Competitive Equilibrium always exists

Discrete Math Days 2022

04 July 2022

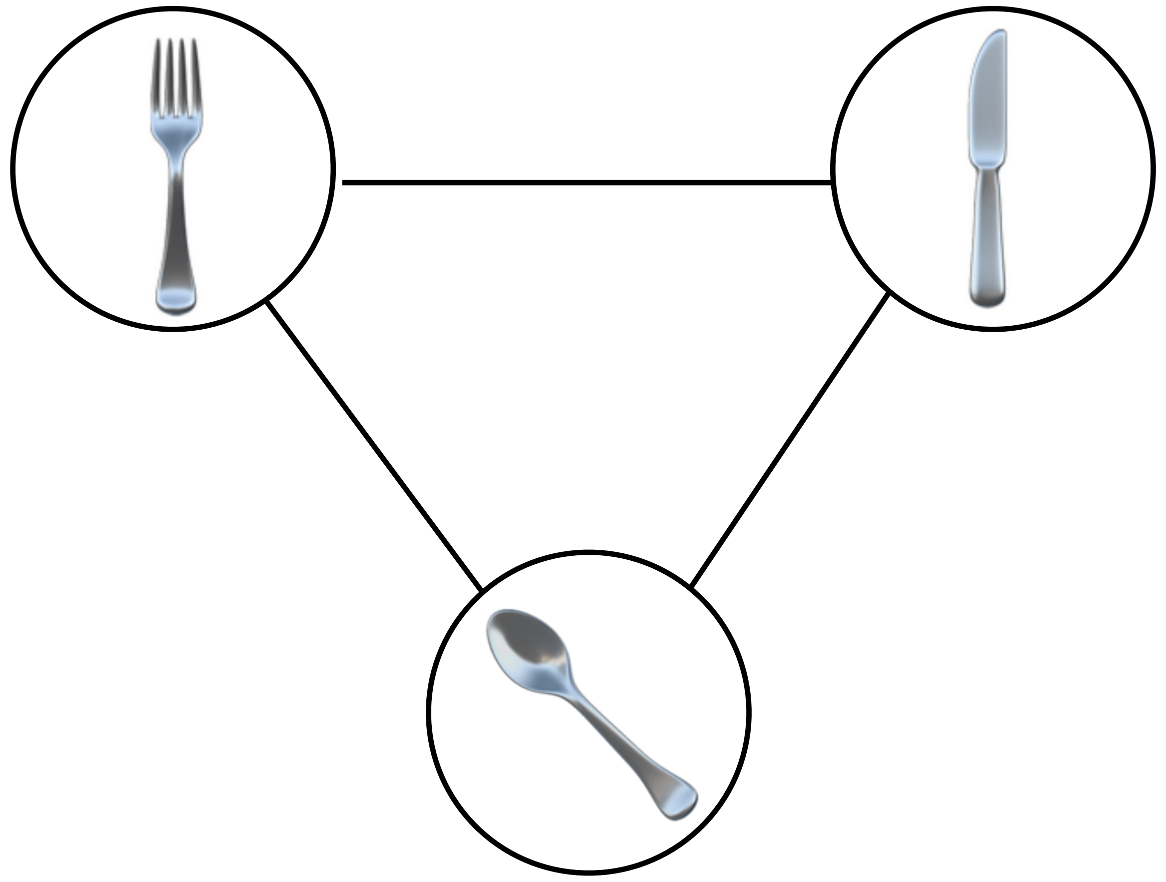
Marie-Charlotte Brandenburg
based on joint work with Christian Haase and Ngoc Mai Tran

Max-Planck-Institut für Mathematik in den Naturwissenschaften

MAX-PLANCK-GESELLSCHAFT

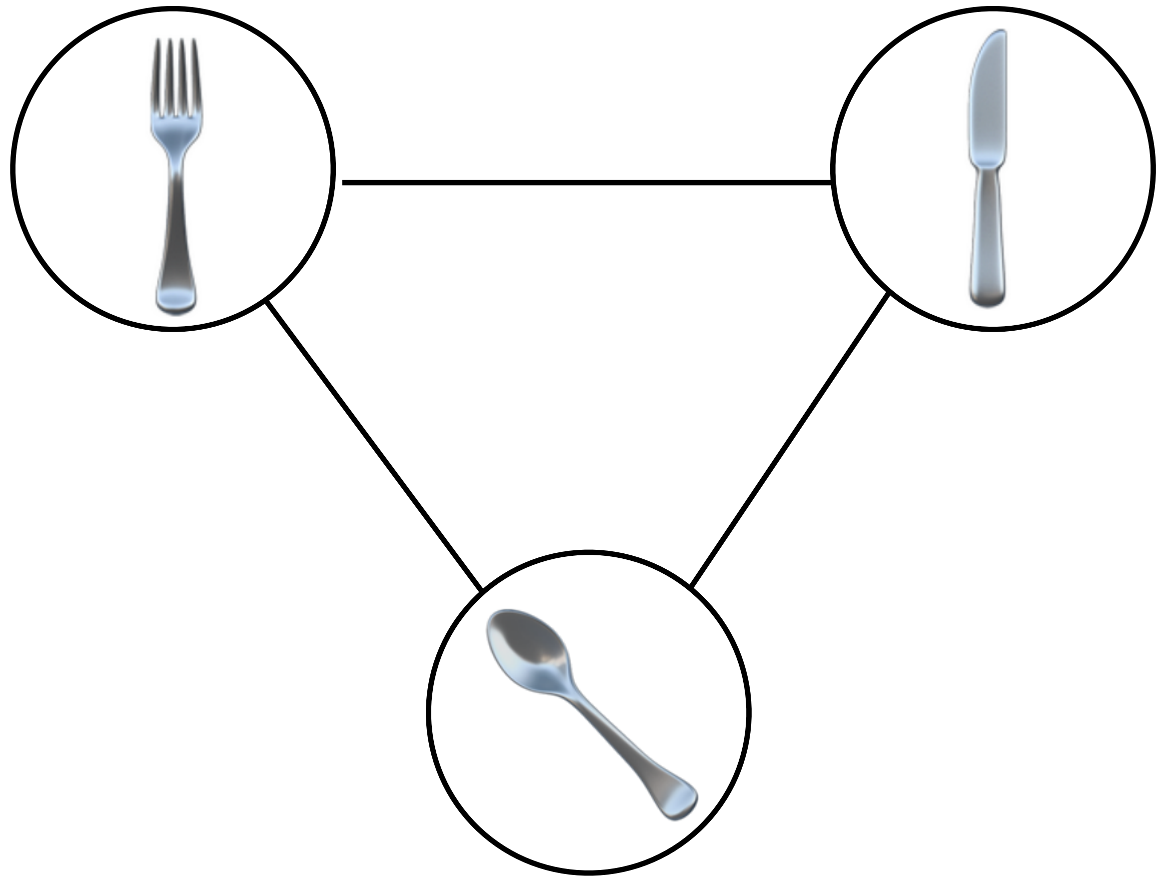
First Example

The cutlery auction at dinner time



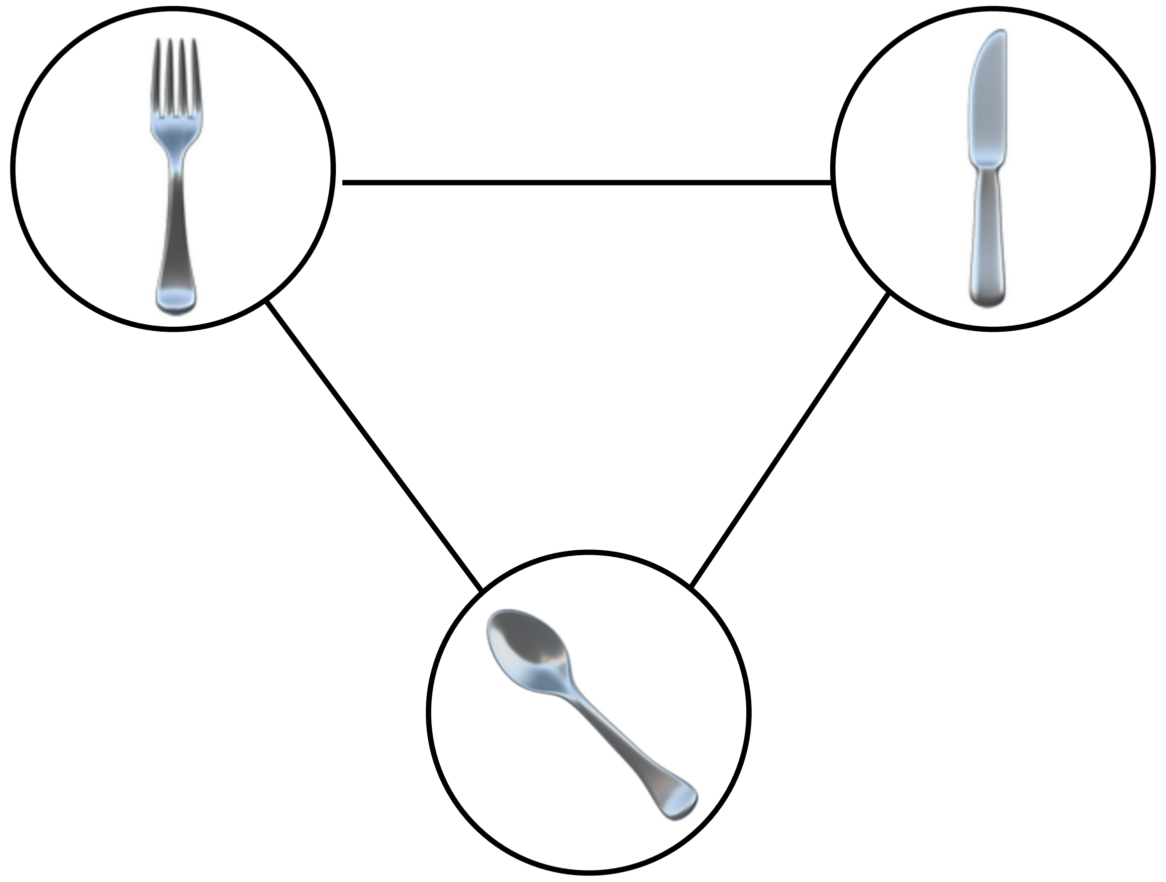
First Example

The cutlery auction at dinner time



First Example

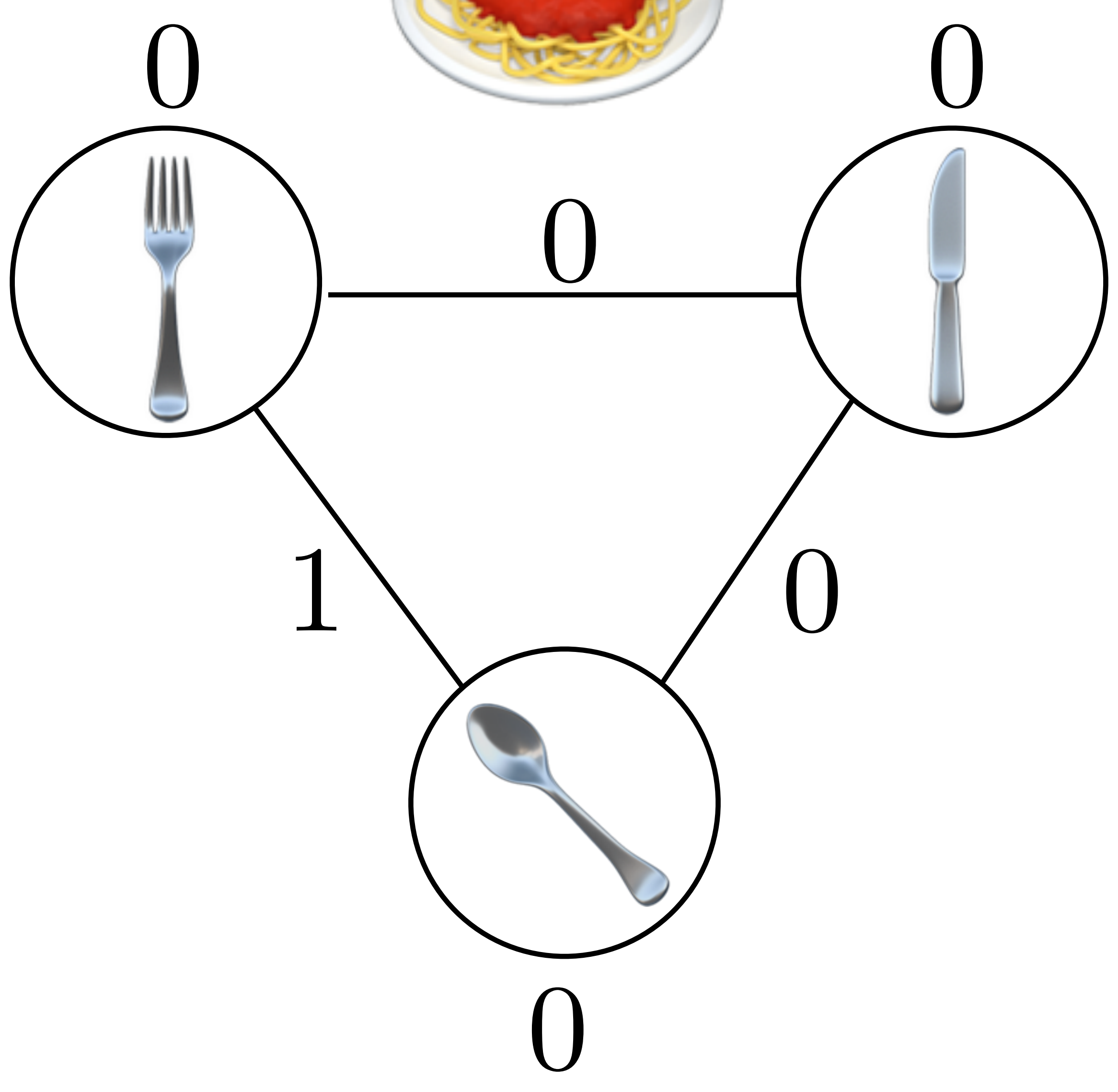
The cutlery auction at dinner time



First Example

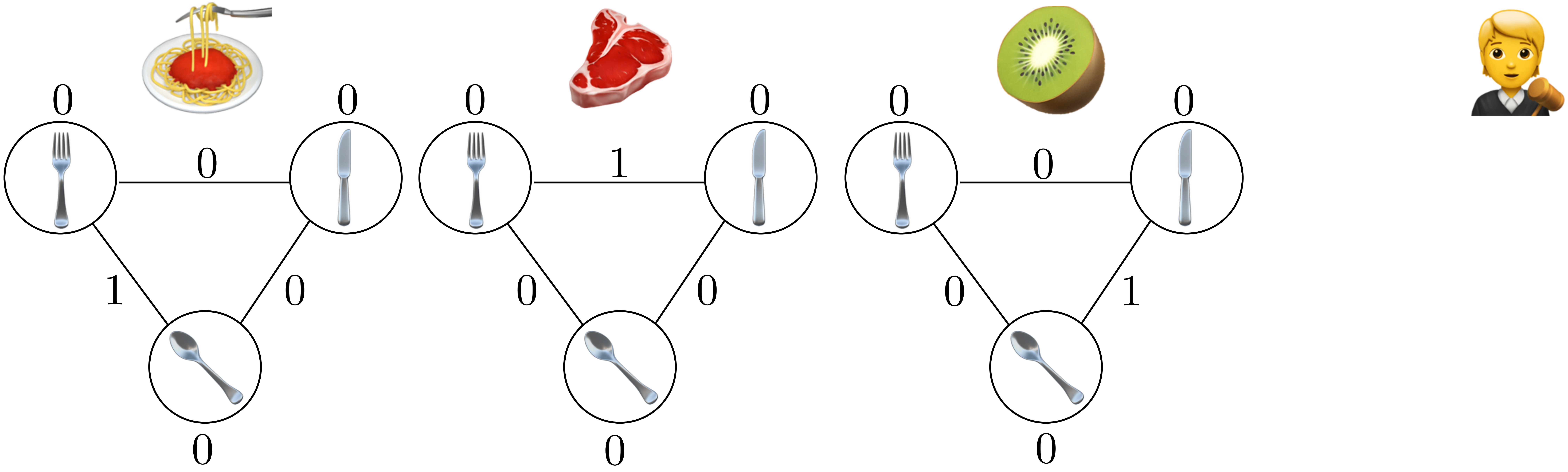
The cutlery auction at dinner time



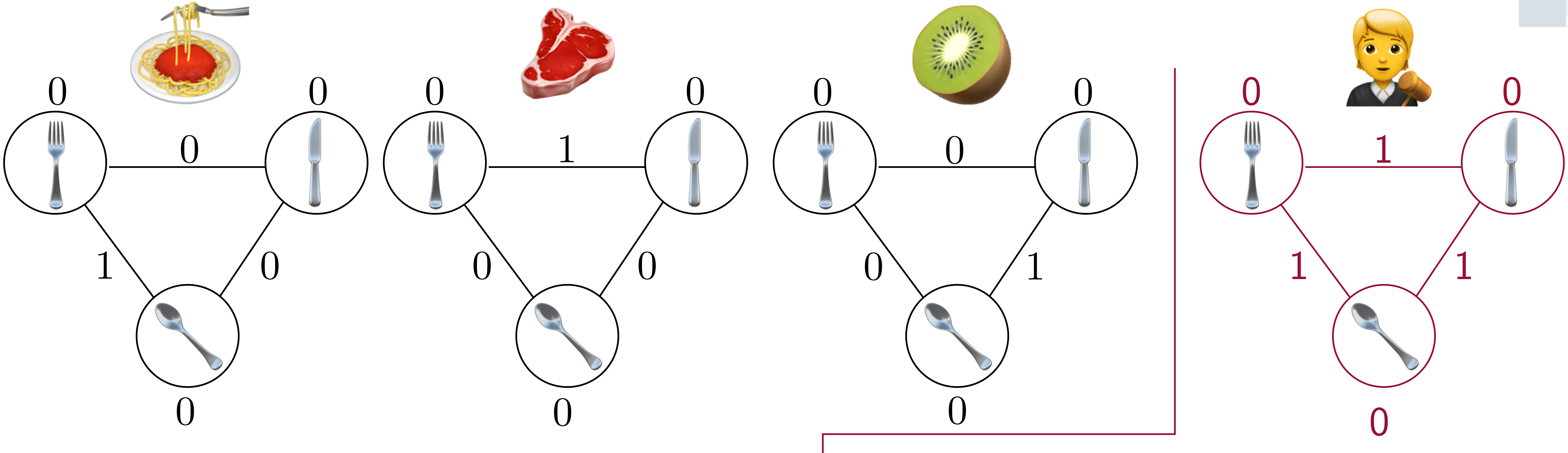
First Example

The cutlery auction at dinner time



First Example

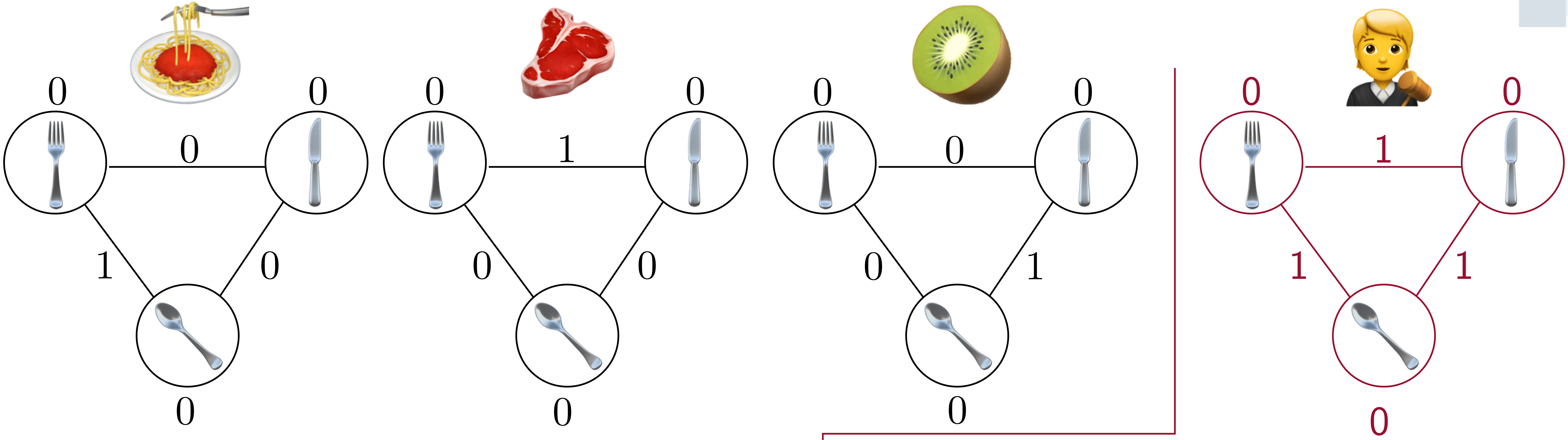
The cutlery auction at dinner time

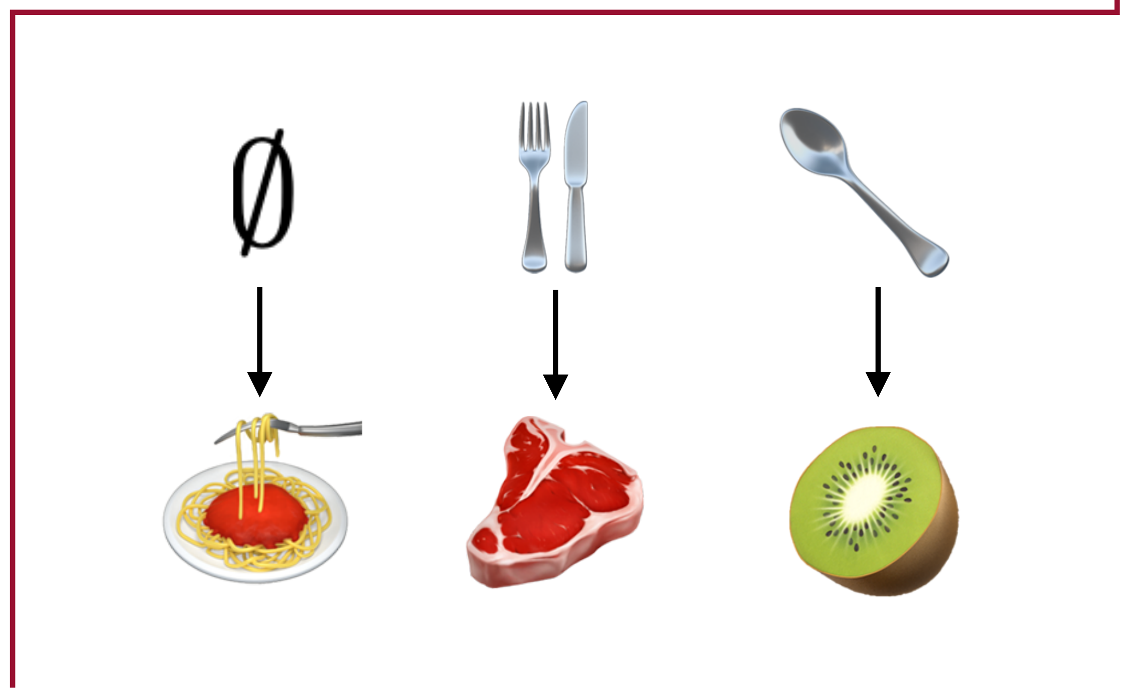


Price for 1 item : 0
Price for 2 items: 1
Price for 3 items: 3

First Example

The cutlery auction at dinner time

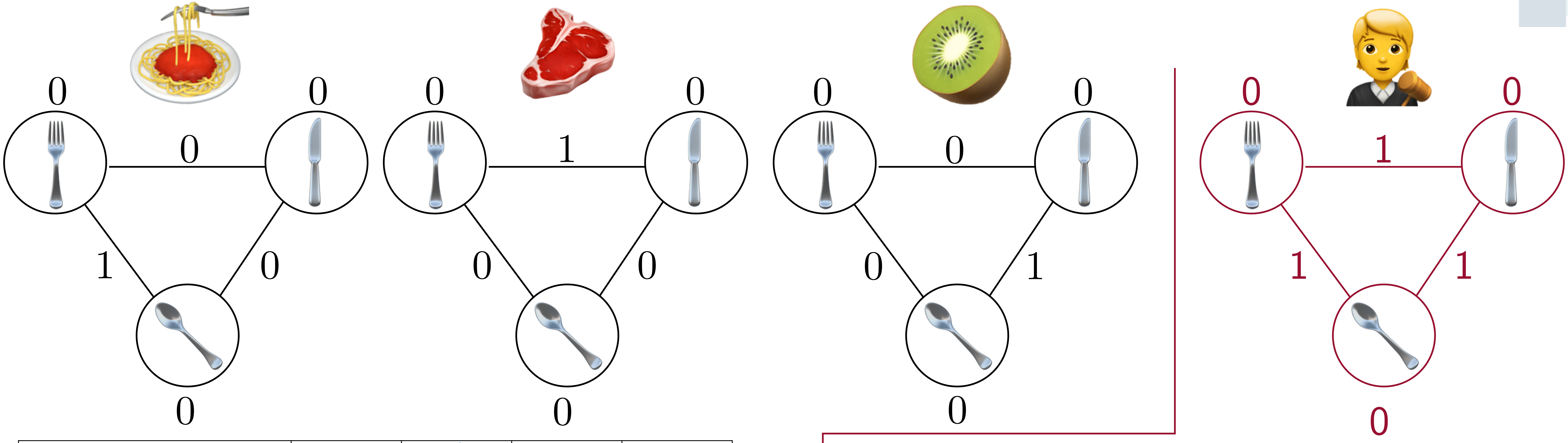




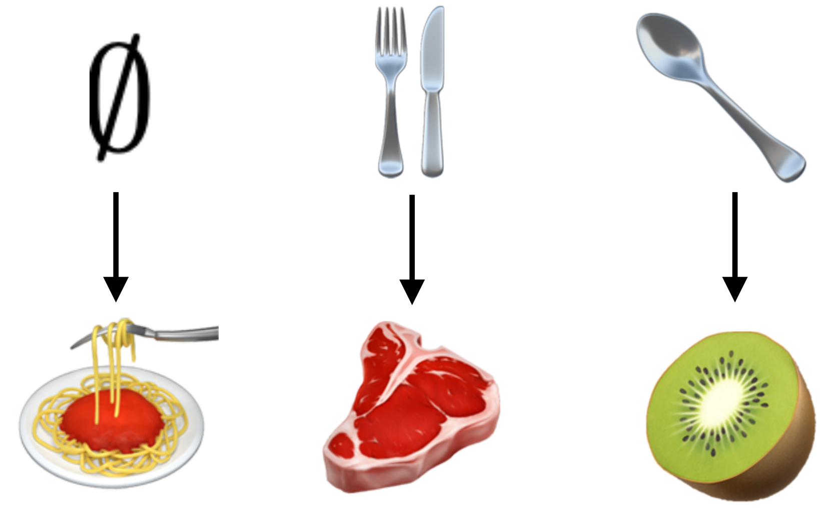
Price for 1 item : 0
Price for 2 items: 1
Price for 3 items: 3

First Example

The cutlery auction at dinner time



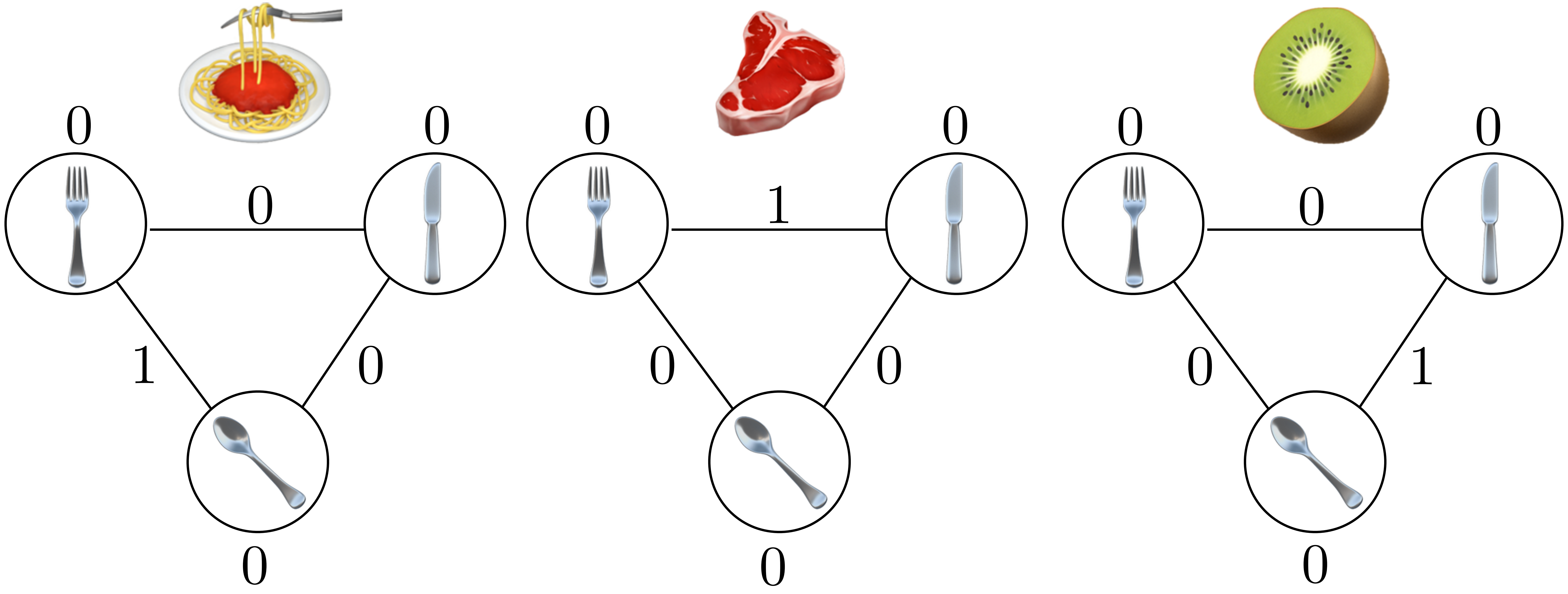
Opinion of 	∅			
Willing to pay	0	0	1	1
Price charged	0	0	1	3
Profit	0	0	0	-2

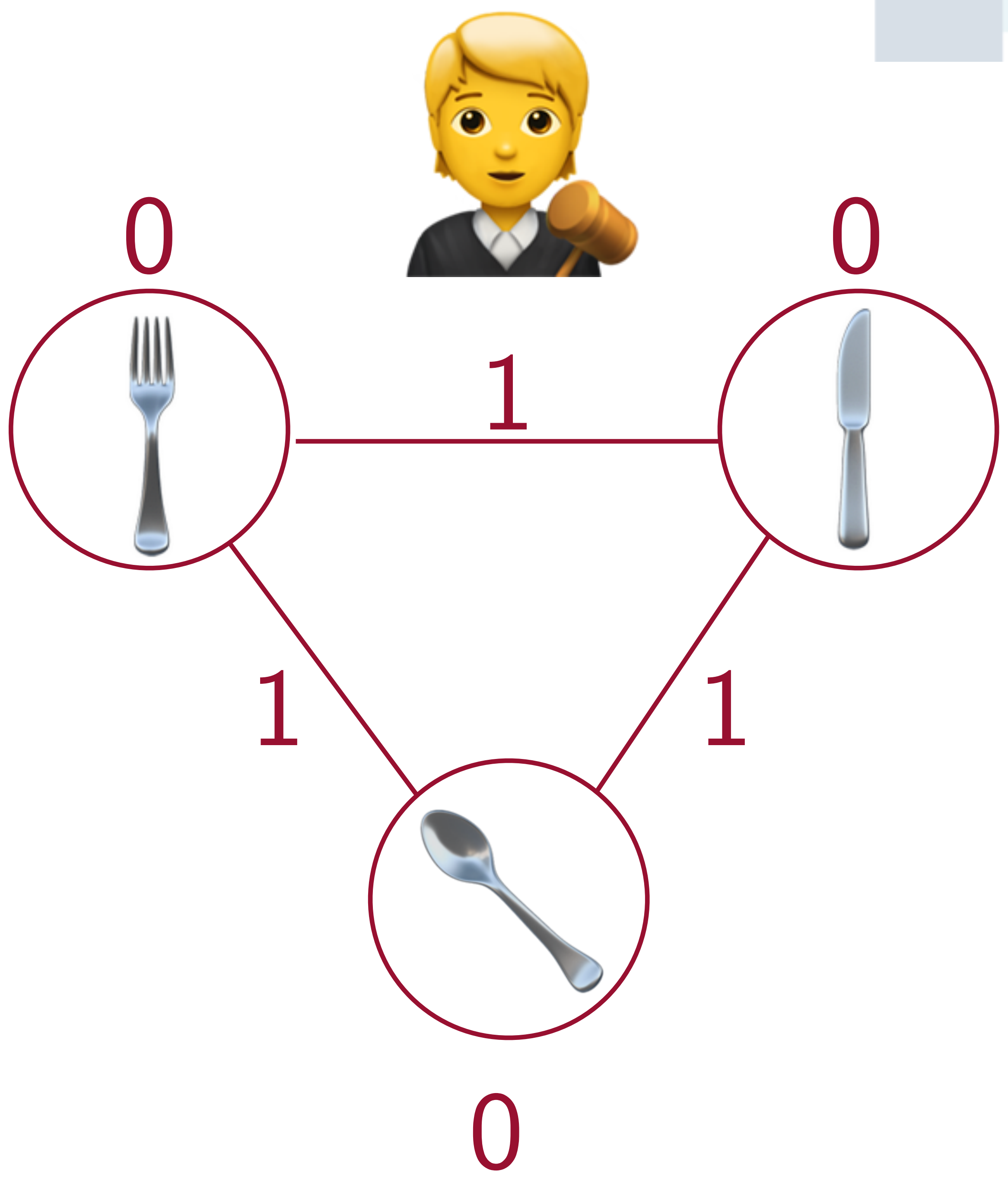


Price for 1 item : 0
Price for 2 items: 1
Price for 3 items: 3

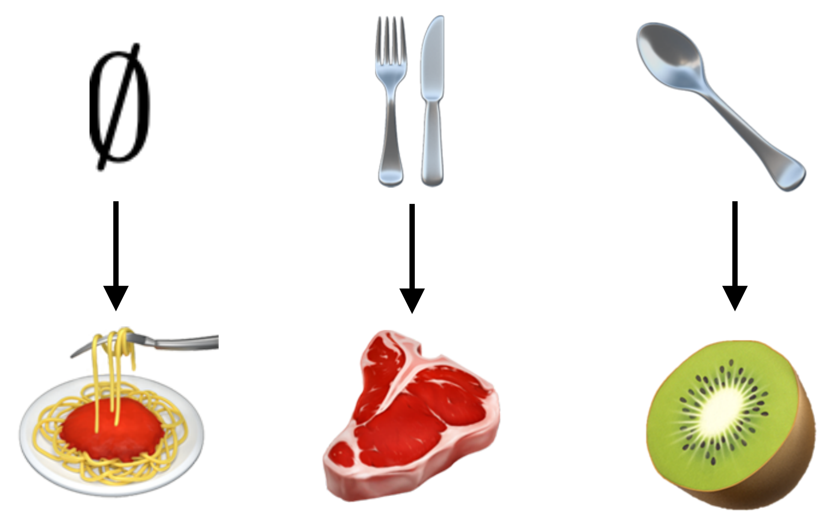
First Example

The cutlery auction at dinner time





Opinion of 🥩	∅	🔪	🍴🔪	🍴🔪🥄
Willing to pay	0	0	1	1
Price charged	0	0	1	3
Profit	0	0	0	-2



Price for 1 item : 0
Price for 2 items: 1
Price for 3 items: 3

The graphical model and its polytope

[Candogan-Ozdaglar-Parillo ’18]

The graphical model and its polytope

[Candogan-Ozdaglar-Parillo ’18]

$n = \#$ types of goods, $a_i^* = \#$ items of type i, $a^* \in \mathbb{Z}_{\geq 0}^n$

The graphical model and its polytope

[Candogan-Ozdaglar-Parillo '18]

$n = \#$ types of goods, $a_i^* = \#$ items of type i, $a^* \in \mathbb{Z}^n_{\geq 0}$

General assumptions:

1. Each bidder wants to buy ≤ 1 item per type.
2. Auctioneer wants to sell everything.

The graphical model and its polytope

[Candogan-Ozdaglar-Parillo ’18]

$n = \#$ types of goods, $a_i^* = \#$ items of type i, $a^* \in \mathbb{Z}_{\geq 0}^n$

General assumptions:

1. Each bidder wants to buy ≤ 1 item per type.
2. Auctioneer wants to sell everything.

$G = ([n], E)$ graph,

The graphical model and its polytope

[Candogan-Ozdaglar-Parillo '18]

$n = \#$ types of goods, $a_i^* = \#$ items of type i, $a^* \in \mathbb{Z}^n_{\geq 0}$

General assumptions:

1. Each bidder wants to buy ≤ 1 item per type.
2. Auctioneer wants to sell everything.

$G = ([n], E)$ graph, $G' \subseteq G$ induced subgraph. Define $\chi_{G'} \in \{0,1\}^{n+|E|}$ as

$$(\chi_{G'})_i = \begin{cases} 1 & \text{if } i \in V(G') \\ 0 & \text{if } i \notin V(G') \end{cases} \qquad (\chi_{G'})_{ij} = \begin{cases} 1 & \text{if } ij \in E(G') \\ 0 & \text{if } ij \notin E(G') \end{cases}$$

The graphical model and its polytope

[Candogan-Ozdaglar-Parillo '18]

$n = \#$ types of goods, $a_i^* = \#$ items of type i, $a^* \in \mathbb{Z}^n_{\geq 0}$

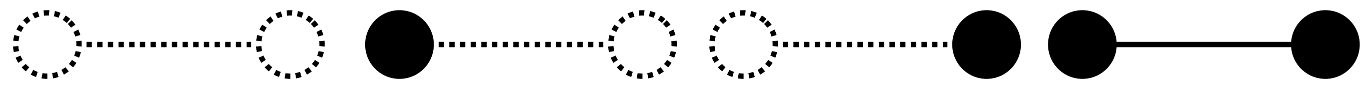
General assumptions:

1. Each bidder wants to buy ≤ 1 item per type.
2. Auctioneer wants to sell everything.

$G = ([n], E)$ graph, $G' \subseteq G$ induced subgraph. Define $\chi_{G'} \in \{0,1\}^{n+|E|}$ as

$$(\chi_{G'})_i = \begin{cases} 1 & \text{if } i \in V(G') \\ 0 & \text{if } i \notin V(G') \end{cases} \qquad (\chi_{G'})_{ij} = \begin{cases} 1 & \text{if } ij \in E(G') \\ 0 & \text{if } ij \notin E(G') \end{cases}$$

$$\begin{matrix} 1 \\ 2 \\ 12 \end{matrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$



The graphical model and its polytope

[Candogan-Ozdaglar-Parillo '18]

$n = \#$ types of goods, $a_i^* = \#$ items of type i, $a^* \in \mathbb{Z}^n_{\geq 0}$

General assumptions:

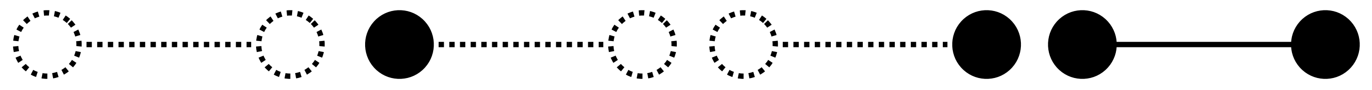
1. Each bidder wants to buy ≤ 1 item per type.
2. Auctioneer wants to sell everything.

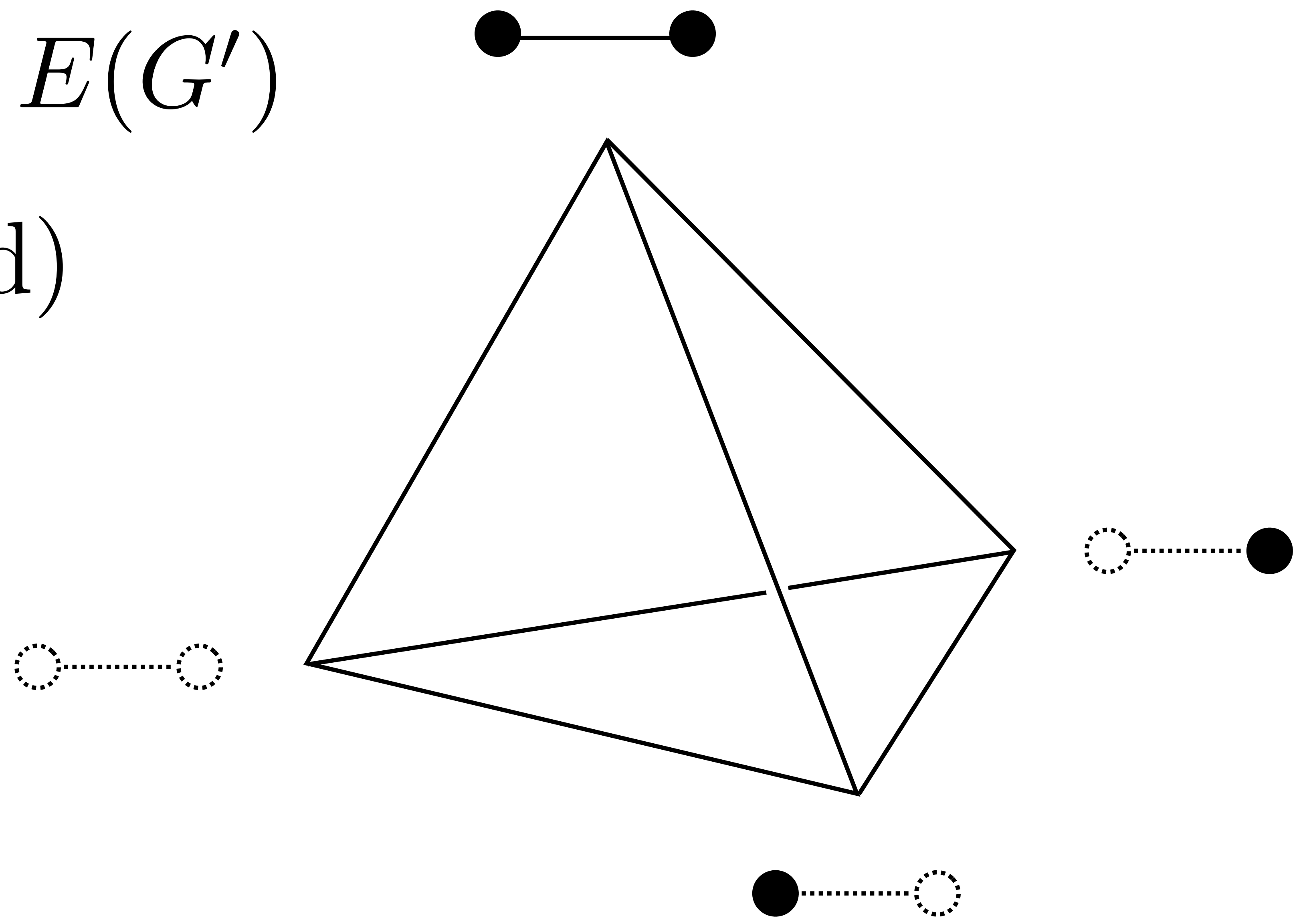
$G = ([n], E)$ graph, $G' \subseteq G$ induced subgraph. Define $\chi_{G'} \in \{0,1\}^{n+|E|}$ as

$$(\chi_{G'})_i = \begin{cases} 1 & \text{if } i \in V(G') \\ 0 & \text{if } i \notin V(G') \end{cases} \qquad (\chi_{G'})_{ij} = \begin{cases} 1 & \text{if } ij \in E(G') \\ 0 & \text{if } ij \notin E(G') \end{cases}$$

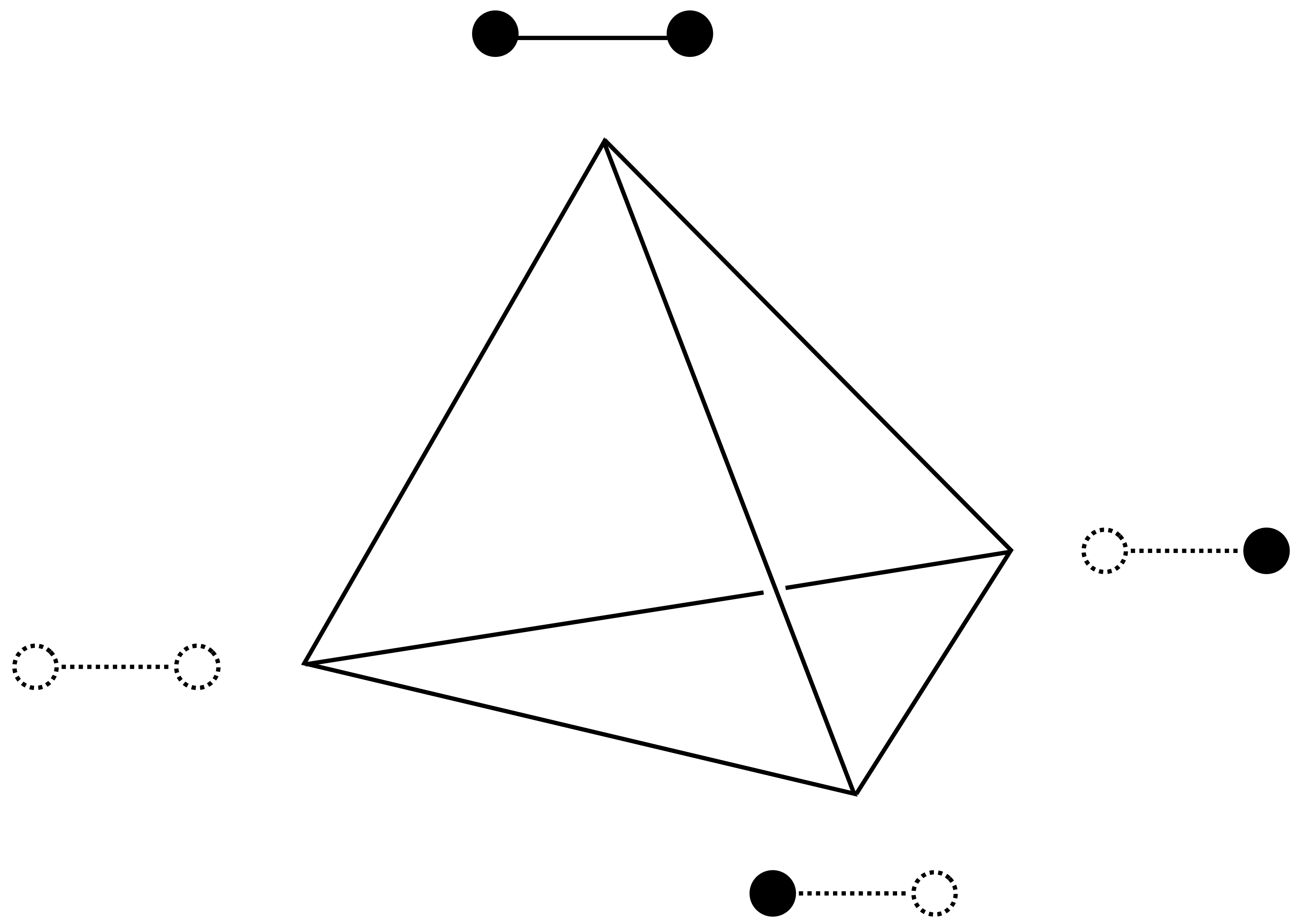
$$P(G) = \operatorname{conv}(\chi_{G'} \mid G' \subseteq G \text{ induced})$$

$$\begin{matrix} 1 \\ 2 \\ 12 \end{matrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$



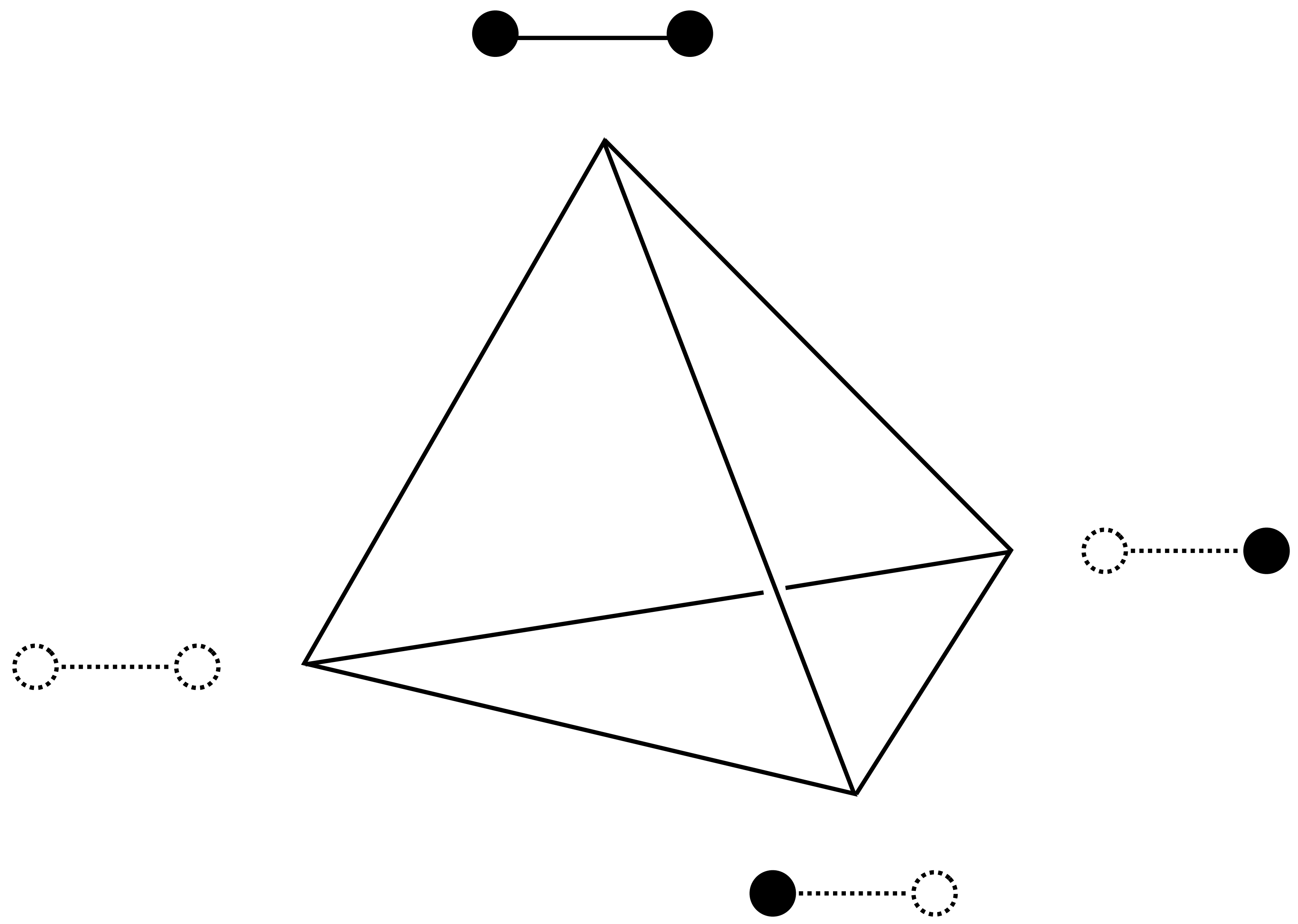


Bidding round and auctioneer's decision



Bidding round and auctioneer's decision

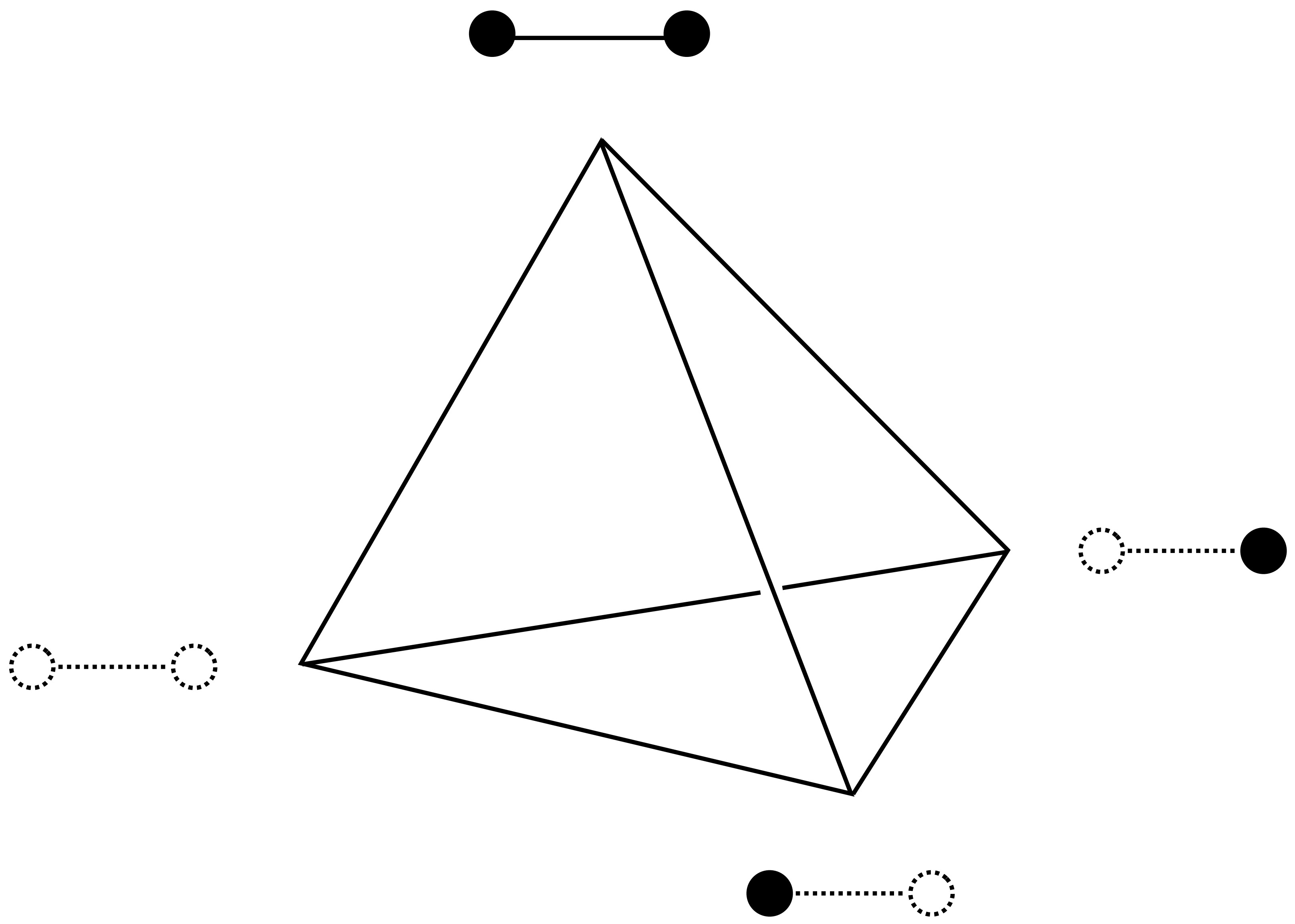
Bidder $b \in [m]$ communicates preferences to auctioneer



Bidding round and auctioneer’s decision

Bidder $b \in [m]$ communicates preferences to auctioneer

Valuation function $v^b : P \cap \mathbb{Z}^{n+|E|} \to \mathbb{R}$, $v^b(a) = \langle w^b, a \rangle$ for some $w^b \in \mathbb{R}^{n+|E|}$

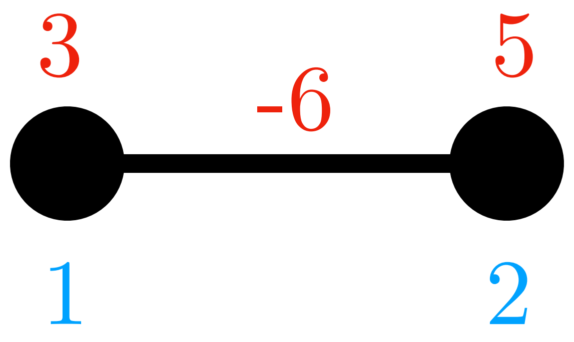


Bidding round and auctioneer's decision

Bidder $b \in [m]$ communicates preferences to auctioneer

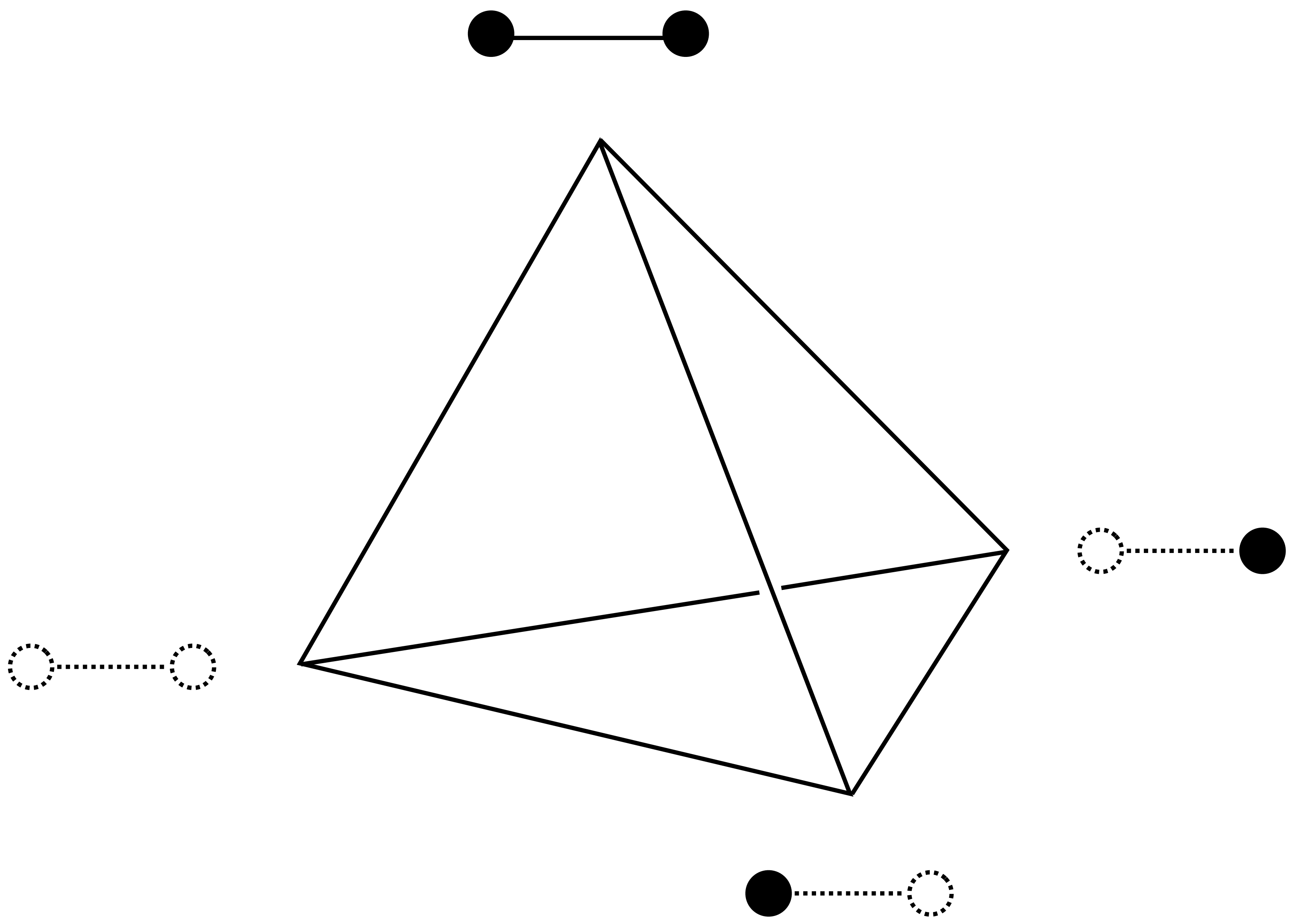
Valuation function $v^b : P \cap \mathbb{Z}^{n+|E|} \to \mathbb{R}$, $v^b(a) = \langle w^b, a \rangle$ for some $w^b \in \mathbb{R}^{n+|E|}$

$$w^b = \begin{pmatrix} 3 \\ 5 \\ -6 \end{pmatrix}$$



$$v^b\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right) = 0, \quad v^b\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) = 3,$$

$$v^b\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) = 5, \quad v^b\left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right) = 2$$

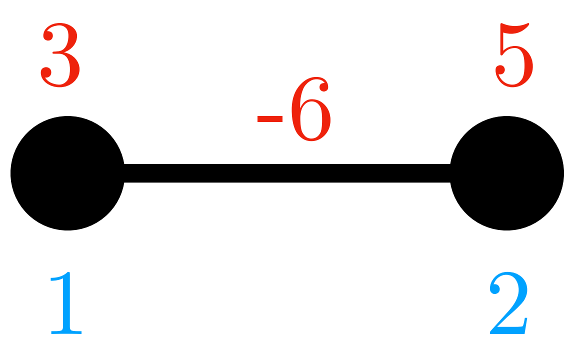


Bidding round and auctioneer's decision

Bidder $b \in [m]$ communicates preferences to auctioneer

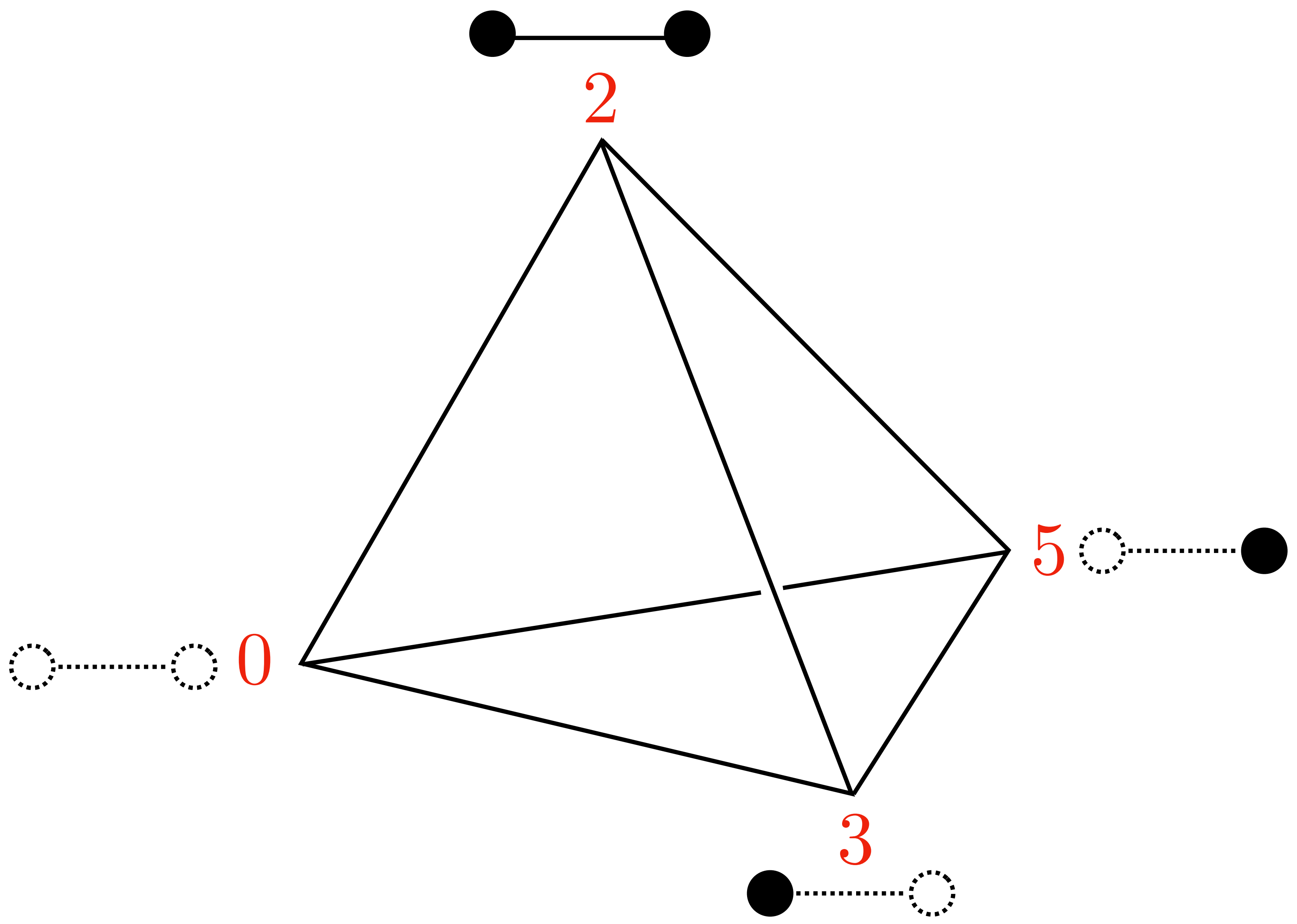
Valuation function $v^b : P \cap \mathbb{Z}^{n+|E|} \to \mathbb{R}$, $v^b(a) = \langle w^b, a \rangle$ for some $w^b \in \mathbb{R}^{n+|E|}$

$$w^b = \begin{pmatrix} 3 \\ 5 \\ -6 \end{pmatrix}$$



$$v^b\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right) = 0, \quad v^b\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) = 3,$$

$$v^b\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) = 5, \quad v^b\left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right) = 2$$

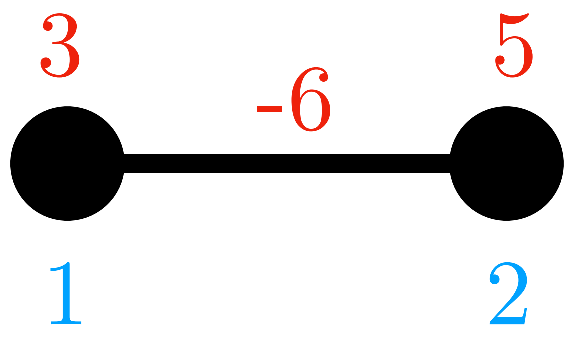


Bidding round and auctioneer's decision

Bidder $b \in [m]$ communicates preferences to auctioneer

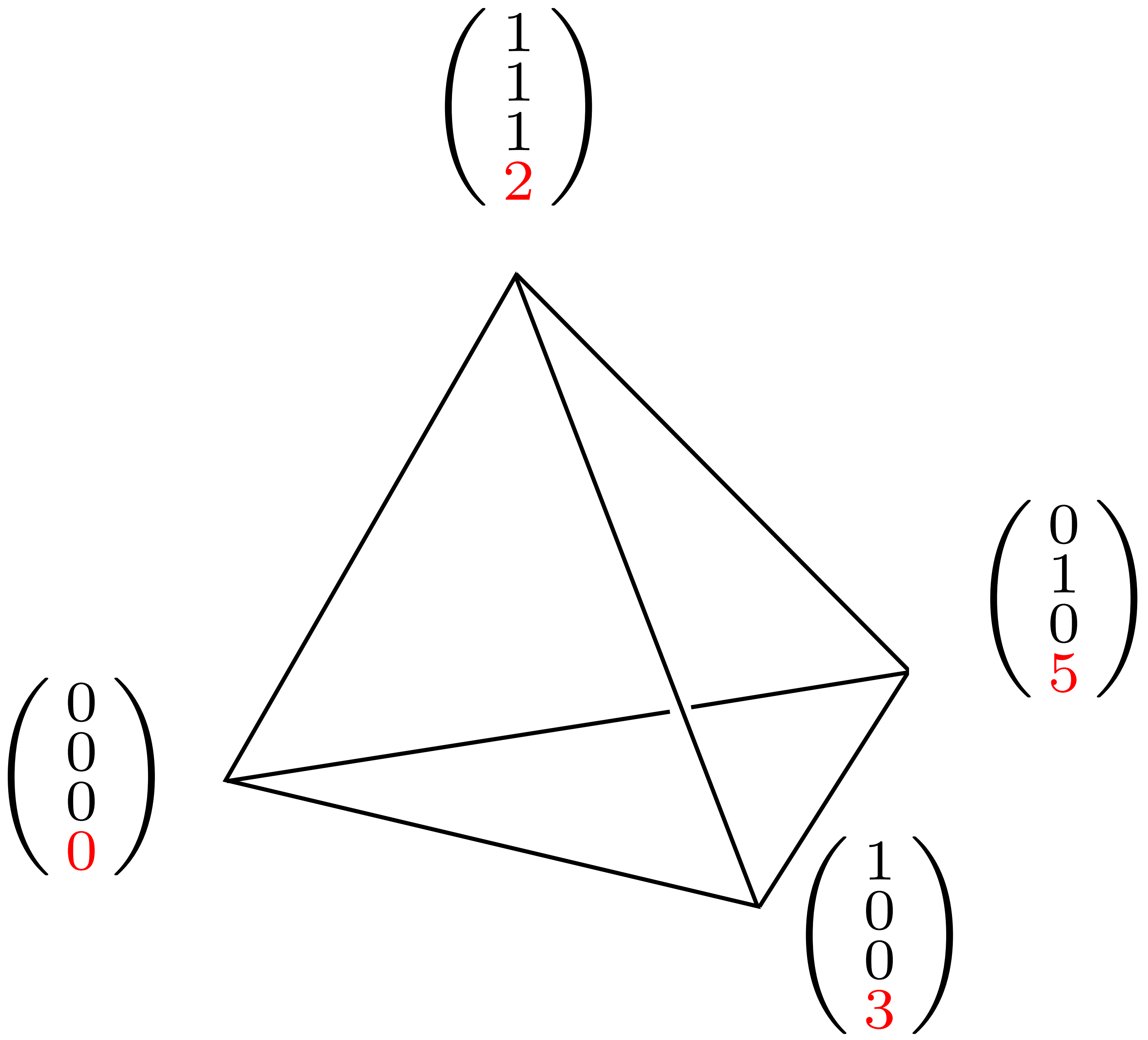
Valuation function $v^b : P \cap \mathbb{Z}^{n+|E|} \to \mathbb{R}, \;\; v^b(a) = \langle w^b, a \rangle$ for some $w^b \in \mathbb{R}^{n+|E|}$

$$w^b = \begin{pmatrix} 3 \\ 5 \\ -6 \end{pmatrix}$$



$$v^b\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right) = 0, \quad v^b\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) = 3,$$

$$v^b\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) = 5, \quad v^b\left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right) = 2$$



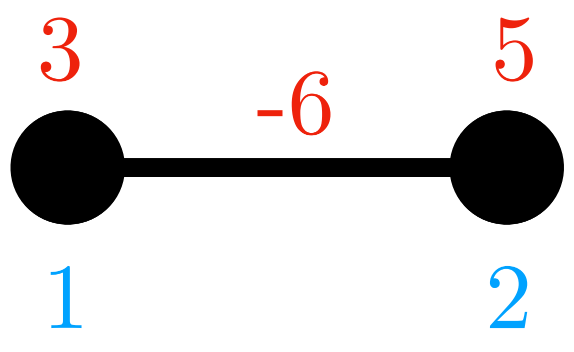
Bidding round and auctioneer's decision

Bidder $b \in [m]$ communicates preferences to auctioneer

Valuation function $v^b : P \cap \mathbb{Z}^{n+|E|} \to \mathbb{R}$, $v^b(a) = \langle w^b, a \rangle$ for some $w^b \in \mathbb{R}^{n+|E|}$

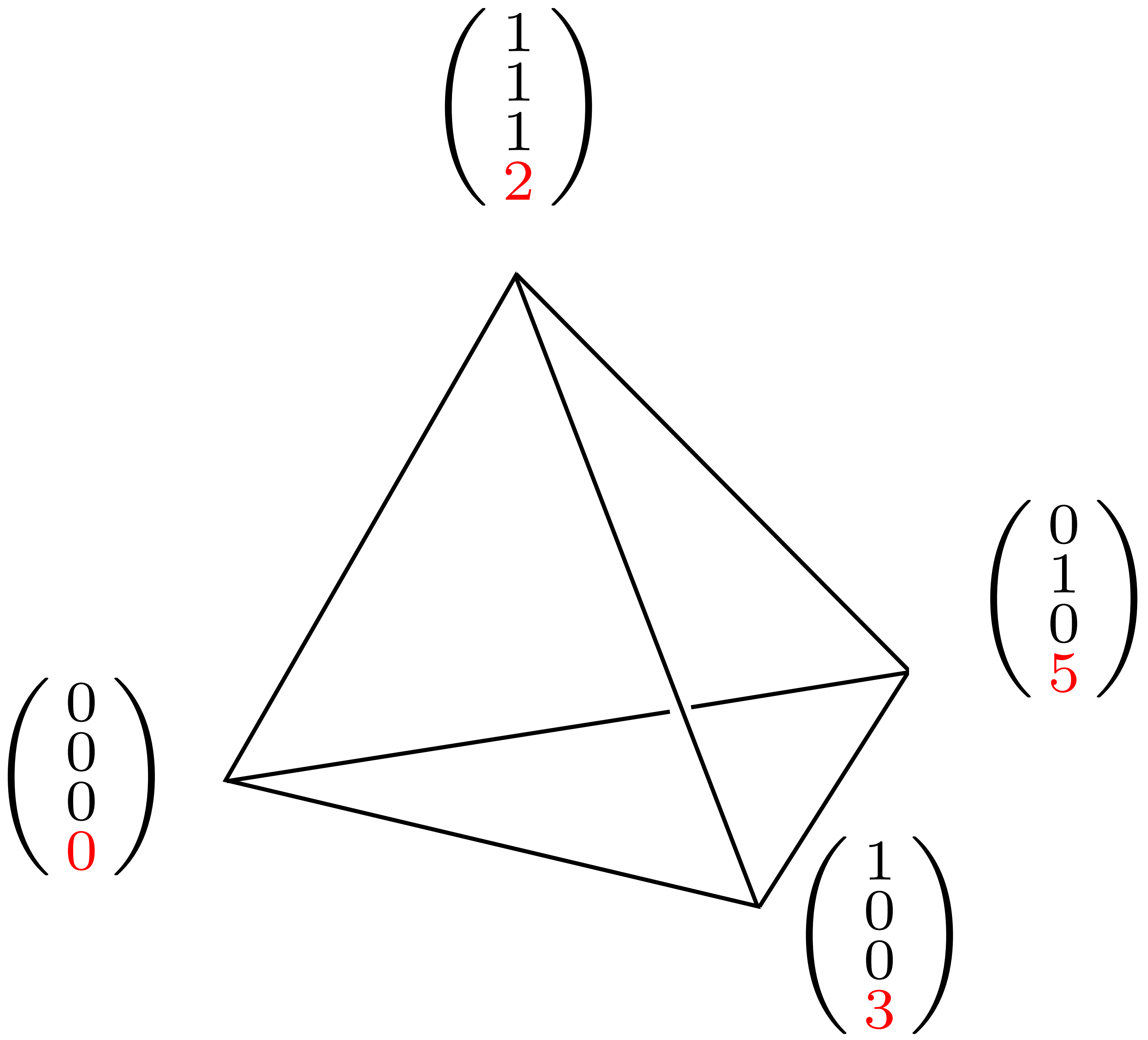
Auctioneer sets a price

$$w^b = \begin{pmatrix} 3 \\ 5 \\ -6 \end{pmatrix}$$



$$v^b(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}) = 0, \quad v^b(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}) = 3,$$

$$v^b(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}) = 5, \quad v^b(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}) = 2$$



Bidding round and auctioneer's decision

Bidder $b \in [m]$ communicates preferences to auctioneer

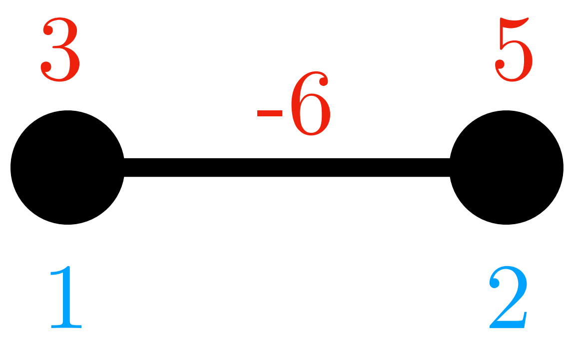
Valuation function $v^b : P \cap \mathbb{Z}^{n+|E|} \to \mathbb{R}$, $v^b(a) = \langle w^b, a \rangle$ for some $w^b \in \mathbb{R}^{n+|E|}$

Auctioneer sets a price

Auctioneer computes the *demand set* of bidder *b* at price $p \in \mathbb{R}^{n+|E|}$:

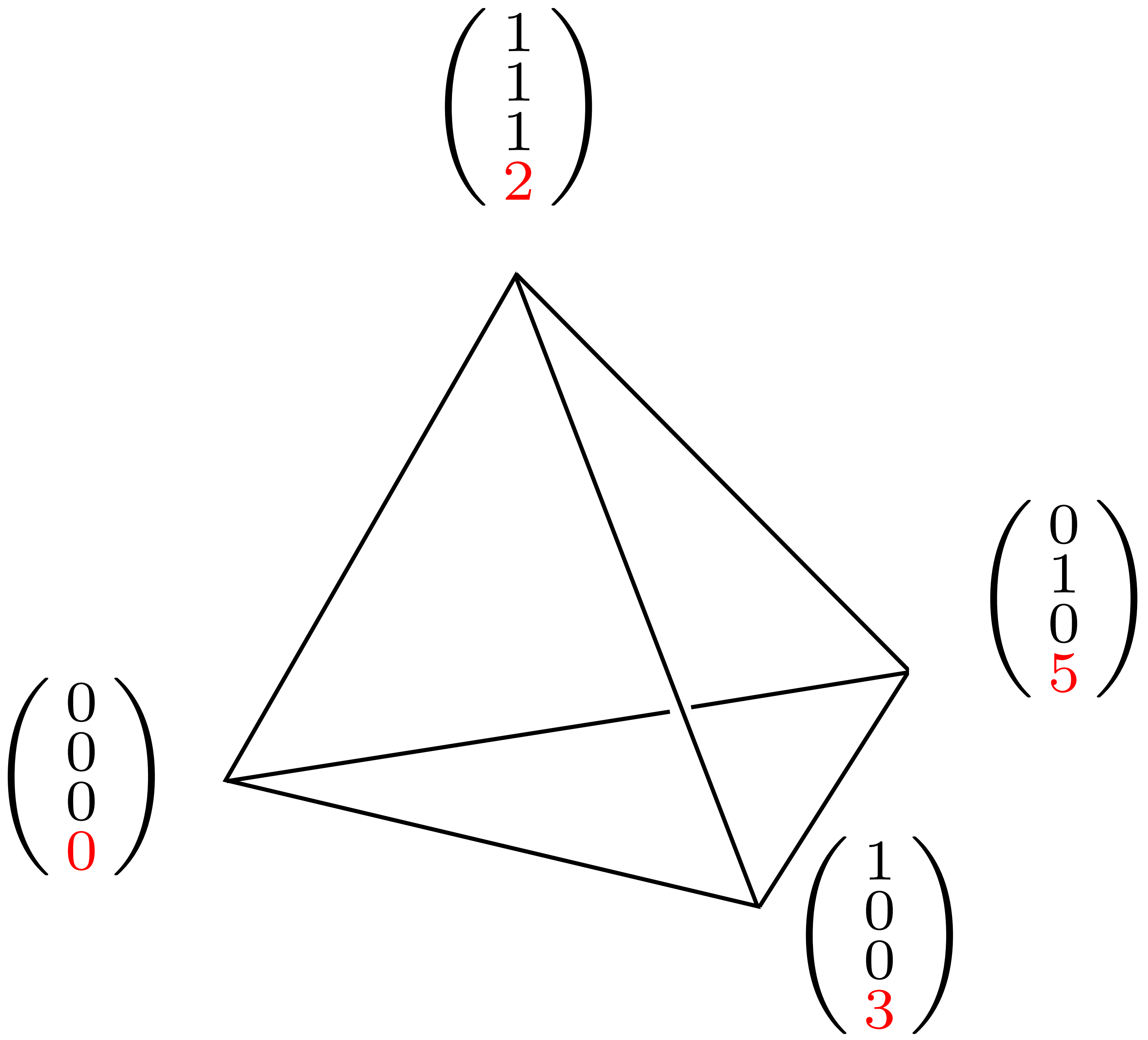
$$D(v^b, p) = \underset{a \in \mathrm{vert}(P(G))}{\mathrm{argmax}} \{v^b(a) - \langle p, a \rangle\}$$

$$w^b = \begin{pmatrix} 3 \\ 5 \\ -6 \end{pmatrix}$$



$$v^b\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right) = 0, \quad v^b\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) = 3,$$

$$v^b\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) = 5, \quad v^b\left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right) = 2$$



Bidding round and auctioneer's decision

Bidder $b \in [m]$ communicates preferences to auctioneer

Valuation function $v^b : P \cap \mathbb{Z}^{n+|E|} \to \mathbb{R}$, $v^b(a) = \langle w^b, a \rangle$ for some $w^b \in \mathbb{R}^{n+|E|}$

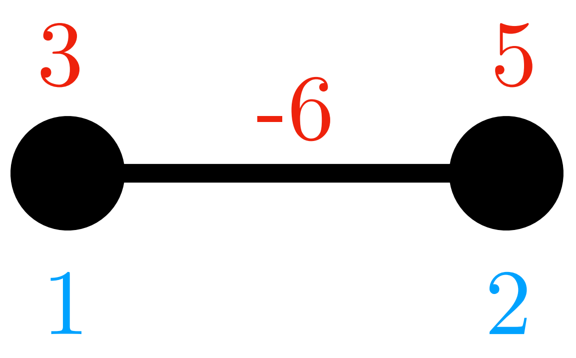
Auctioneer sets a price

Auctioneer computes the *demand set* of bidder b at price $p \in \mathbb{R}^{n+|E|}$:

$$D(v^b, p) = \underset{a \in \text{vert}(P(G))}{\text{argmax}} \{ v^b(a) - \langle p, a \rangle \}$$

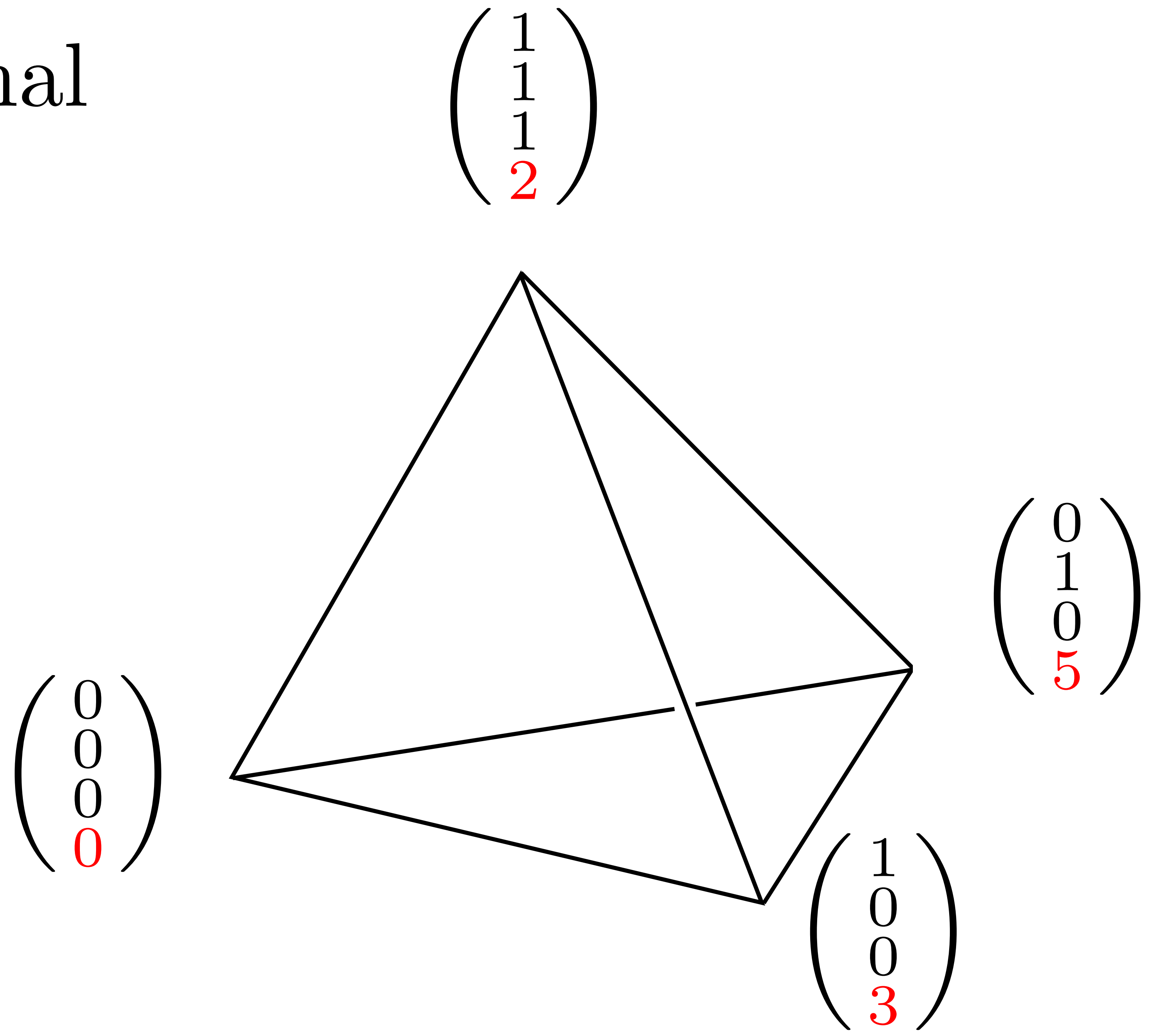
$$a \in D(v^b, p) \iff \langle \begin{pmatrix} a \\ v^b(a) \end{pmatrix}, \begin{pmatrix} -p \\ 1 \end{pmatrix} \rangle \text{ maximal}$$

$$w^b = \begin{pmatrix} 3 \\ 5 \\ -6 \end{pmatrix}$$



$$v^b(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}) = 0, \quad v^b(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}) = 3,$$

$$v^b(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}) = 5, \quad v^b(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}) = 2$$



Bidding round and auctioneer's decision

Bidder $b \in [m]$ communicates preferences to auctioneer

Valuation function $v^b : P \cap \mathbb{Z}^{n+|E|} \to \mathbb{R}$, $v^b(a) = \langle w^b, a \rangle$ for some $w^b \in \mathbb{R}^{n+|E|}$

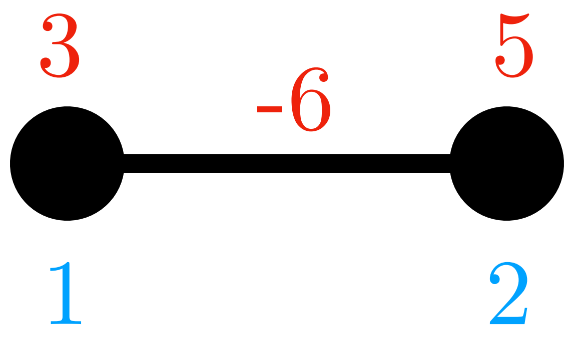
Auctioneer sets a price

Auctioneer computes the *demand set* of bidder *b* at price $p \in \mathbb{R}^{n+|E|}$:

$$D(v^b, p) = \underset{a \in \mathrm{vert}(P(G))}{\mathrm{argmax}} \{v^b(a) - \langle p, a \rangle\} = \mathrm{vert}(F^b) \text{ for some } F^b \preceq P(G)$$

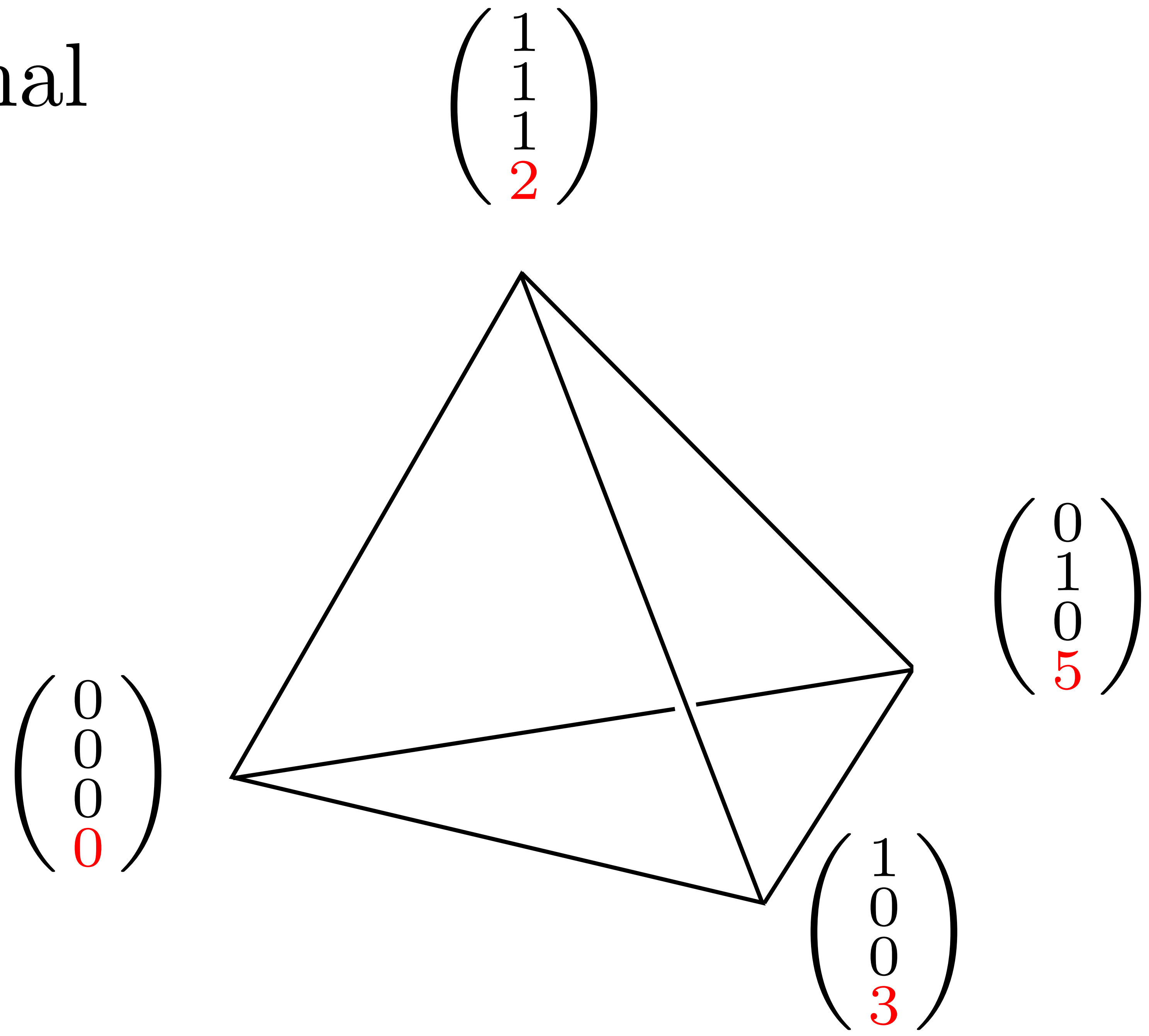
$$a \in D(v^b, p) \iff \left\langle \begin{pmatrix} a \\ v^b(a) \end{pmatrix}, \begin{pmatrix} -p \\ 1 \end{pmatrix} \right\rangle \text{ maximal}$$

$$w^b = \begin{pmatrix} 3 \\ 5 \\ -6 \end{pmatrix}$$



$$v^b\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right) = 0, \quad v^b\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) = 3,$$

$$v^b\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) = 5, \quad v^b\left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right) = 2$$



Competitive equilibrium

and lattice polytopes

Competitive equilibrium

and lattice polytopes

Definition.

A competitive equilibrium is *guaranteed to exist* if for any set of valuations $\{v^b \mid b \in [m]\}$ there exists $p \in \mathbb{R}^{n+|E|}, a \in \sum_{b \in [m]} D(v^b, p)$ such that $a_i^* = a_i \; \forall i \in [n]$.

Competitive equilibrium

and lattice polytopes

Definition.

A competitive equilibrium is *guaranteed to exist* if for any set of valuations $\{v^b \mid b \in [m]\}$ there exists $p \in \mathbb{R}^{n+|E|}, a \in \sum_{b \in [m]} D(v^b, p)$ such that $a_i^* = a_i \ \forall i \in [n]$.

Lemma (B.-Haase-Tran, '21⁺).

Let $a^* \in \mathbb{Z}_{\geq 0}^n$ and $a \in \pi^{-1}(a^*)$. Then TFAE:

Competitive equilibrium

and lattice polytopes

Definition.

A competitive equilibrium is *guaranteed to exist* if for any set of valuations $\{v^b \mid b \in [m]\}$ there exists $p \in \mathbb{R}^{n+|E|}, a \in \sum_{b\in[m]} D(v^b, p)$ such that $a_i^* = a_i \; \forall i \in [n]$.

Lemma (B.-Haase-Tran, ’21+).

Let $a^* \in \mathbb{Z}^n_{\geq 0}$ and $a \in \pi^{-1}(a^*)$. Then TFAE:

a) $\forall \; \{v^b \mid b \in [m]\} \; \exists \; p \in \mathbb{R}^{n+|E|} : \; a \in \sum_{b\in[m]} D(v^b, p)$

Competitive equilibrium

and lattice polytopes

Definition.

A competitive equilibrium is *guaranteed to exist* if for any set of valuations $\{v^b \mid b \in [m]\}$ there exists $p \in \mathbb{R}^{n+|E|}, a \in \sum_{b\in[m]} D(v^b, p)$ such that $a_i^* = a_i \ \forall i \in [n]$.

Lemma (B.-Haase-Tran, '21+).

Let $a^* \in \mathbb{Z}^n_{\geq 0}$ and $a \in \pi^{-1}(a^*)$. Then TFAE:

a) $\forall \ \{v^b \mid b \in [m]\} \ \exists \ p \in \mathbb{R}^{n+|E|} : \ a \in \sum_{b\in[m]} D(v^b, p)$

b) $\forall \ F^1, \ldots, F^m \preceq P(G) : \ \text{if } a \in \sum_{b\in[m]} F^b \text{ then } a \in \sum_{b\in[m]} \operatorname{vert}(F^b)$

Competitive equilibrium

and lattice polytopes

Definition.

A competitive equilibrium is *guaranteed to exist* if for any set of valuations $\{v^b \mid b \in [m]\}$ there exists $p \in \mathbb{R}^{n+|E|}, a \in \sum_{b\in[m]} D(v^b, p)$ such that $a_i^* = a_i \; \forall i \in [n]$.

Lemma (B.-Haase-Tran, '21+).

Let $a^* \in \mathbb{Z}^n_{\geq 0}$ and $a \in \pi^{-1}(a^*)$. Then TFAE:

a) $\forall \; \{v^b \mid b \in [m]\} \; \exists \; p \in \mathbb{R}^{n+|E|} : \; a \in \sum_{b\in[m]} D(v^b, p)$

b) $\forall \; F^1, \ldots, F^m \preceq P(G) : \; \text{if } a \in \sum_{b\in[m]} F^b \text{ then } a \in \sum_{b\in[m]} \text{vert}(F^b)$

In particular, then a CE is guaranteed to exist.

Competitive equilibrium

and lattice polytopes

Definition.
A competitive equilibrium is *guaranteed to exist* if for any set of valuations $\{v^b \mid b \in [m]\}$ there exists $p \in \mathbb{R}^{n+|E|}, a \in \sum_{b\in[m]} D(v^b, p)$ such that $a_i^* = a_i \; \forall i \in [n]$.

Lemma (B.-Haase-Tran, '21$^+$).
Let $a^* \in \mathbb{Z}^n_{\geq 0}$ and $a \in \pi^{-1}(a^*)$. Then TFAE:

a) $\forall \, \{v^b \mid b \in [m]\} \; \exists \, p \in \mathbb{R}^{n+|E|} : \; a \in \sum_{b\in[m]} D(v^b, p)$

b) $\forall \; F^1, \ldots, F^m \preceq P(G) : \;$ if $a \in \sum_{b\in[m]} F^b$ then $a \in \sum_{b\in[m]} \mathrm{vert}(F^b)$

In particular, then a CE is guaranteed to exist.

face in mixed regular subdivision

Points that are always in the upper convex hull of the lifted $mP(G)$

Results

Recall: a_i^* = # items of type i

Results

Recall: $a_i^* = \#$ items of type i

Theorem (B.-Haase-Tran, ’21$^+$)

Let $G = K_n$ be the complete graph.

a) If $a^* \in \{0,1\}^n$ then CE is guaranteed to exist for all distributions of goods.

b) If $a^* \in \mathbb{Z}^n_{\geq 0}$. then CE is guaranteed to exist.

Results

Recall: a_i^* = # items of type i

Theorem (B.-Haase-Tran, '21$^+$)

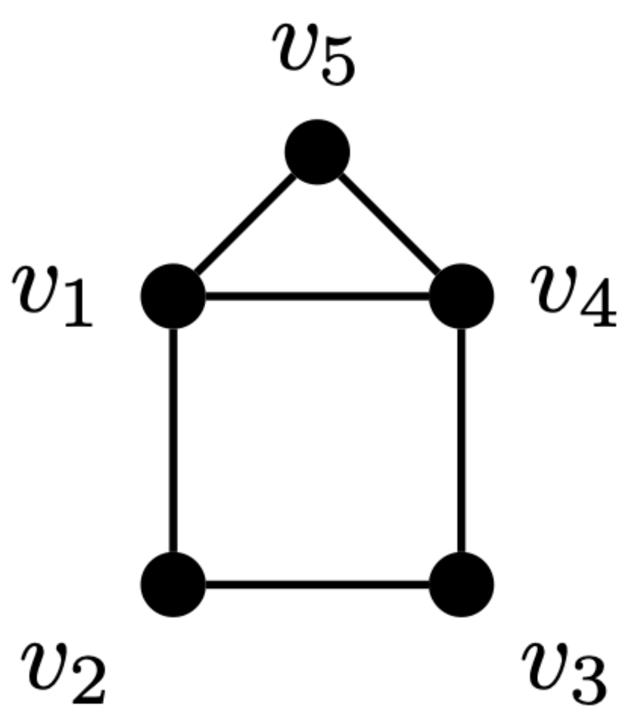
Let $G = K_n$ be the complete graph.

a) If $a^* \in \{0,1\}^n$ then CE is guaranteed to exist for all distributions of goods.

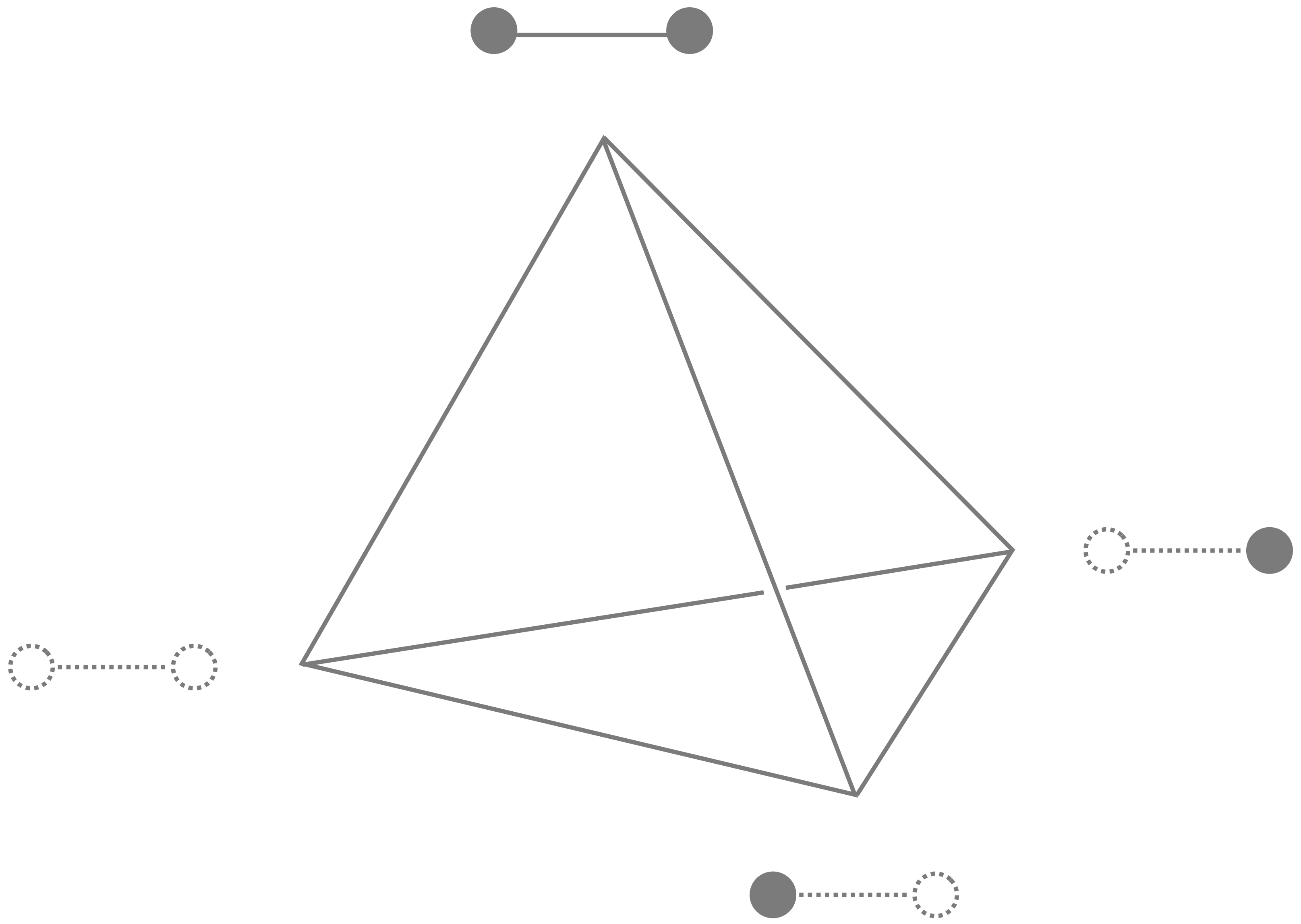
b) If $a^* \in \mathbb{Z}^n_{\geq 0}$. then CE is guaranteed to exist.

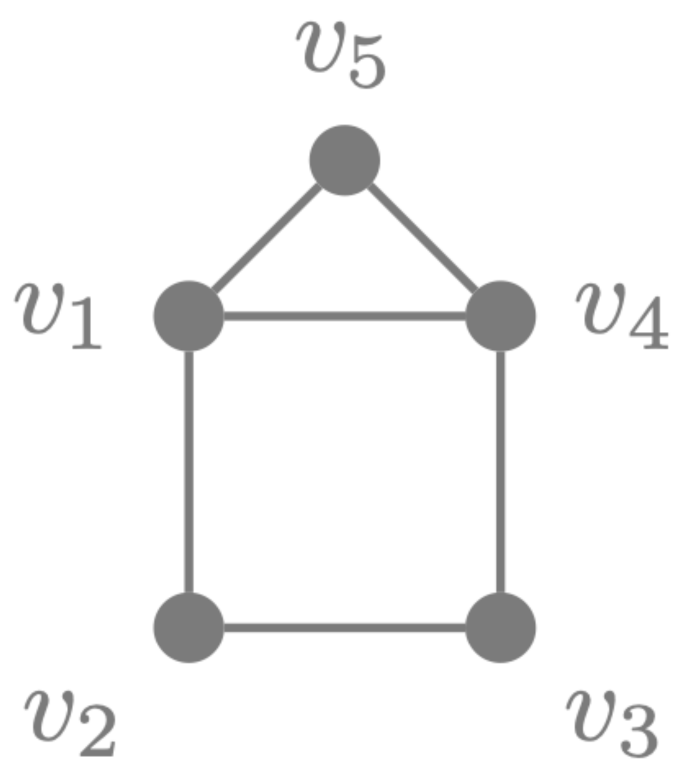
Theorem (B.-Haase-Tran, '21$^+$)

Let $G =$



. Then CE is not guaranteed to exist.





Thank you!