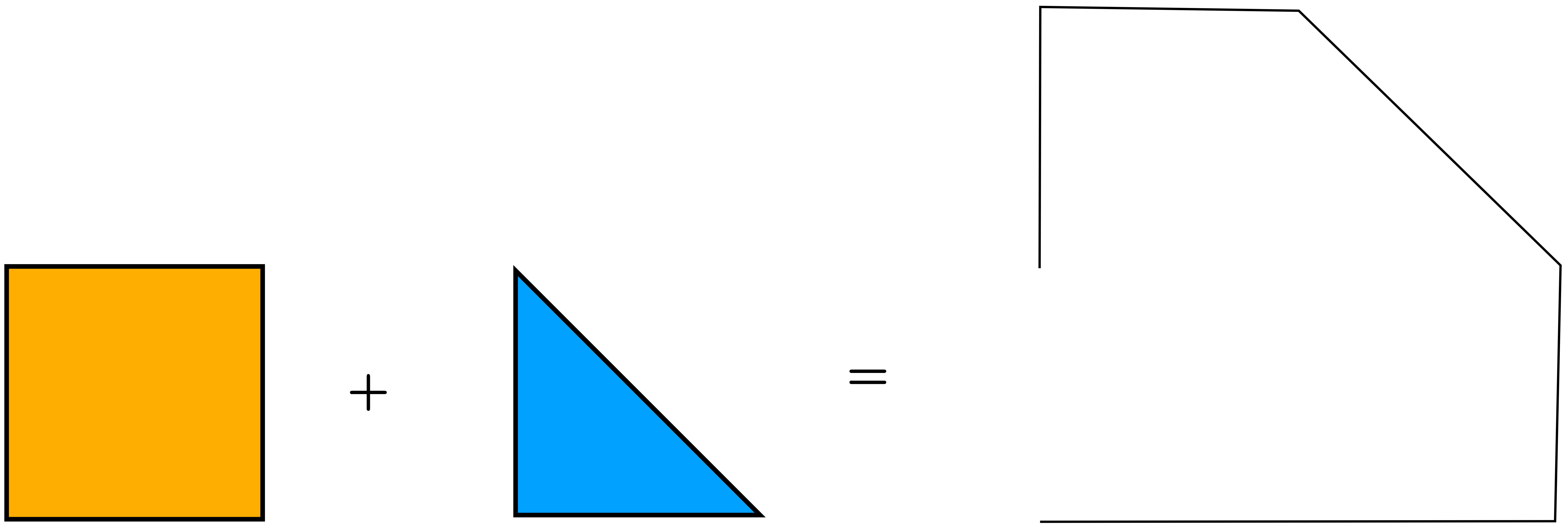

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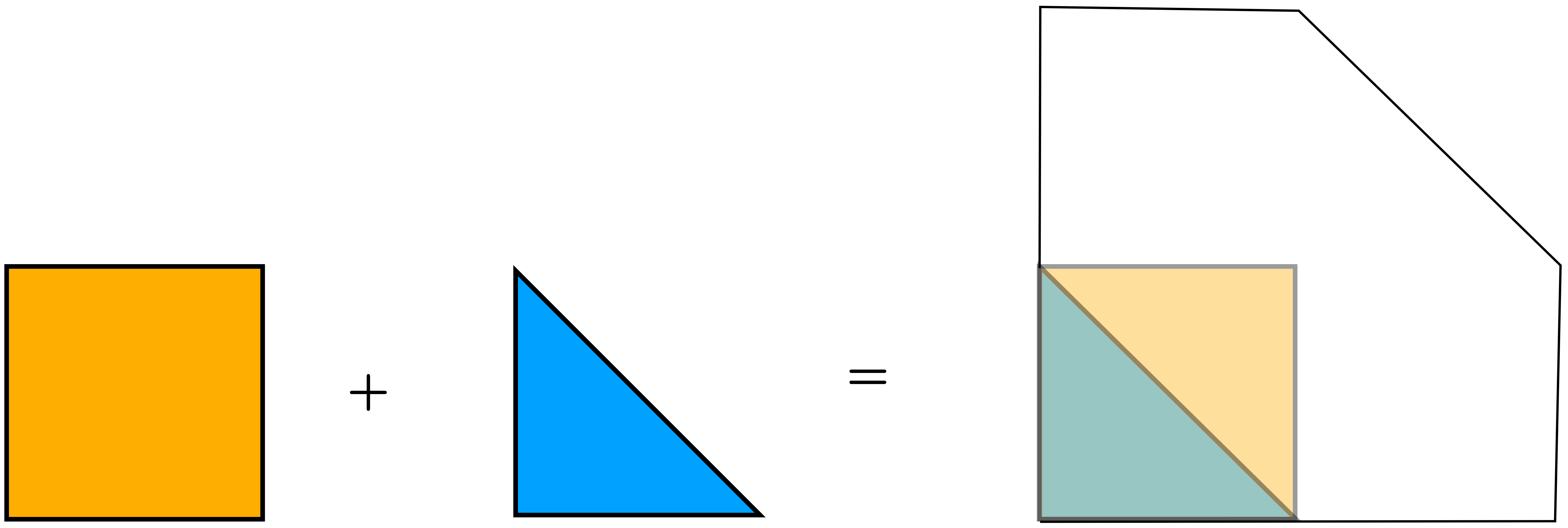

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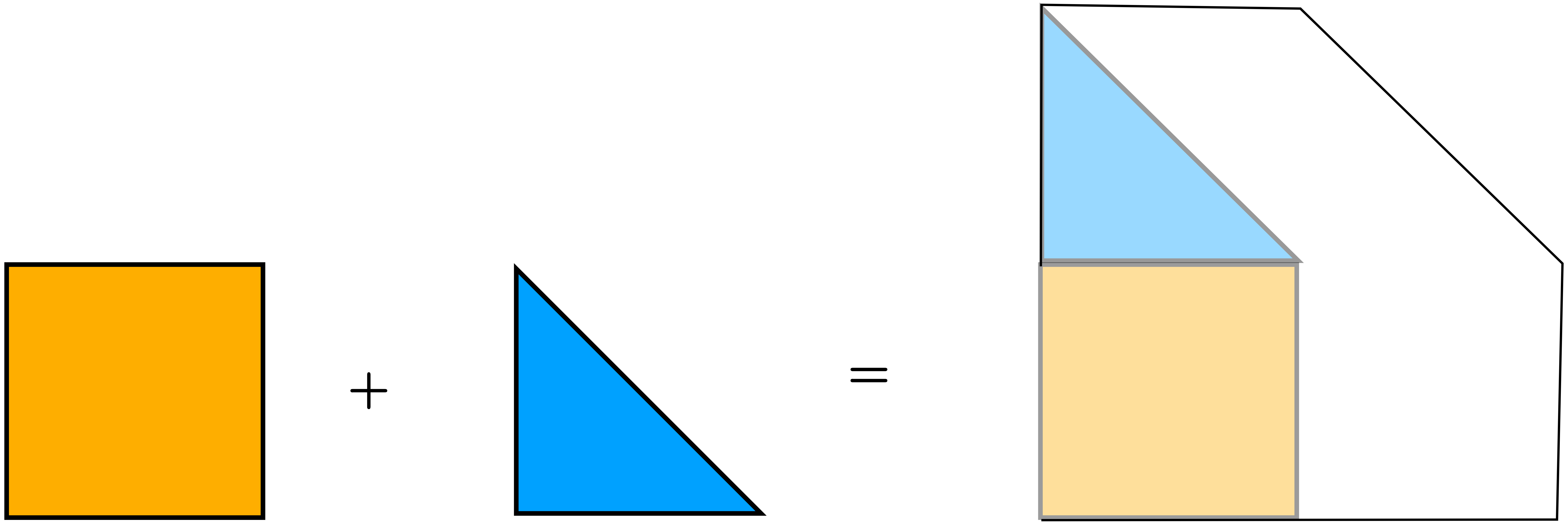

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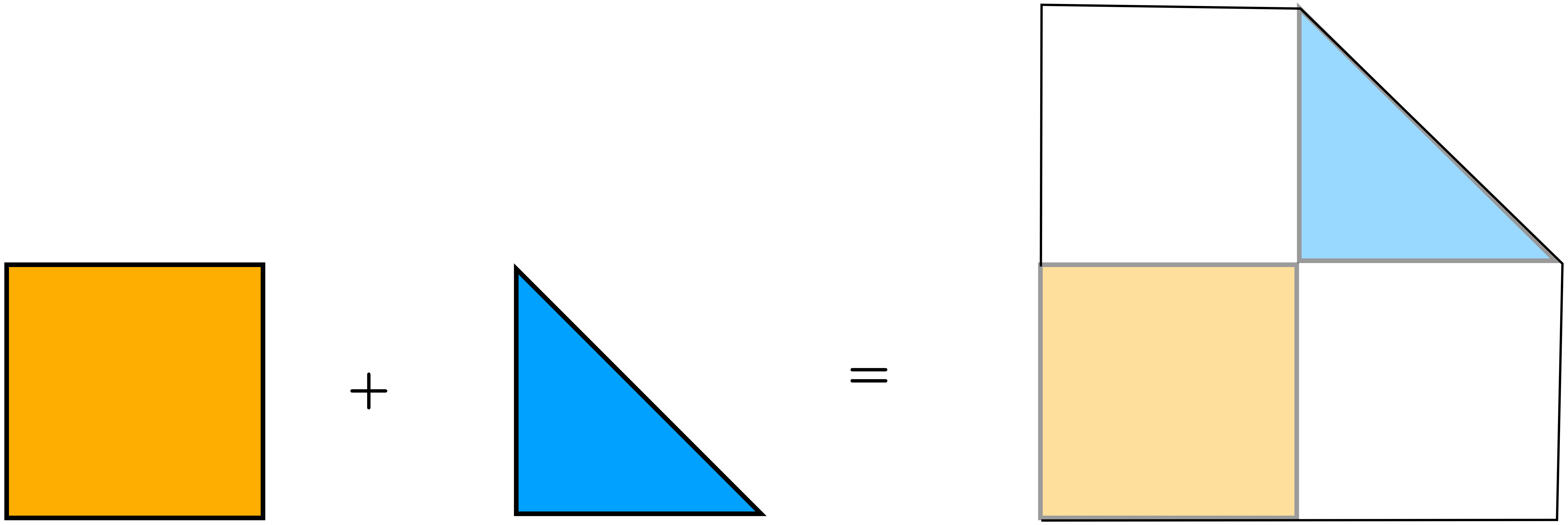

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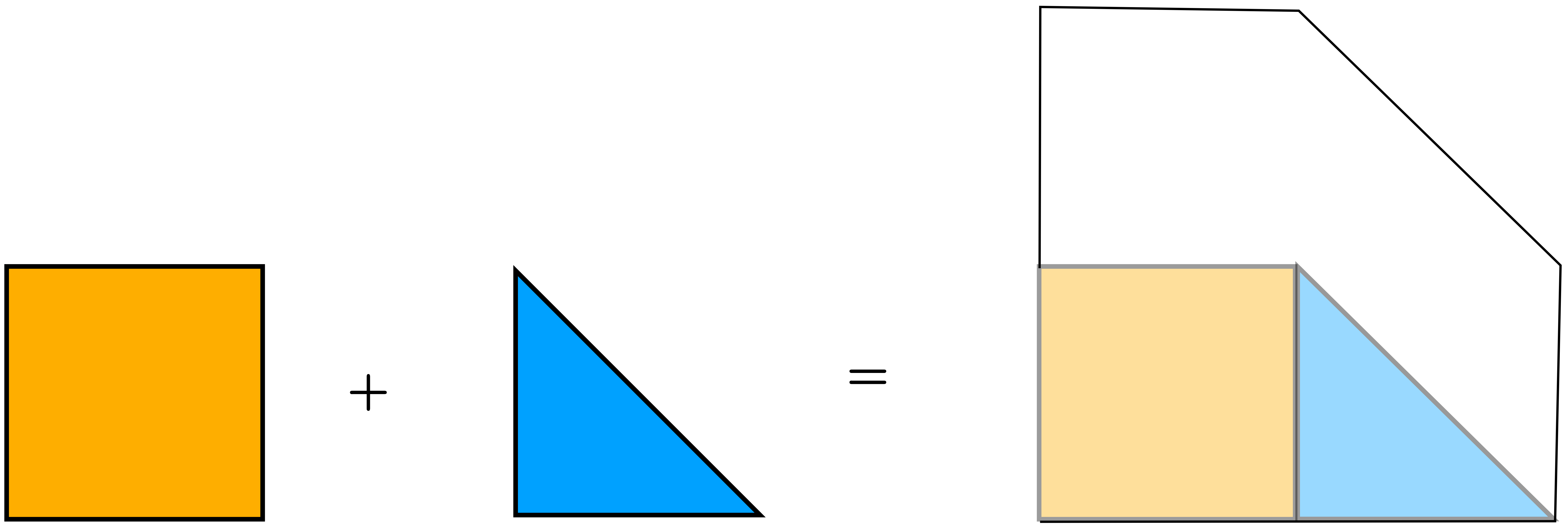

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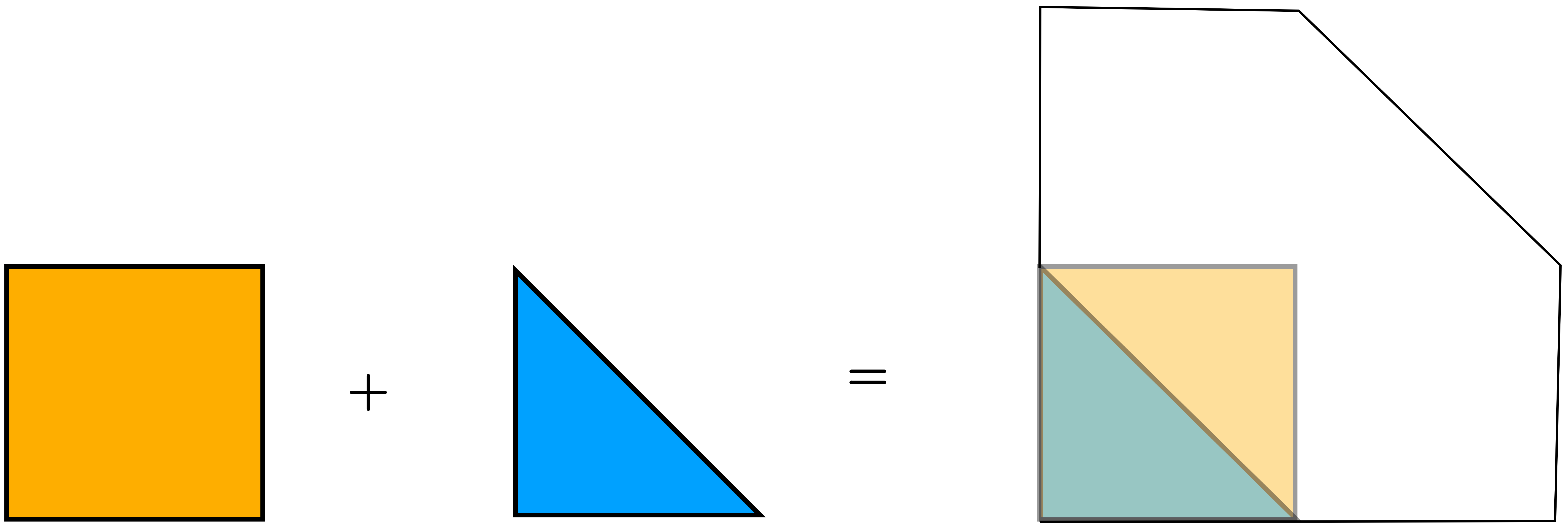

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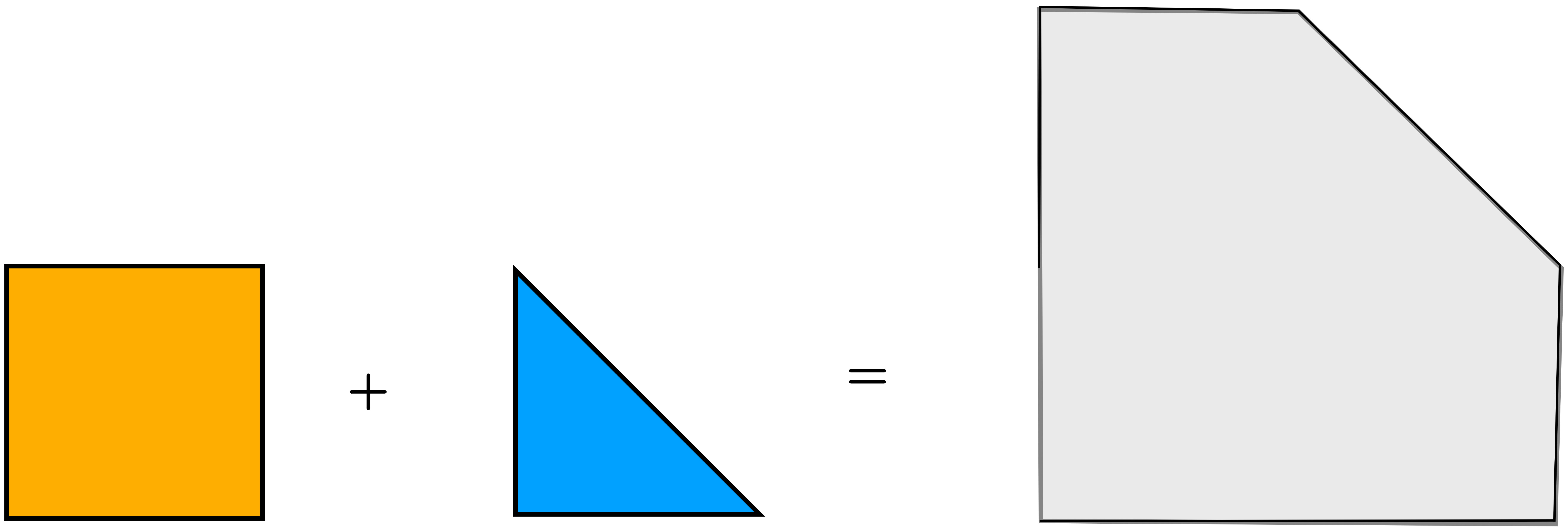

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General assumptions:

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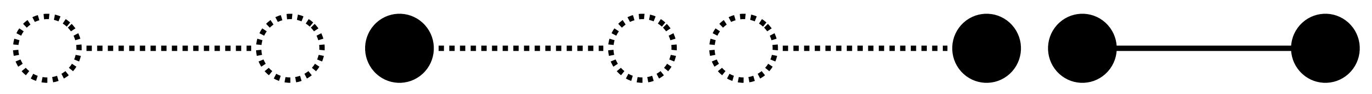

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$$P(G) = \mathrm{conv}(\chi_{G'} \mid G' \subseteq G \text{ induced})$$

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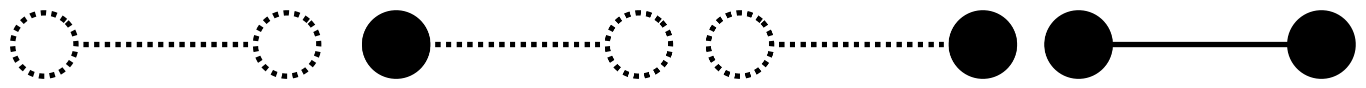

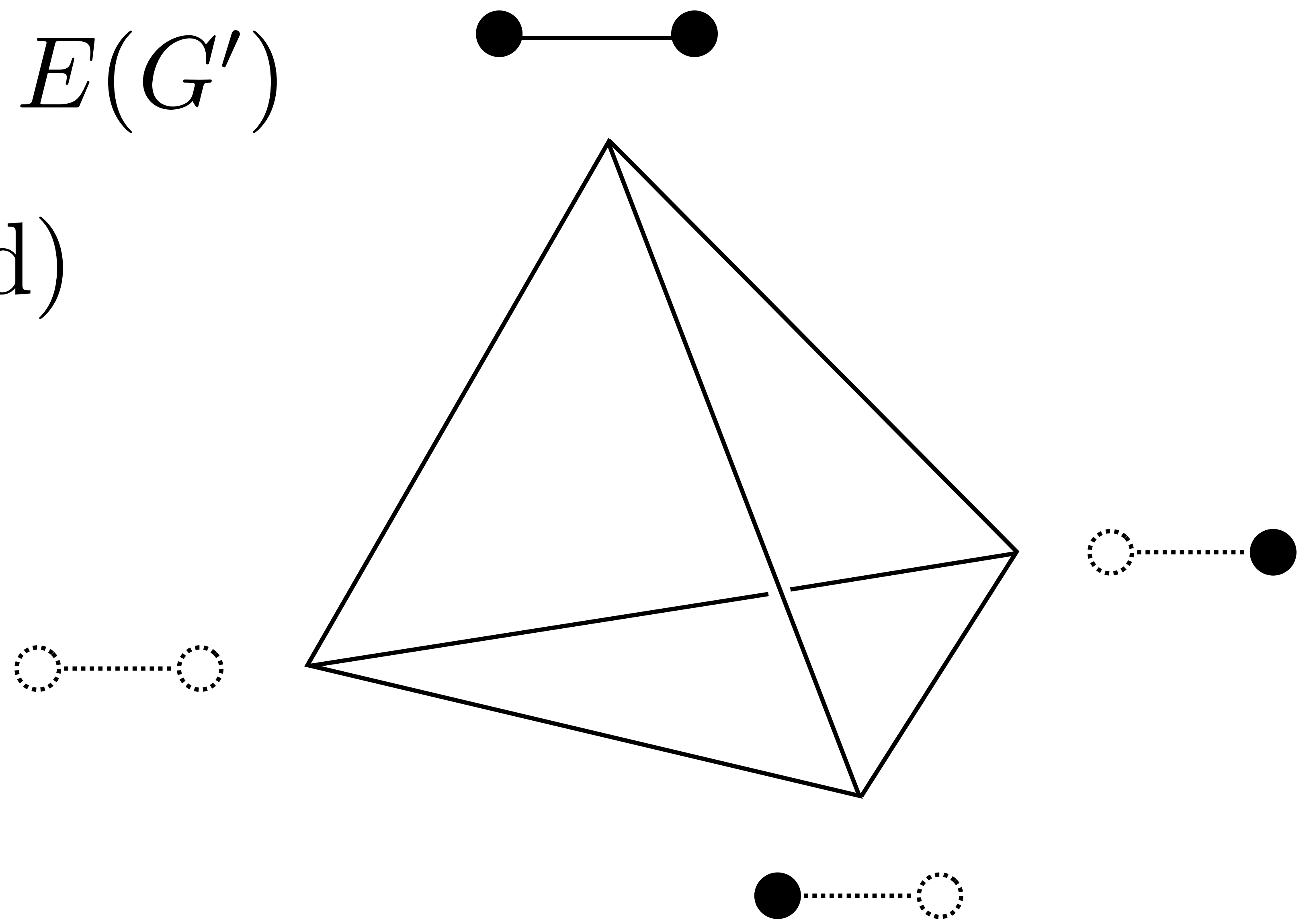

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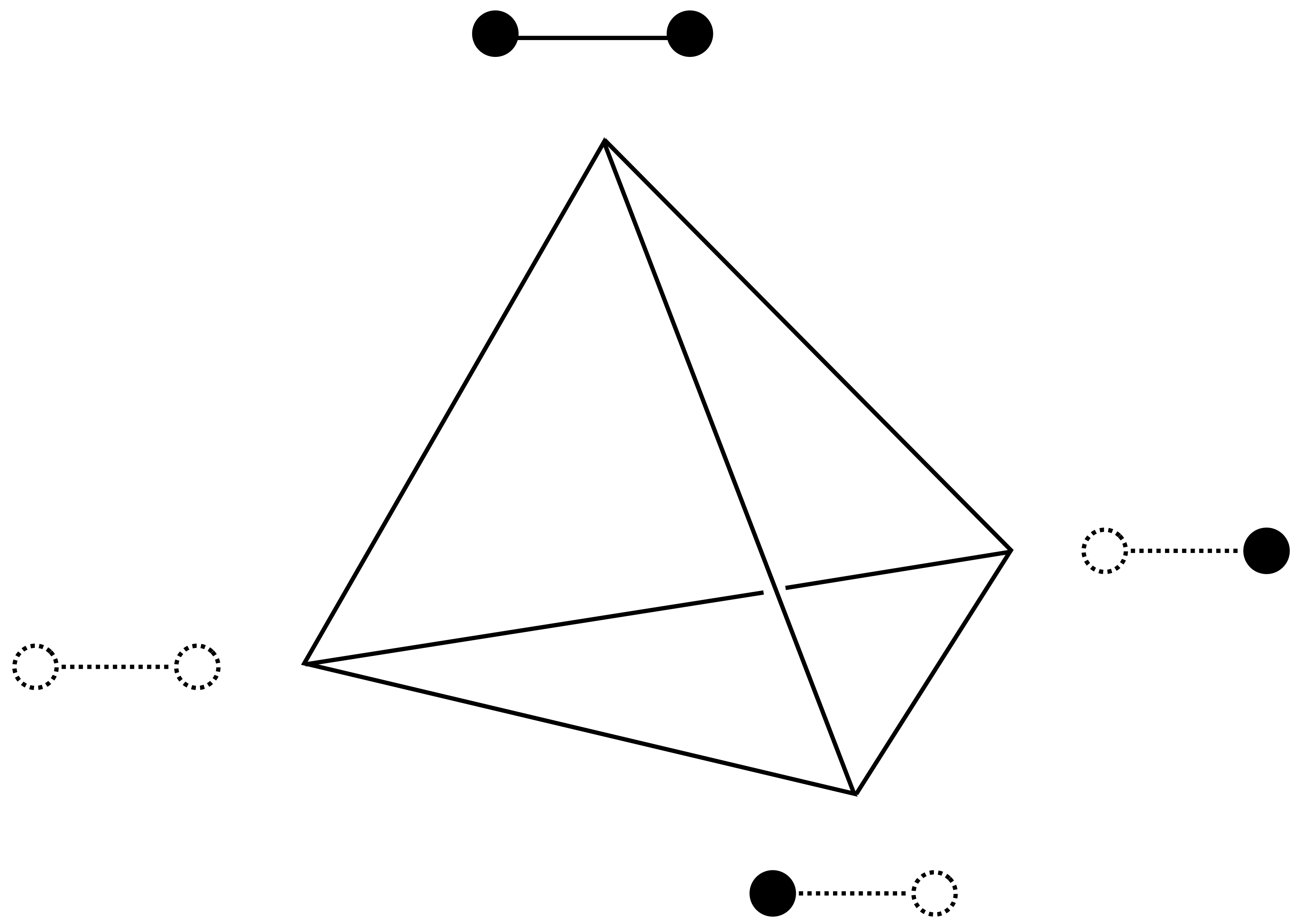

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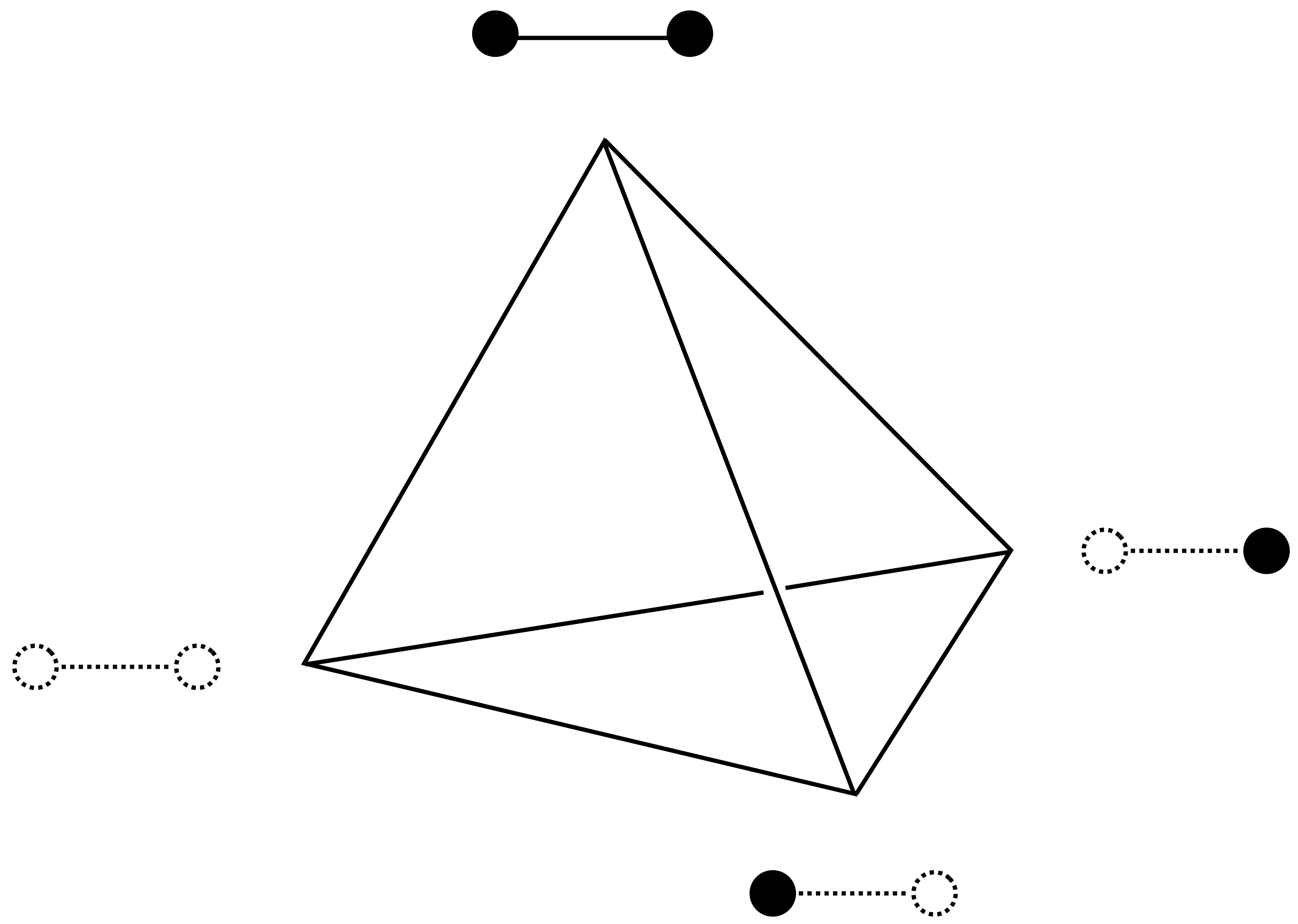

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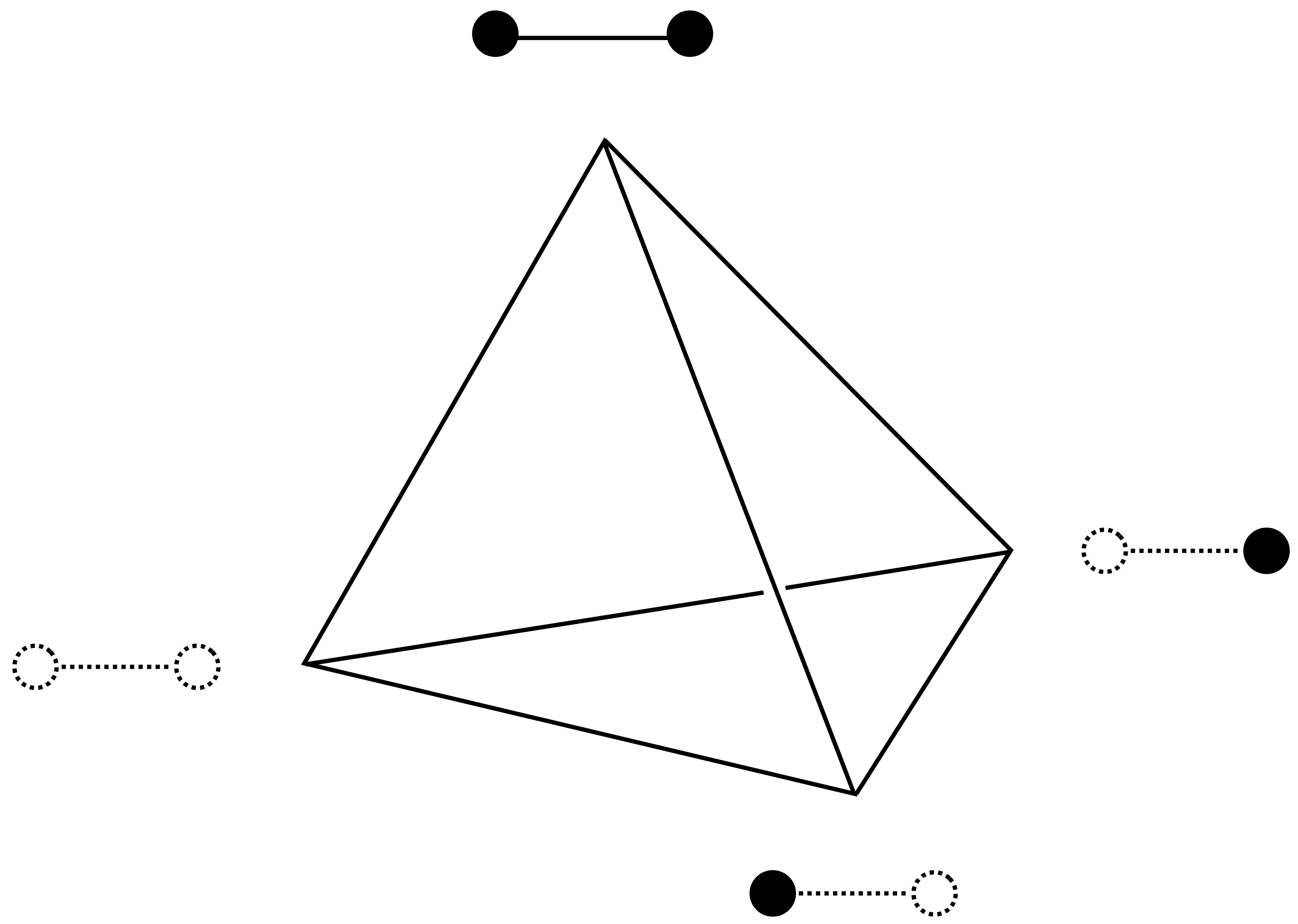
Valuation function $v^b : P \cap \mathbb{Z}^{n+|E|} \to \mathbb{R}$, $v^b(a) = \langle w^b, a \rangle$ for some $w^b \in \mathbb{R}^{n+|E|}$



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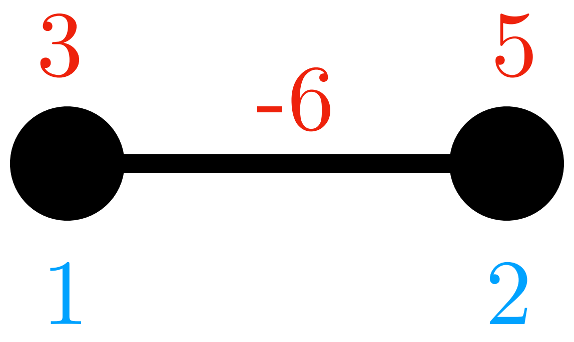
Valuation function $v^b : P \cap \mathbb{Z}^{n+|E|} \to \mathbb{R}$, $v^b(a) = \langle w^b, a \rangle$ for some $w^b \in \mathbb{R}^{n+|E|}$

$$w^b = \begin{pmatrix} 3 \\ 5 \\ -6 \end{pmatrix}$$



$$v^b\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right) = 0, \quad v^b\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) = 3,$$

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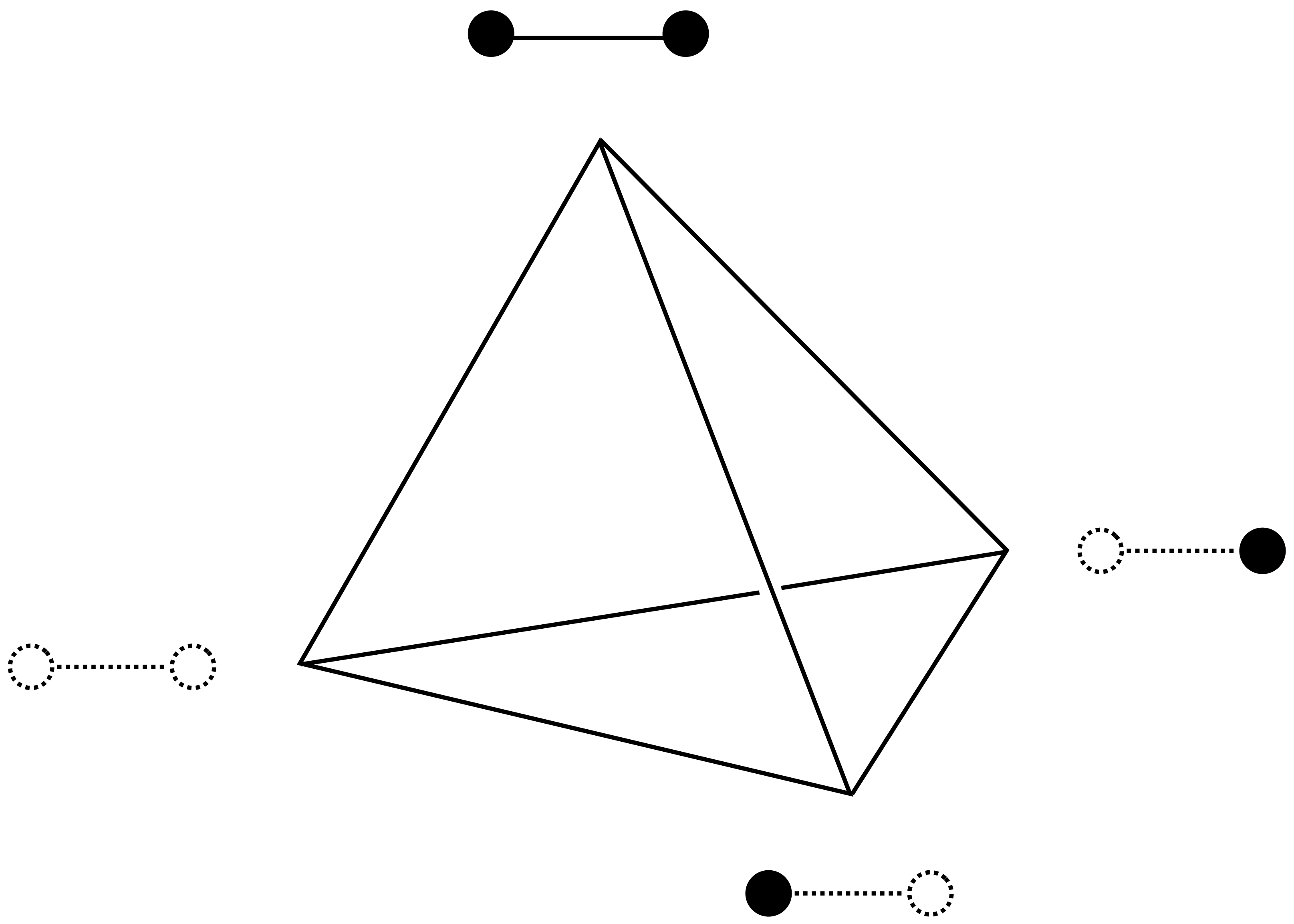

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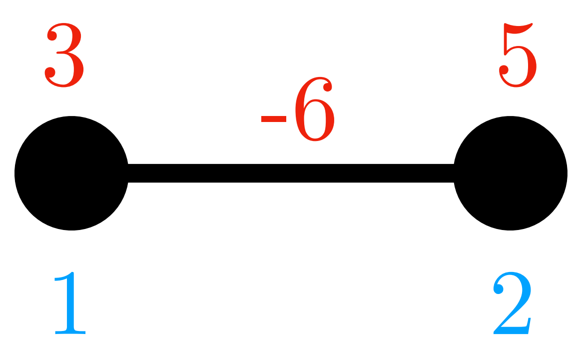

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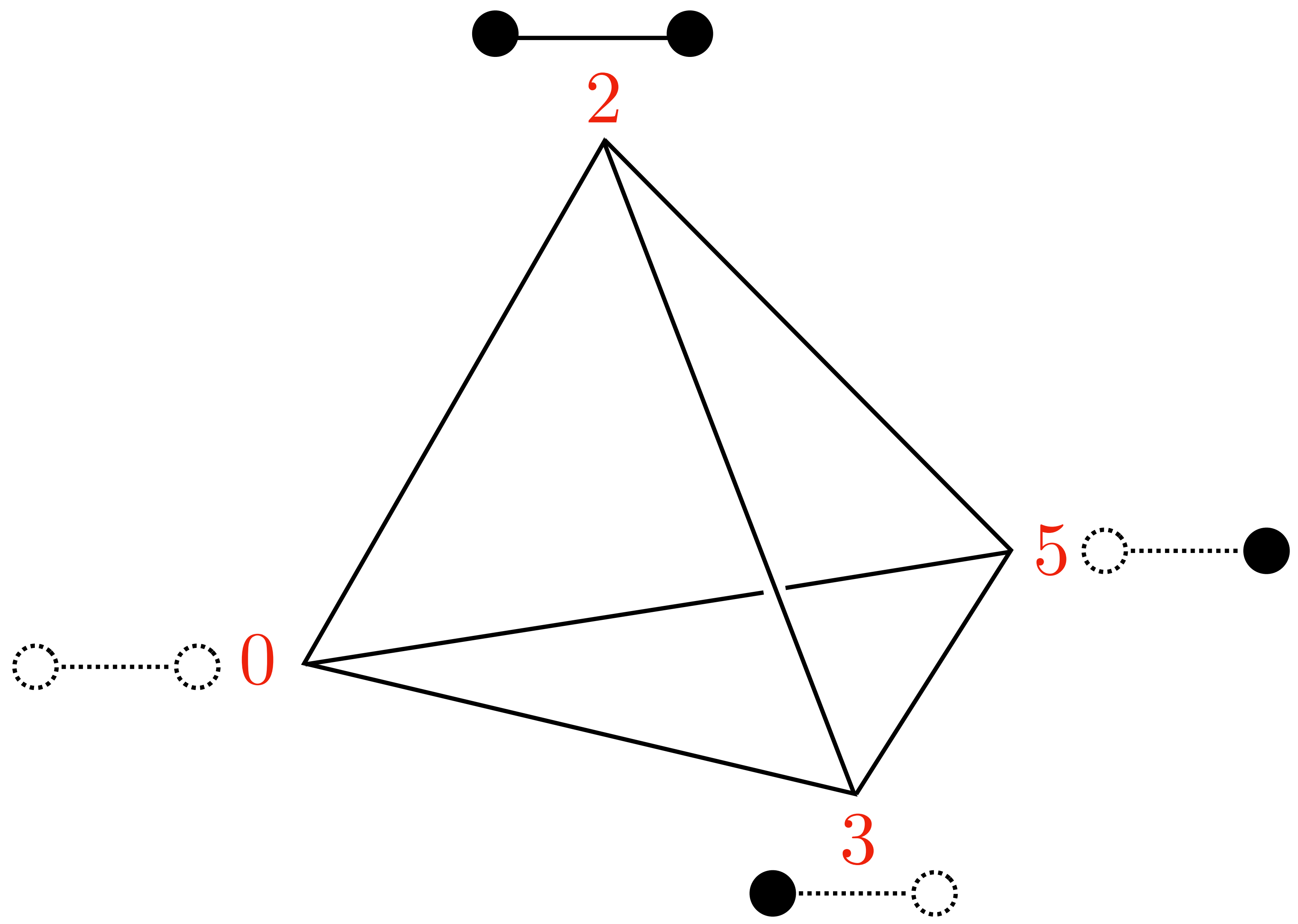

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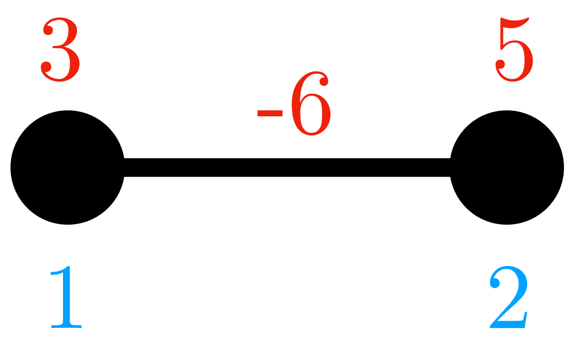

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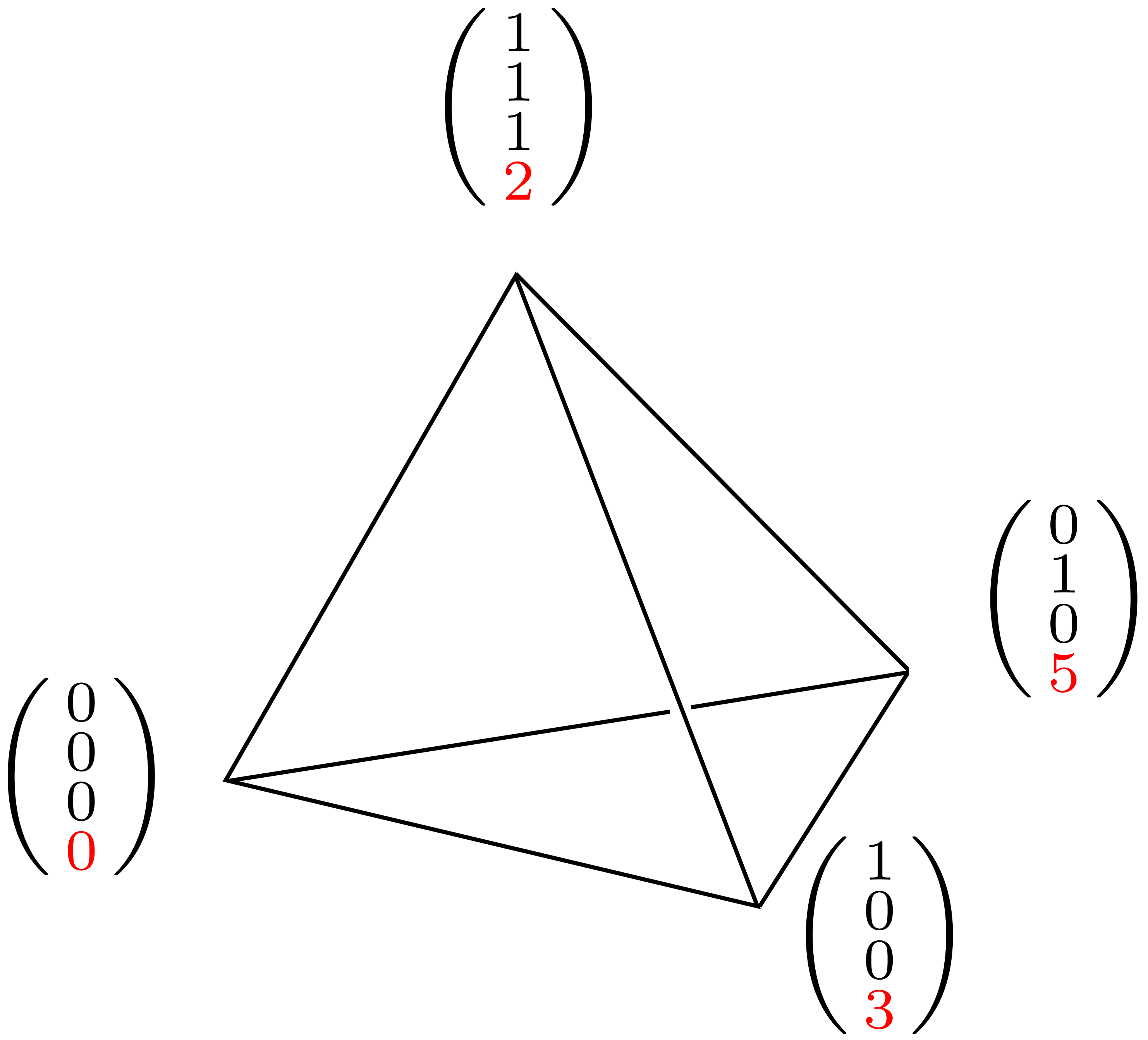

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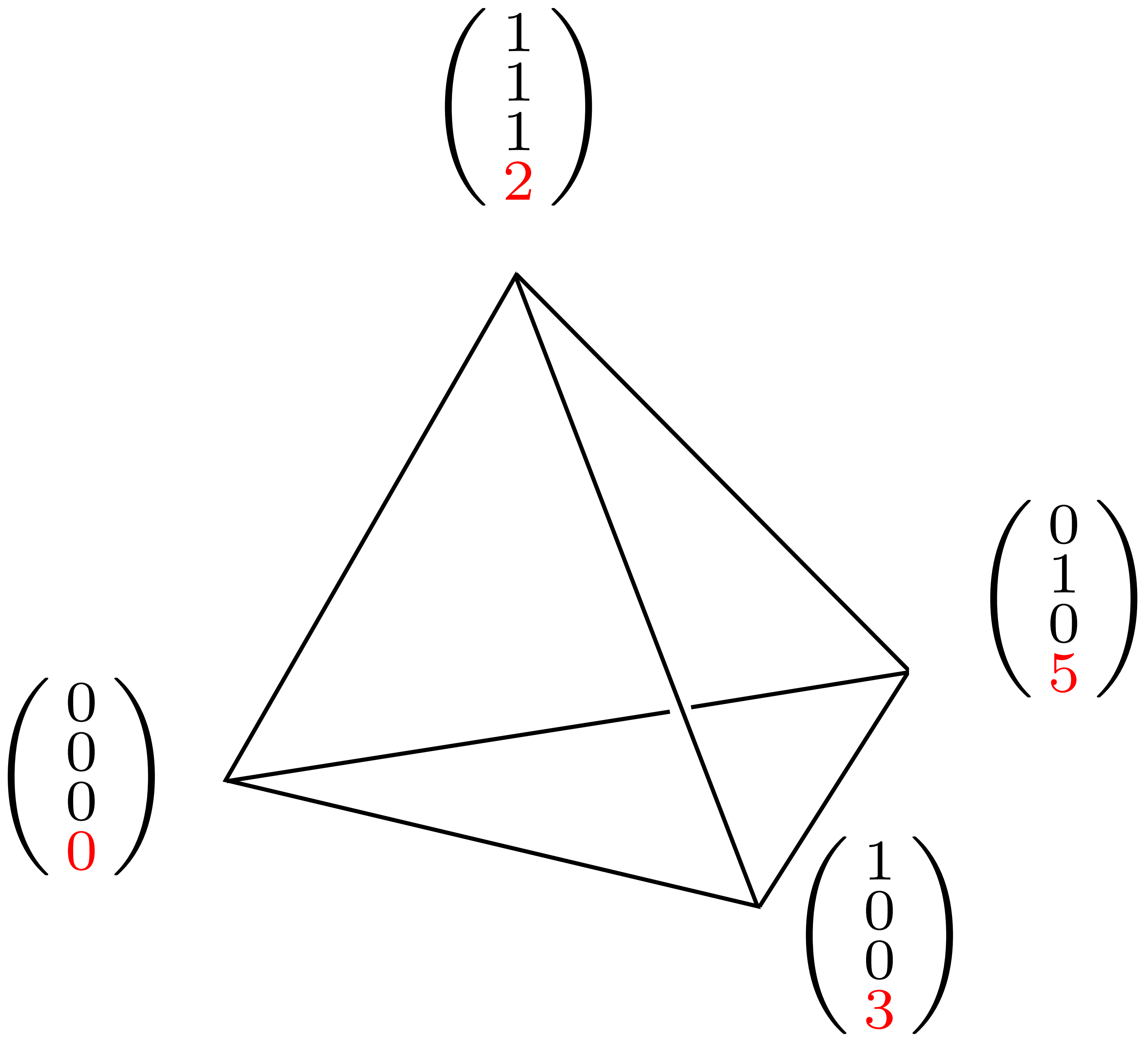
Auctioneer sets a price



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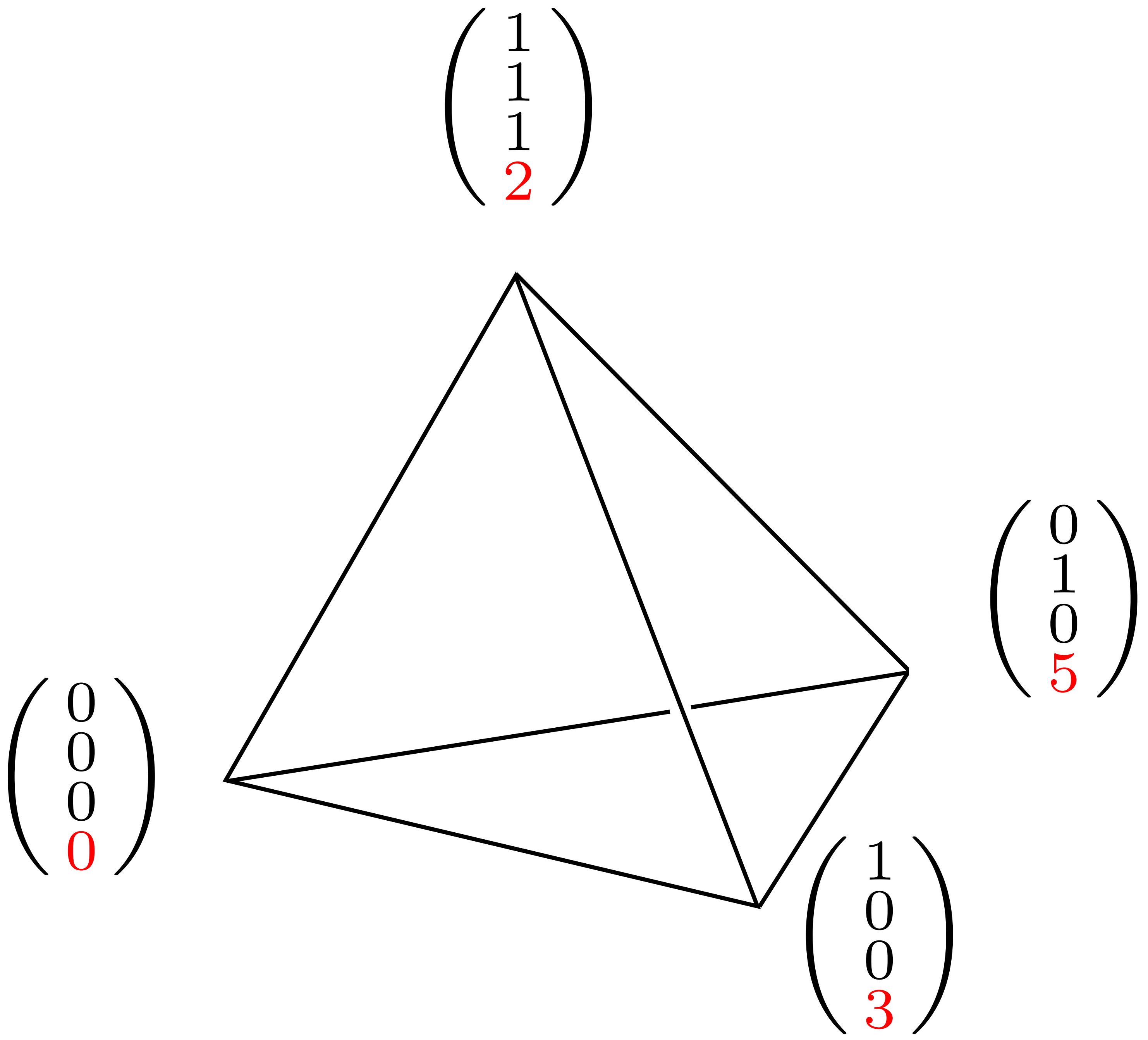

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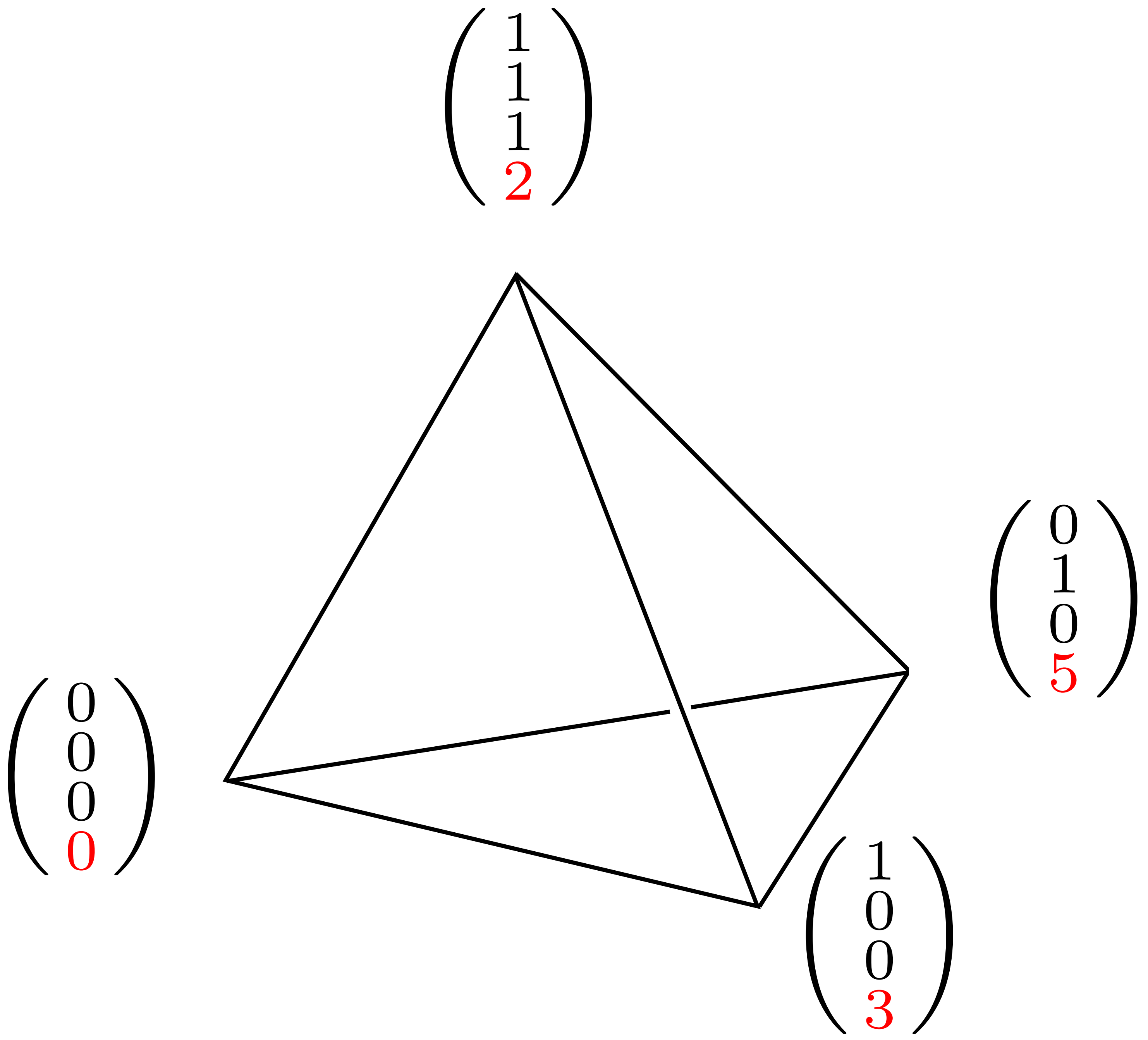

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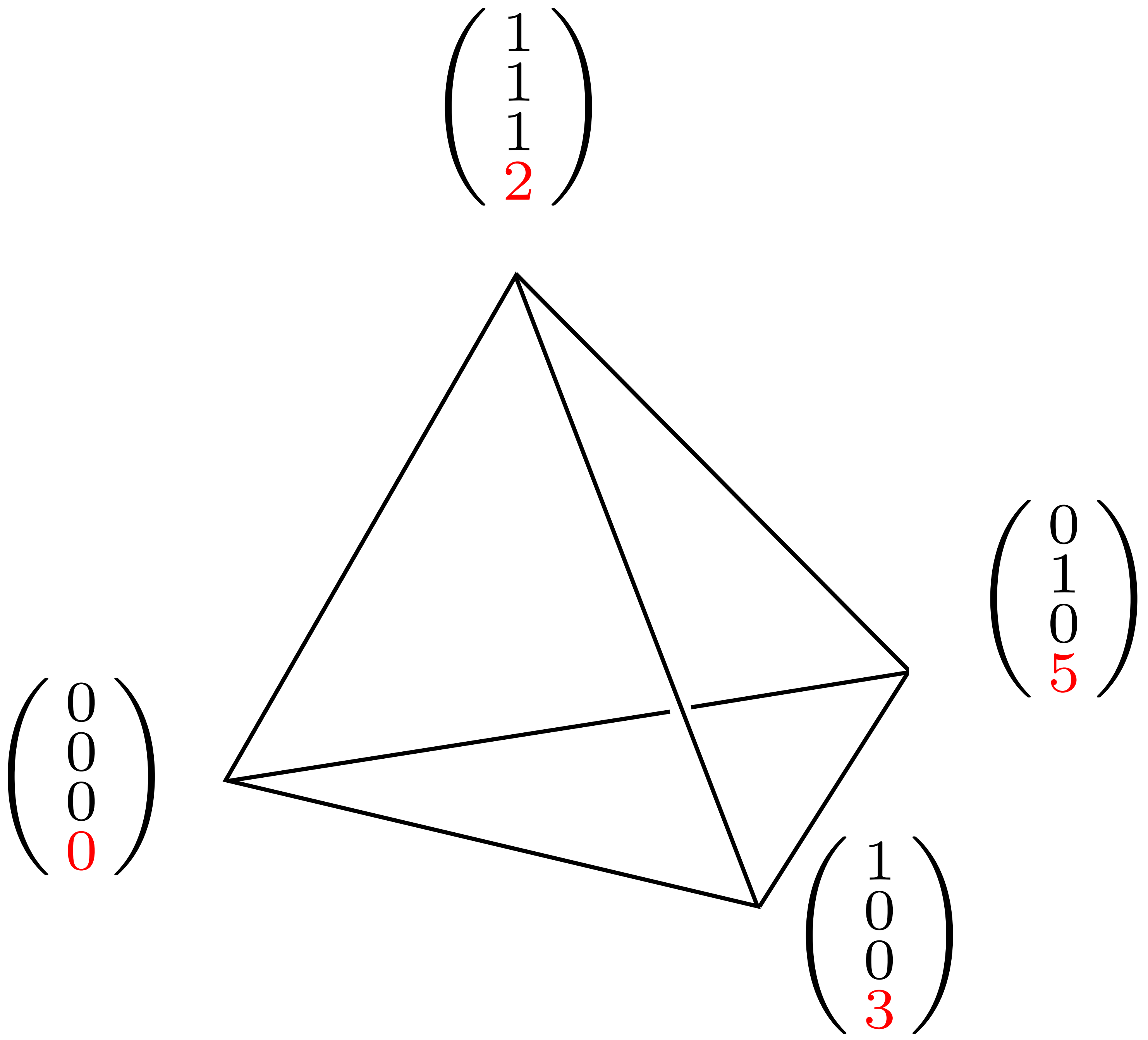
$v^b(a) = \langle w^b, a \rangle$	0	3	5	2



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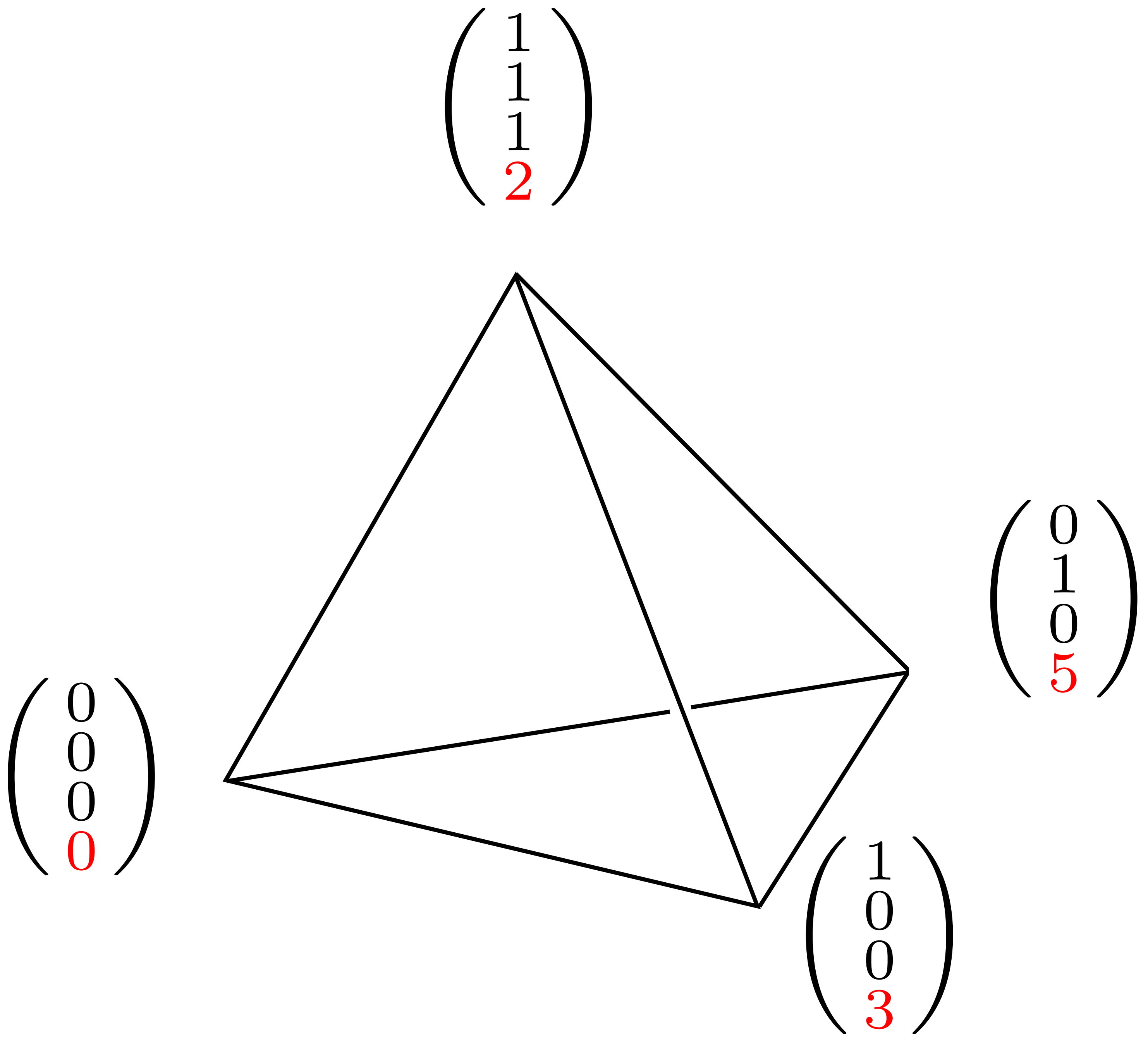

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a	$(0,0,0)^T$	$(1,0,0)^T$	$(0,1,0)^T$	$(1,1,1)^T$
$v^b(a) = \langle w^b, a \rangle$	0	3	5	2
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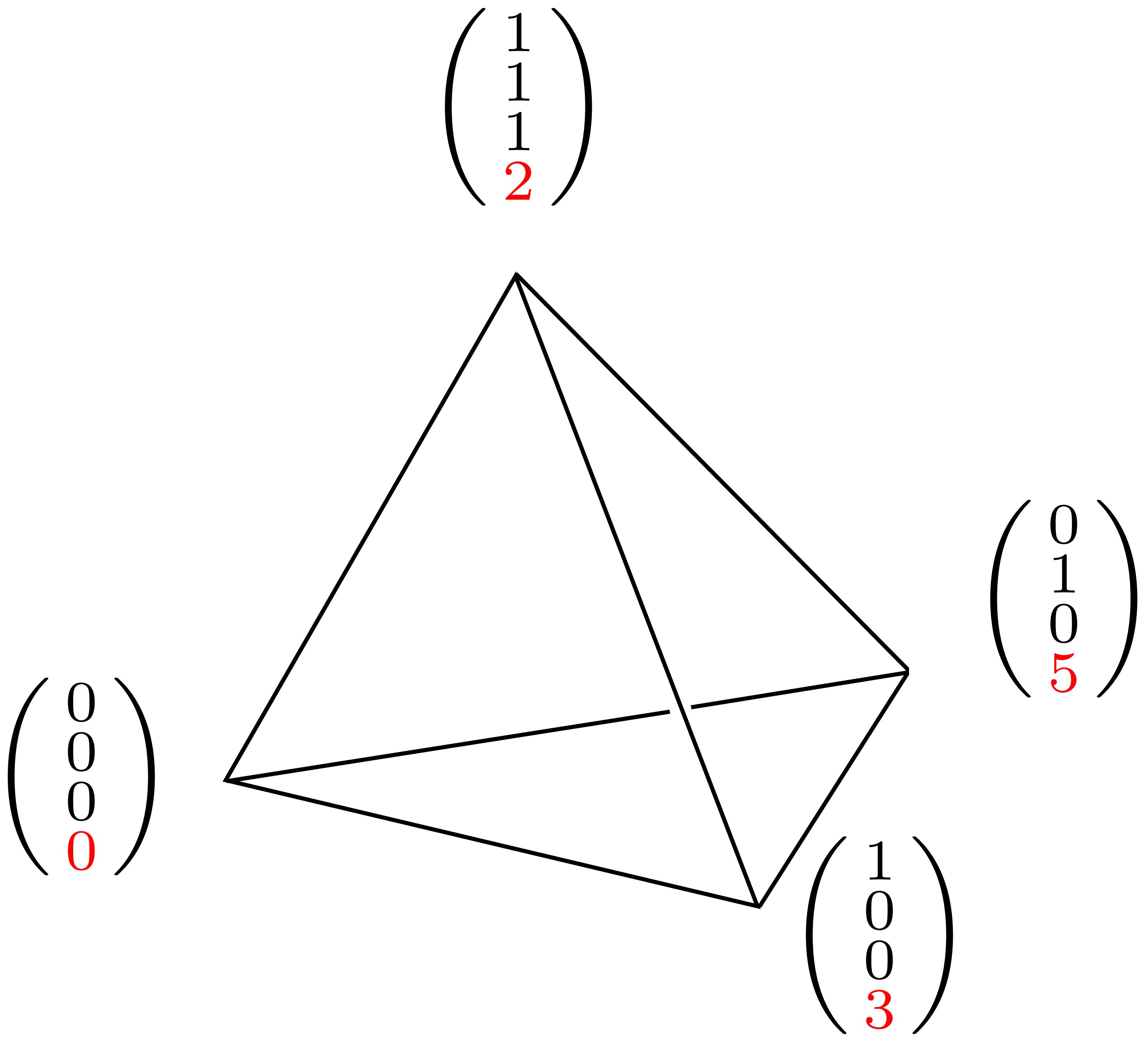

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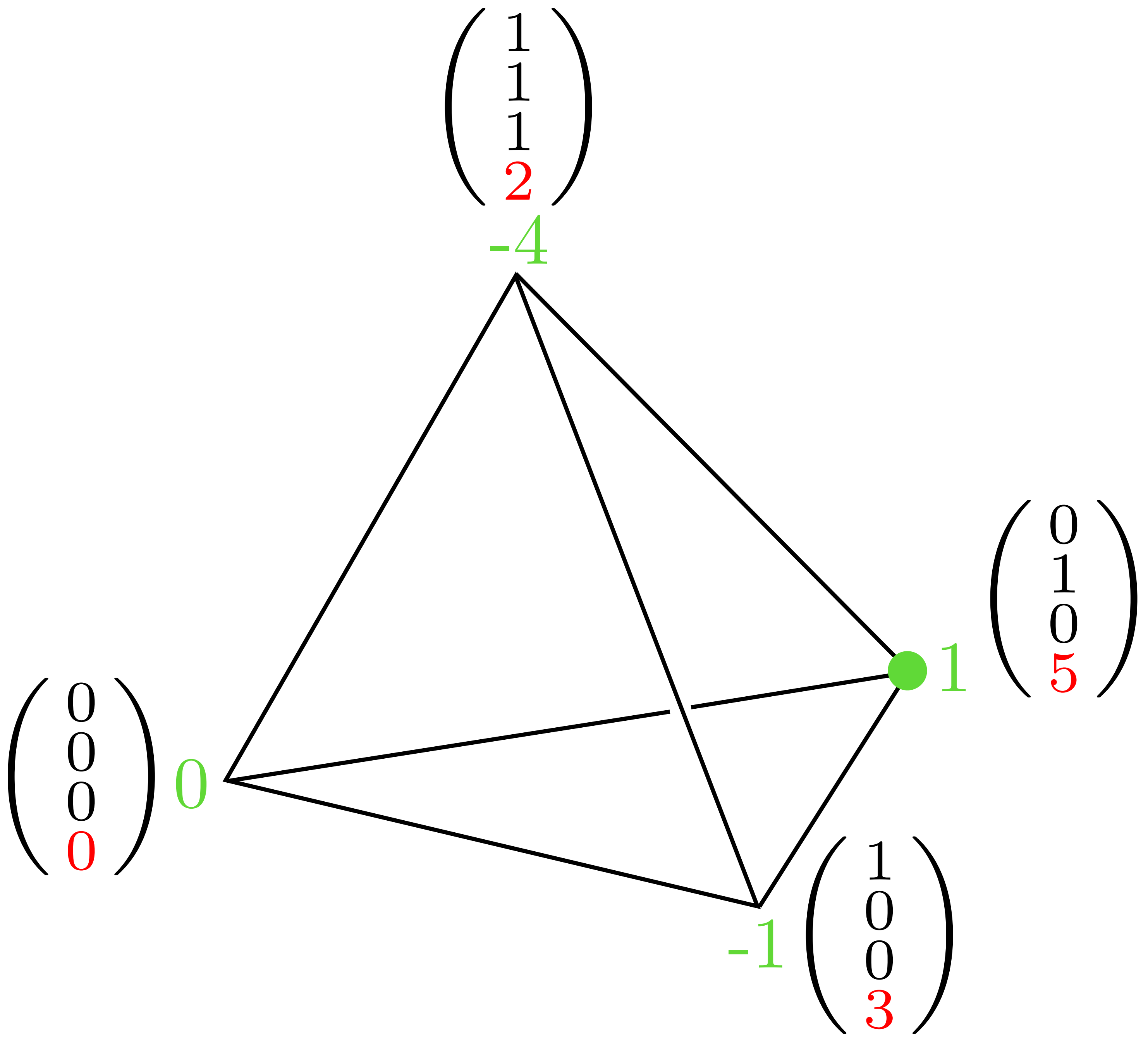

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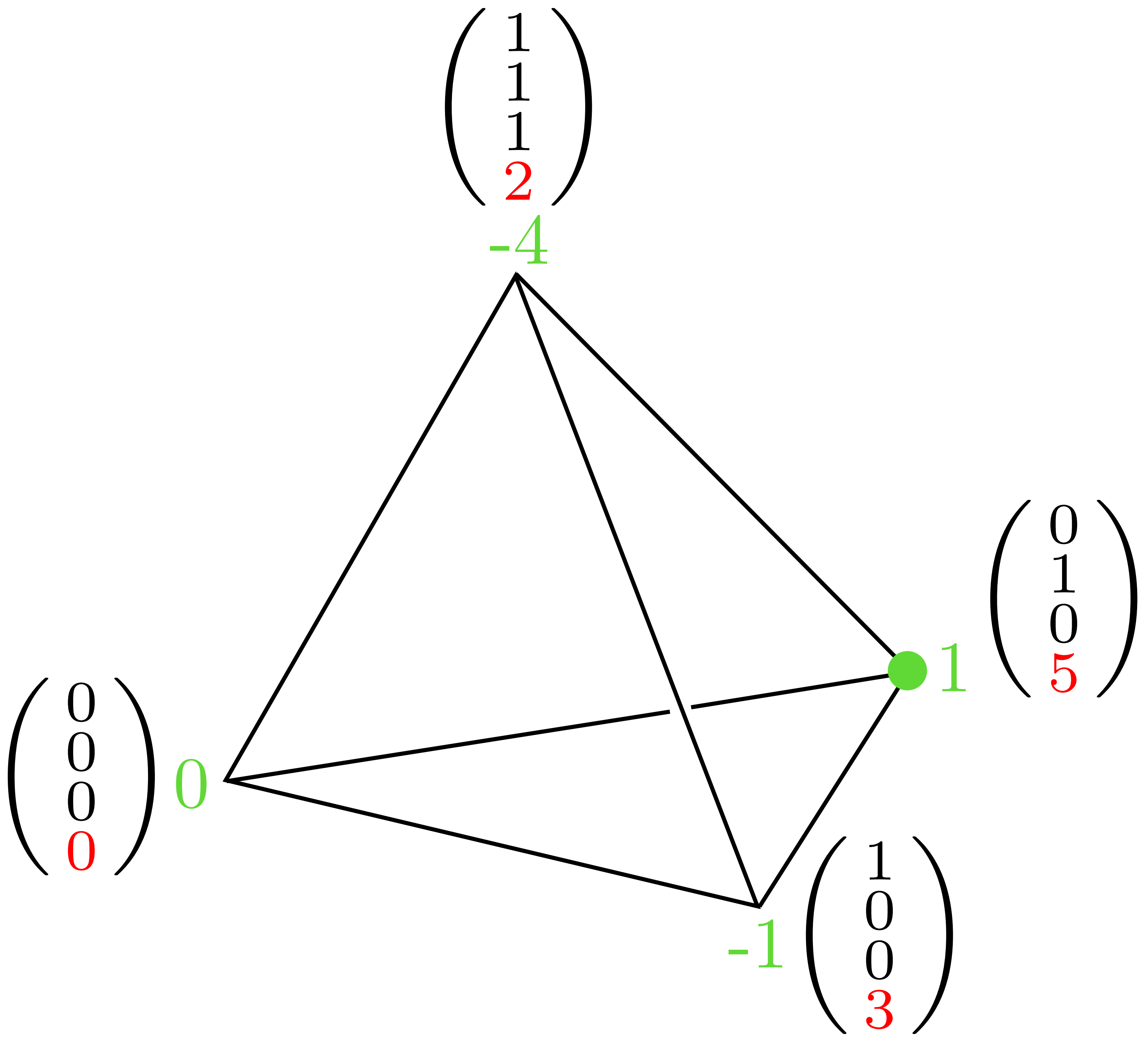

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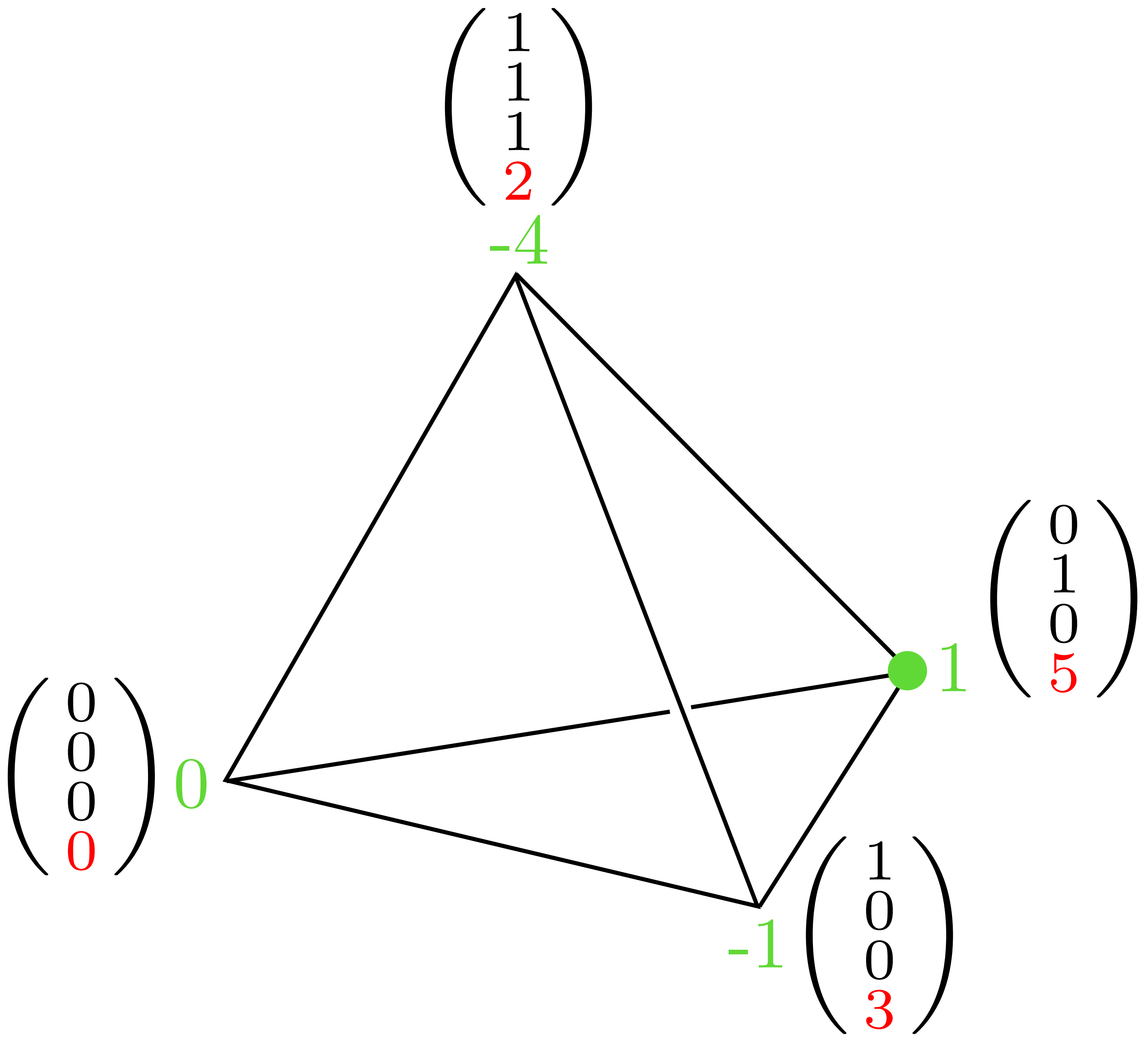

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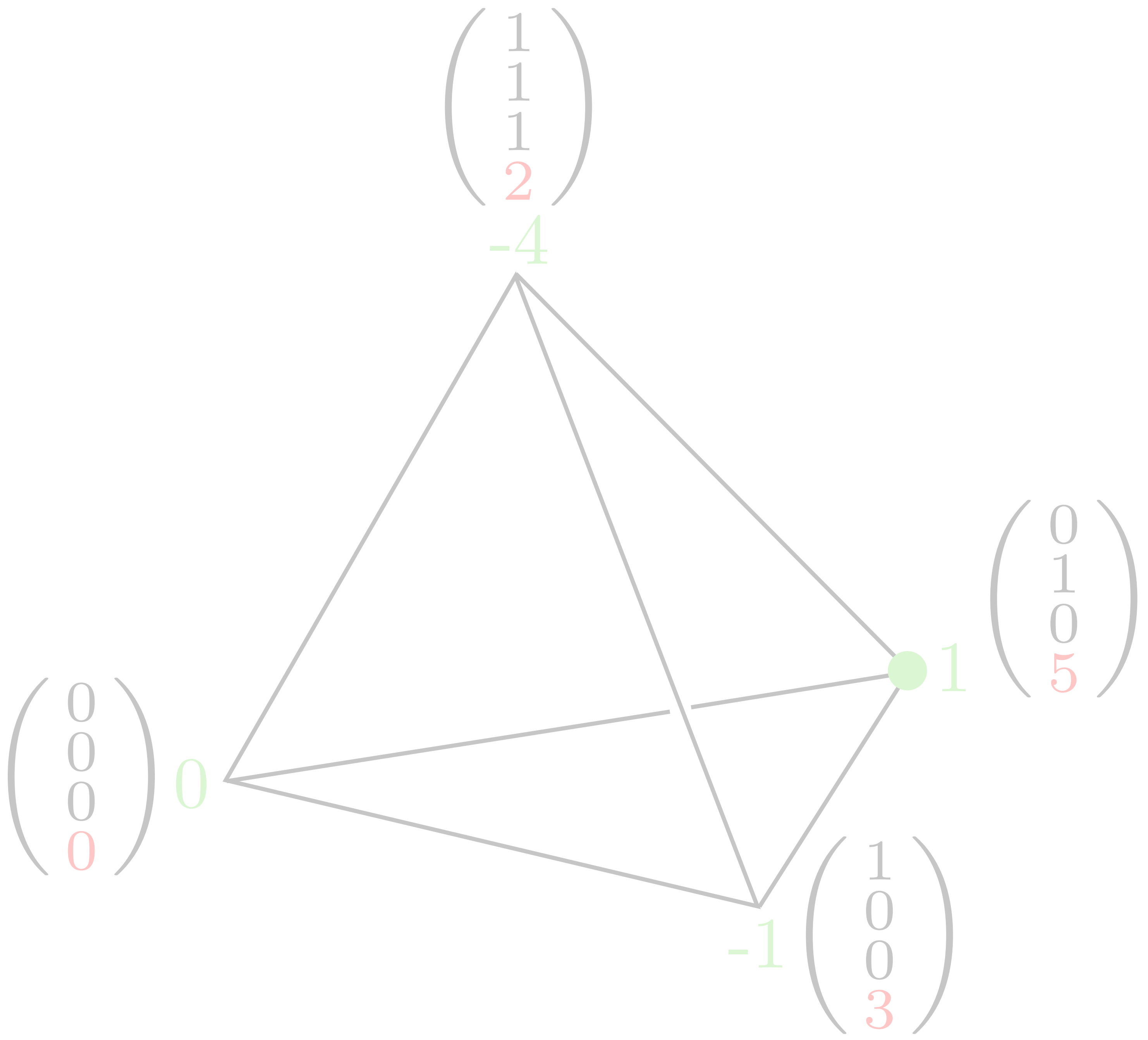

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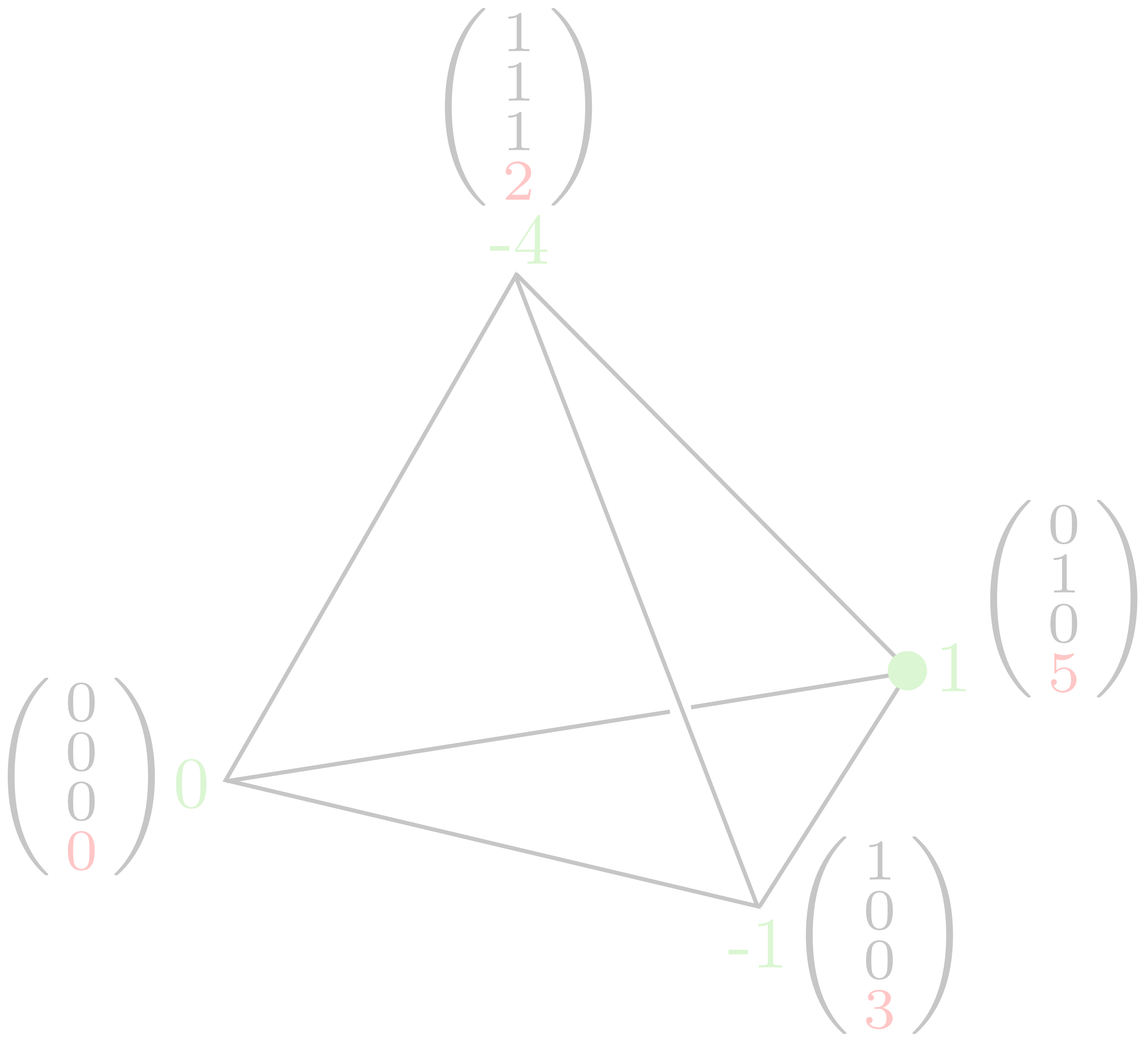

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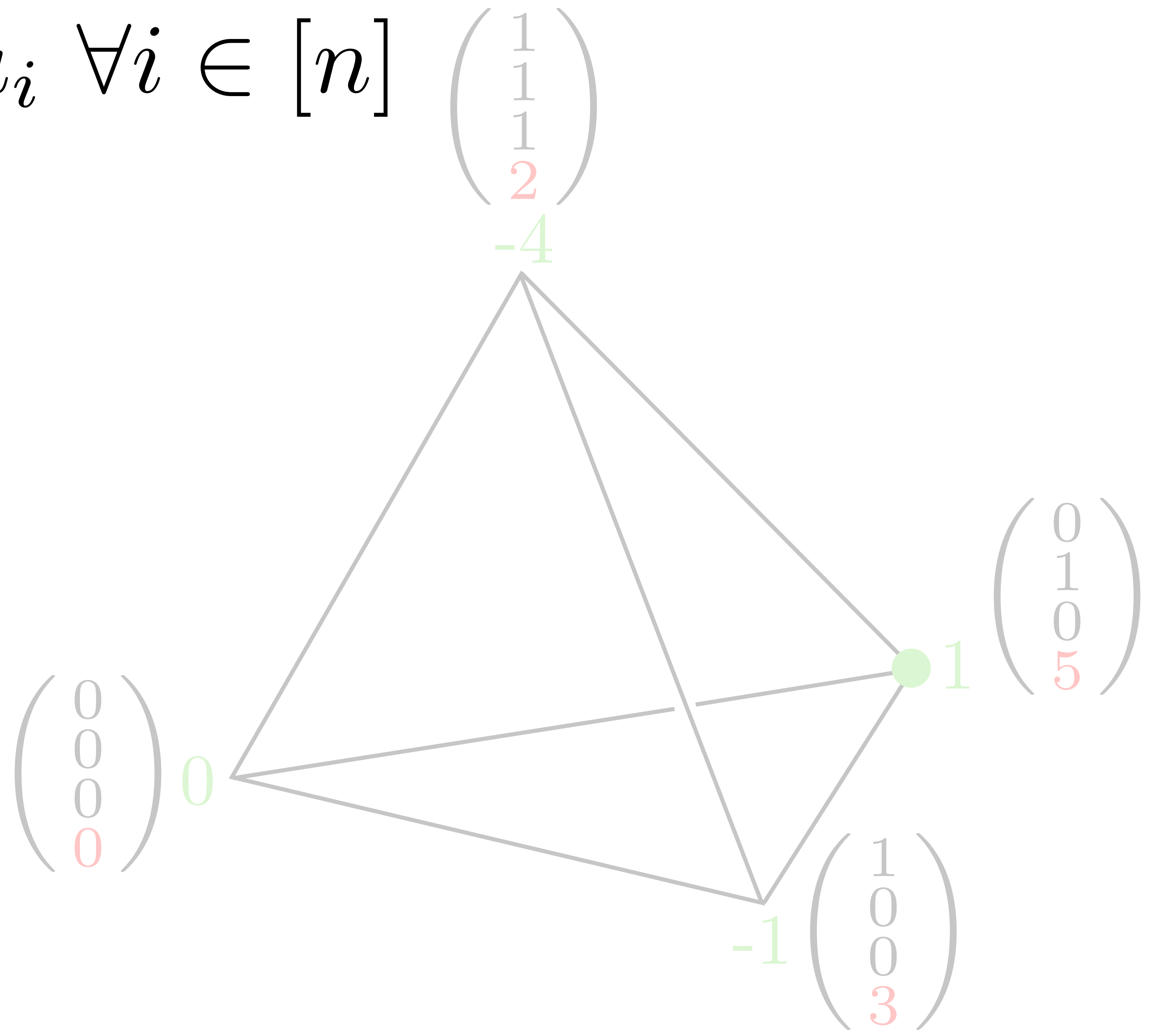

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$$\underbrace{\forall\, b \in [m]\ \exists\, a^b \in D(v^b, p)}_{\text{all bidders are happy}} : \underbrace{a = \sum_{b \in [m]} a^b \text{ and } a_i^* = a_i\ \forall i \in [n]}_{\text{all items are sold}}$$

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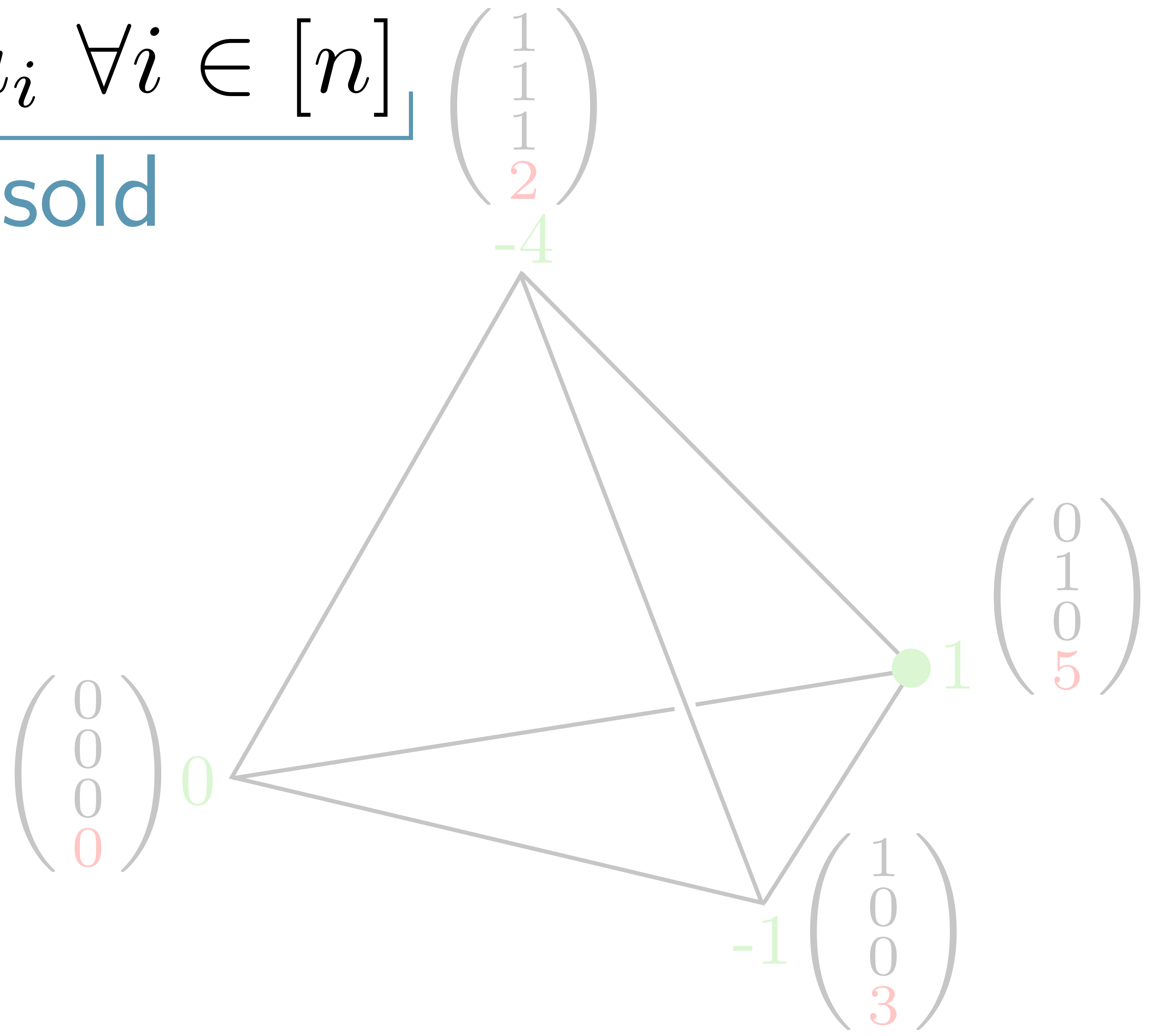

Competitive equilibrium

Definitions

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Definition.

Given valuations $\{v^b \mid b \in [m]\}$, a *competitive equilibrium exists* if there exist $p \in \mathbb{R}^{n+|E|}, a \in \sum_{b\in[m]} D(v^b, p)$ such that $a \in \pi^{-1}(a^*)$ (i.e. $a_i^* = a_i \; \forall i \in [n]$).

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A competitive equilibrium is *guaranteed to exist* if for any set of valuations

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Competitive equilibrium
and lattice polytopes

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b) $\forall \, F^1, \ldots, F^m \preceq P(G) : \; \text{if } a \in \sum_{b\in[m]} F^b \text{ then } a \in \sum_{b\in[m]} \text{vert}(F^b)$

In particular, then a CE is guaranteed to exist.

Results for the complete graph K_n

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Corollary
Let $G = K_n$ be the complete graph. For every auction* with quantities $a^* \in \mathbb{Z}^n_{\geq 0}$ of items, a competitive equilibrium is guaranteed to exists!

*with graphical valuations and graphical pricing on K_n

Other graphs

where CE might not exist

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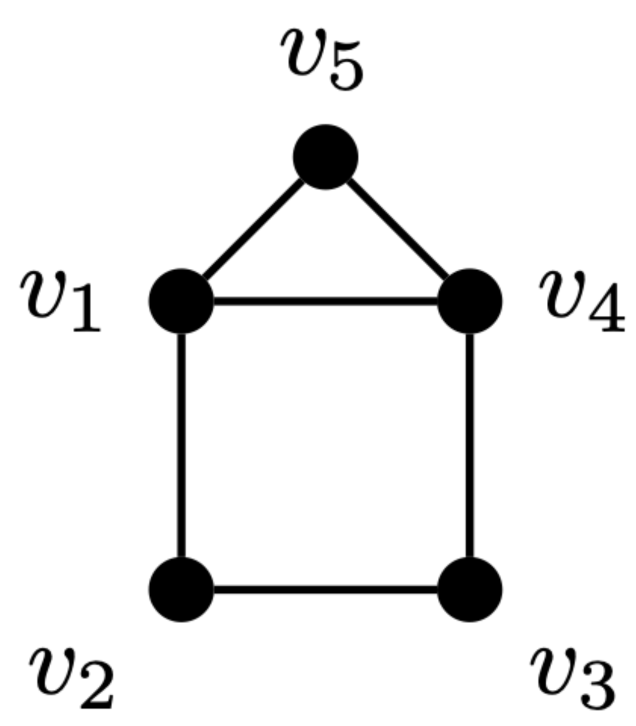
Example.



Other graphs

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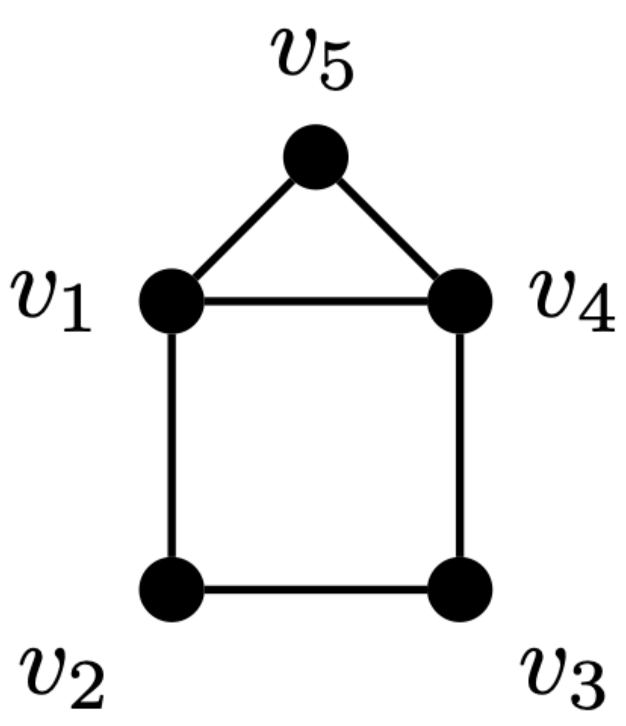


$a^* = (1,1,1,1,1)$. There are edges e_1, e_2, e_3, e_4 of $P(G)$ s.t.

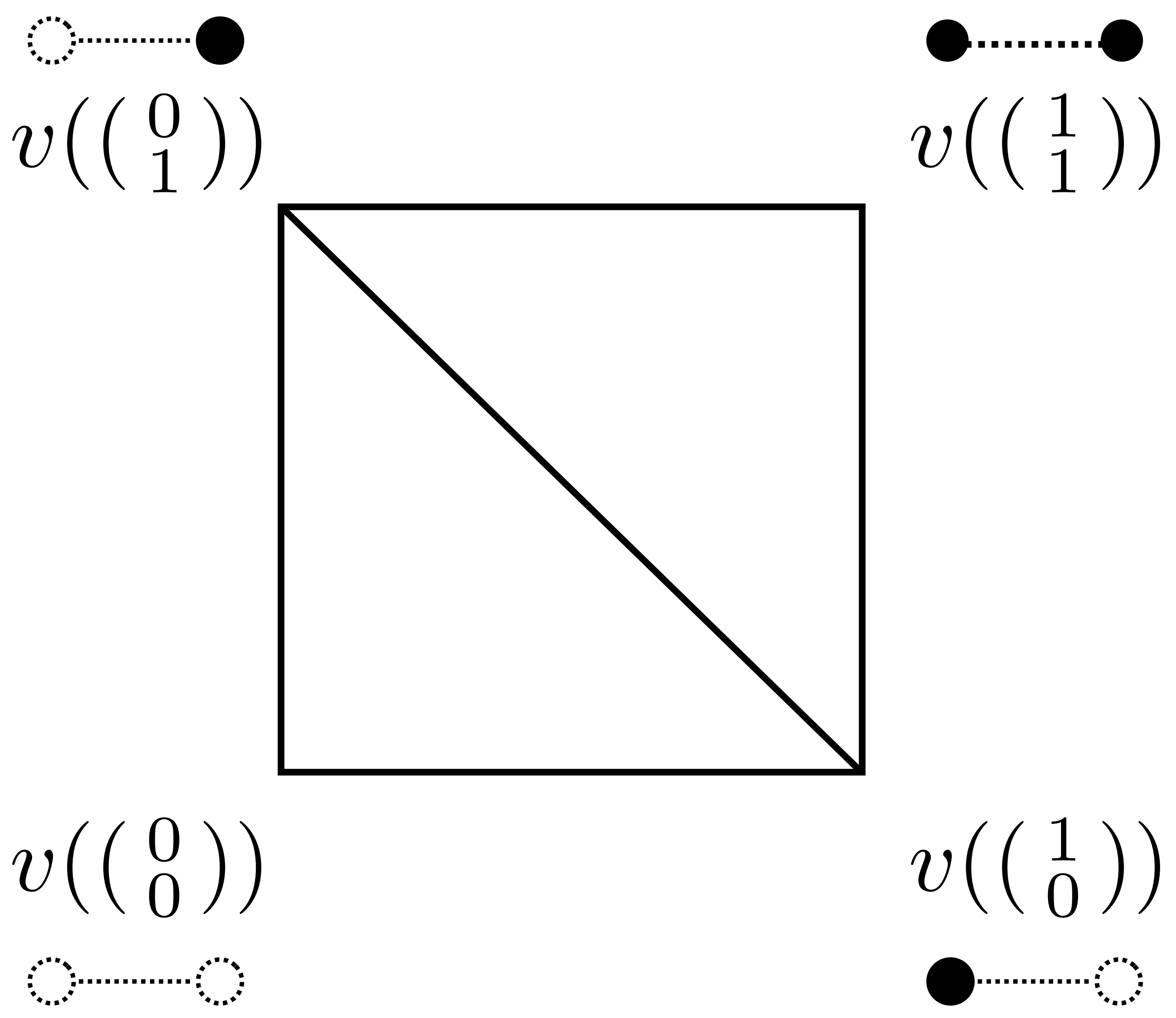
$$\pi^{-1}(a^*) \cap \sum_{i=1}^{4} e_i = \{(1,1,1,1,\ 0,0,0,0,0,0)\}$$

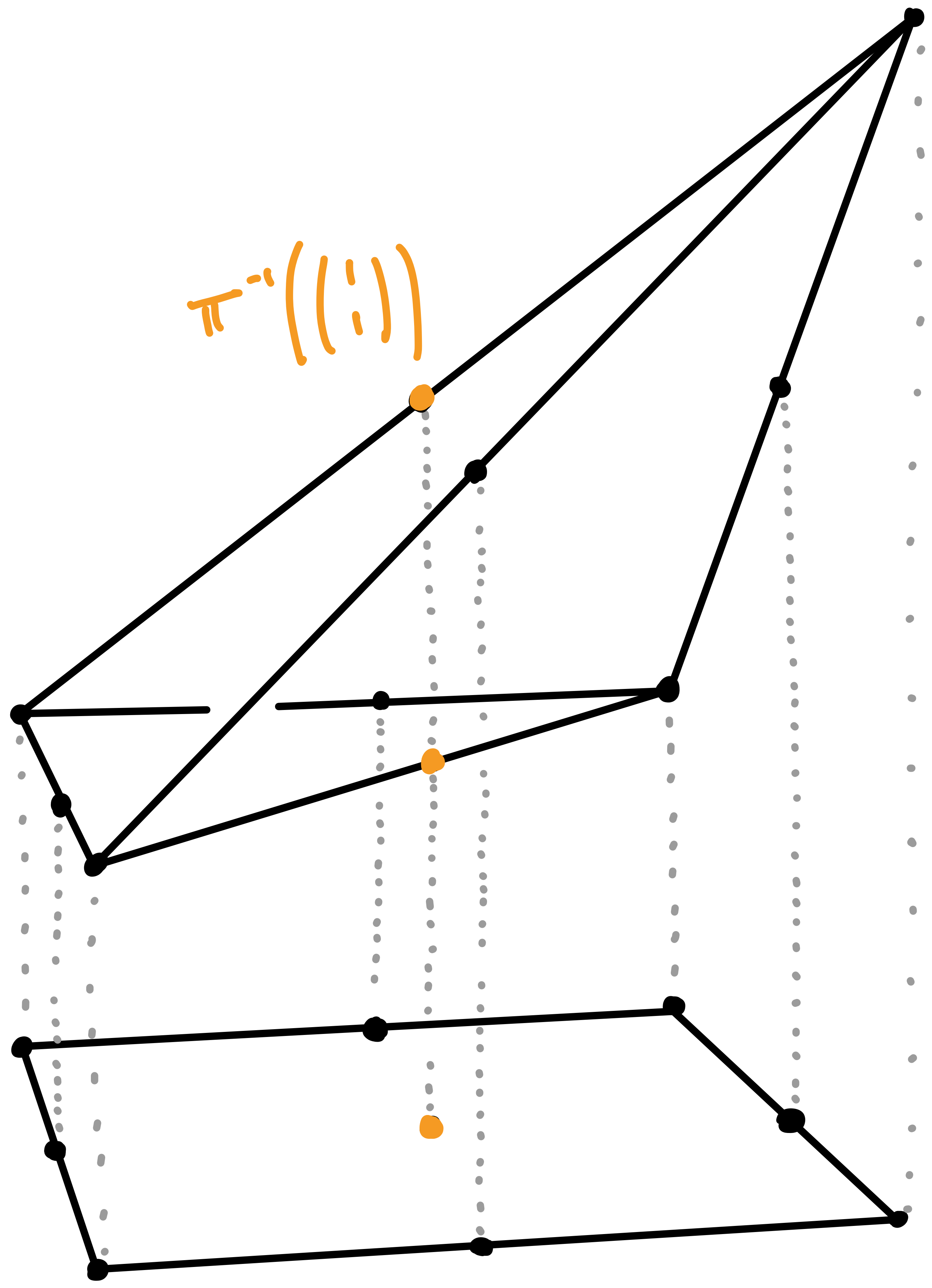
and

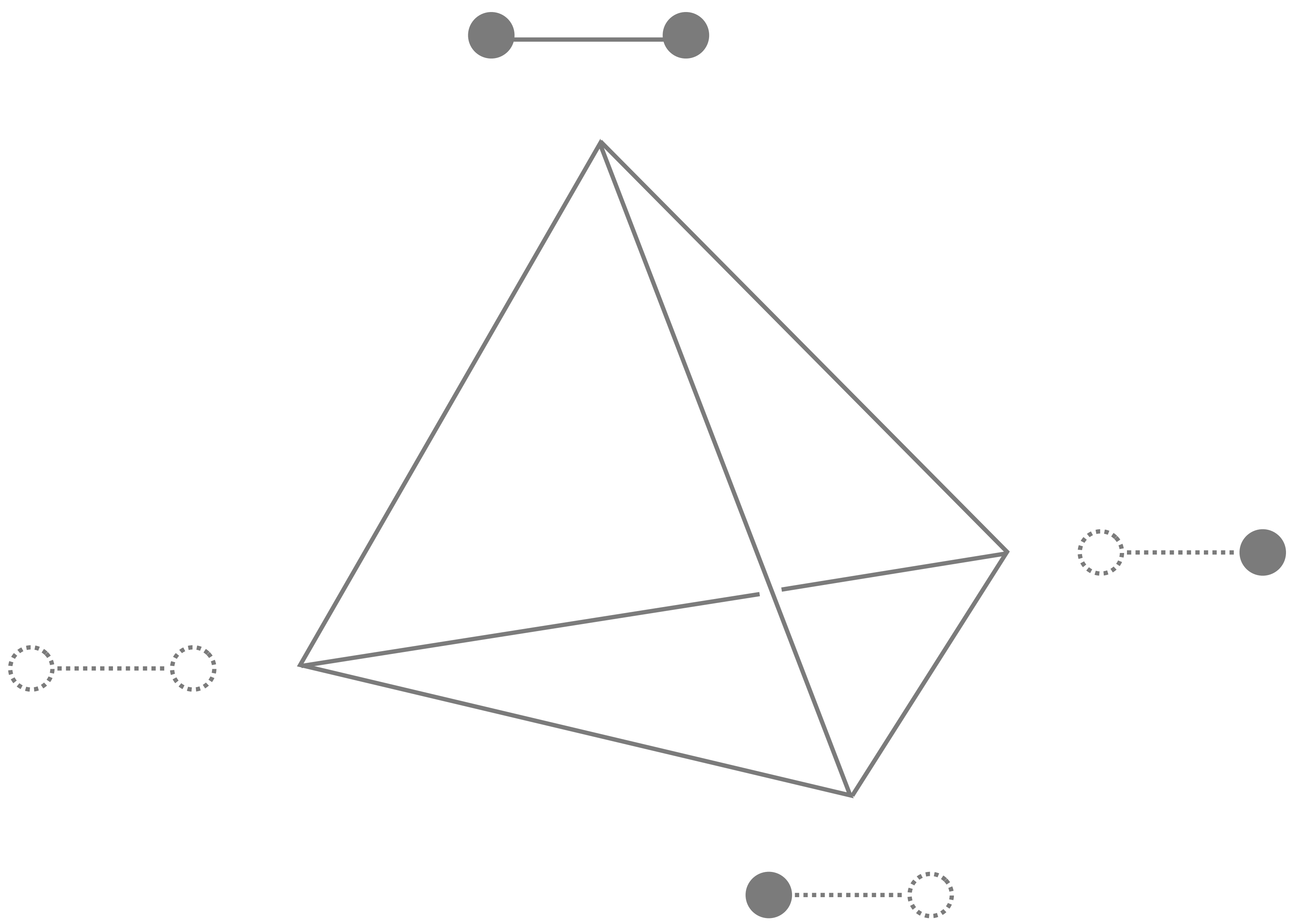
$$\pi^{-1}(a^*) \cap \sum_{i=1}^{4} \mathrm{vert}(e_i) = \emptyset.$$

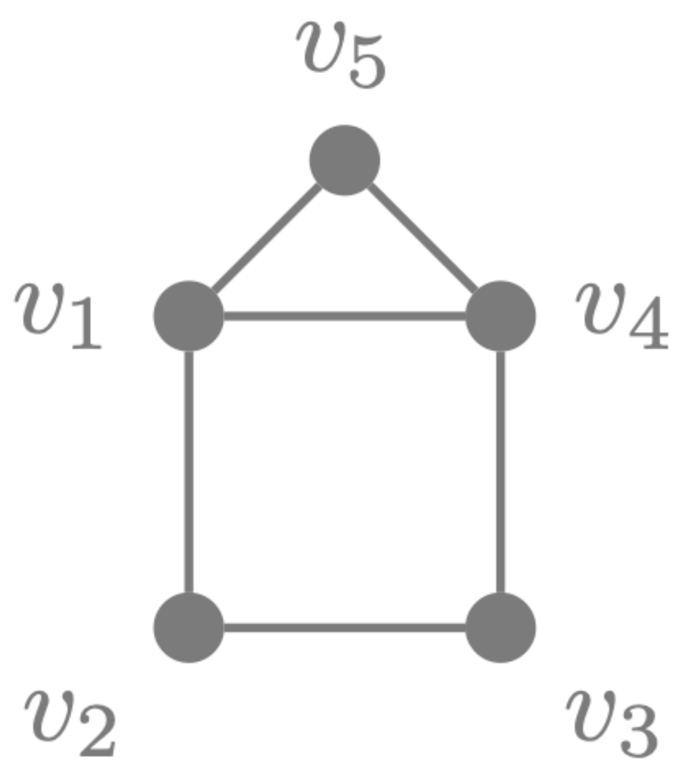
Comparison: classical approach

Non-linear valuations on the cube









Thank you!