Competitive Equilibrium and Lattice Polytopes

Mini-Symposium on Lattice Polytopes

02 March 2022

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joint work with Christian Haase and Ngoc Mai Tran

Max-Planck-Institut für Mathematik in den Naturwissenschaften

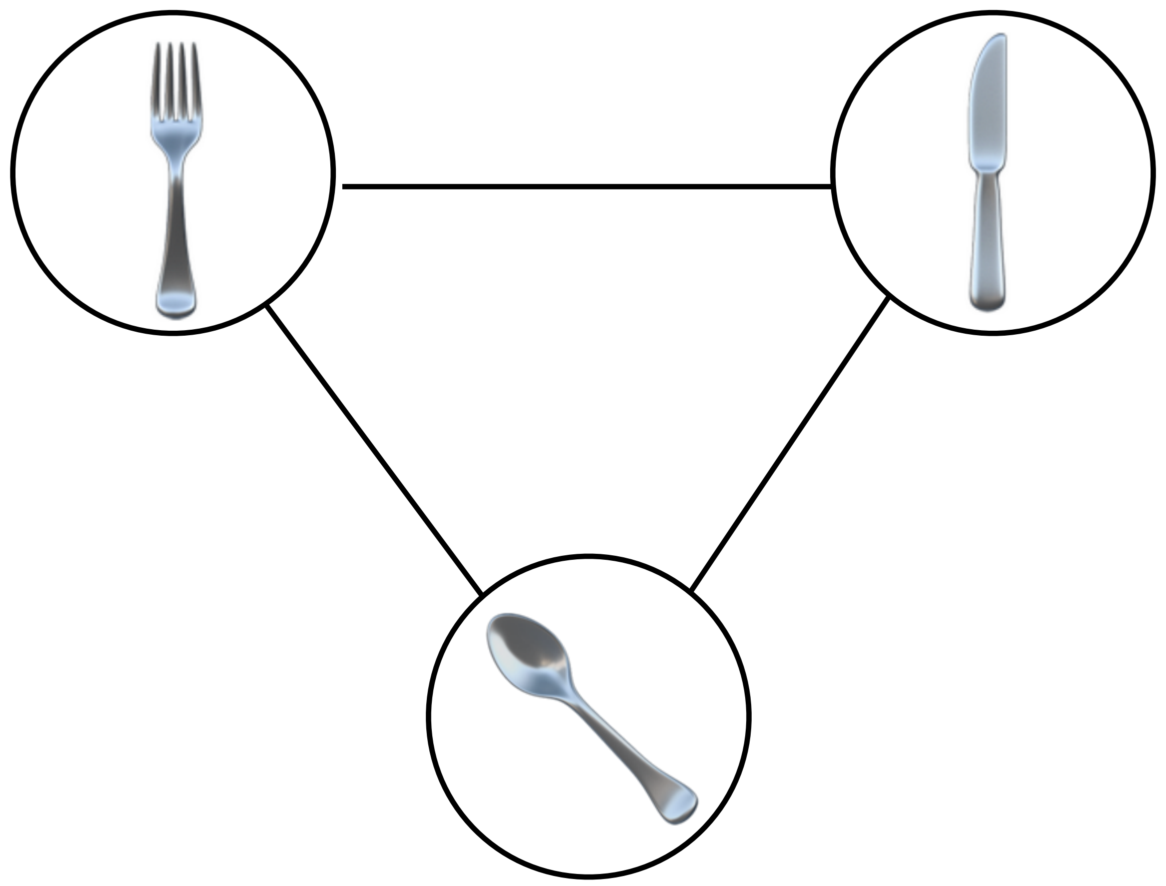
MAX-PLANCK-GESELLSCHAFT

Overview

1. **First Example**
2. **History | Motivation**
3. **Mathematical Model | Connections to Polytopes**
4. **Can we guarantee the existence of a competitive equilibrium?**
   (Answer: yes, if $G = K_n$)

First Example

The cutlery auction at dinner time



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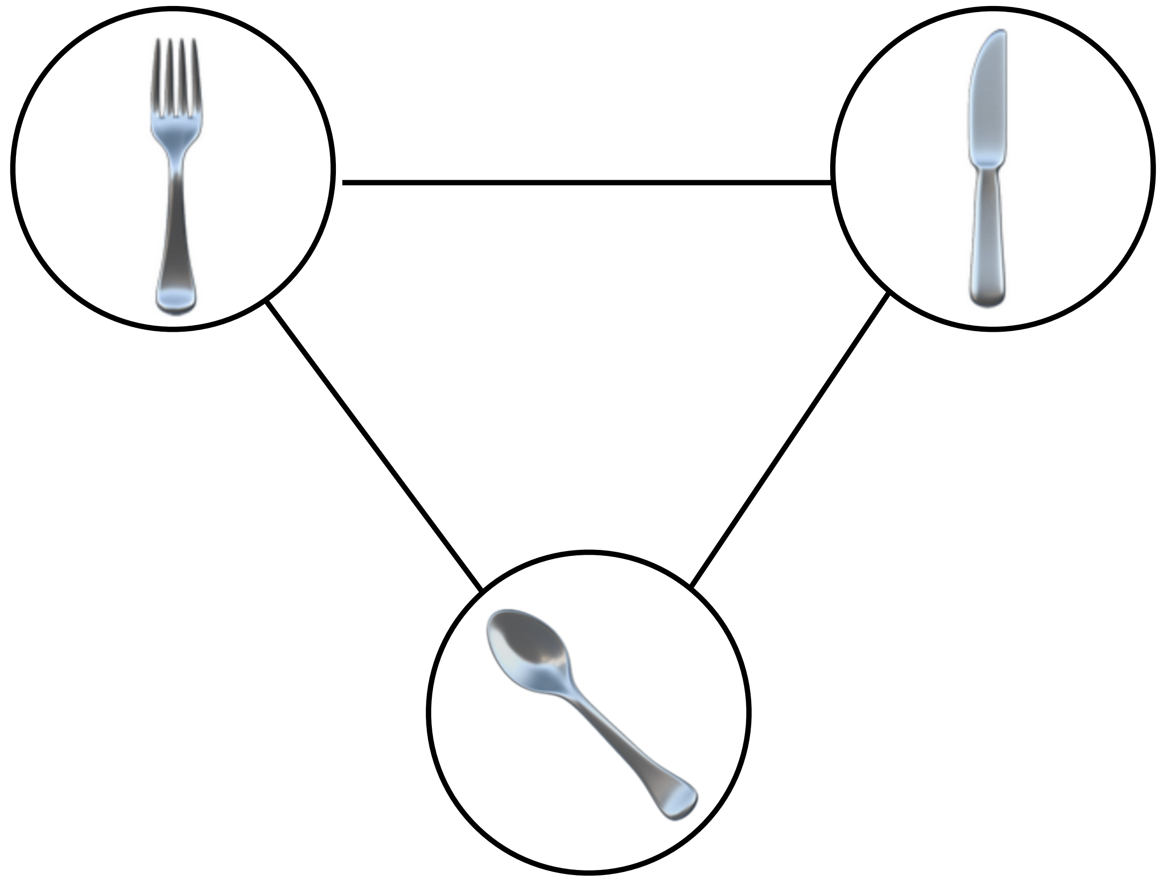

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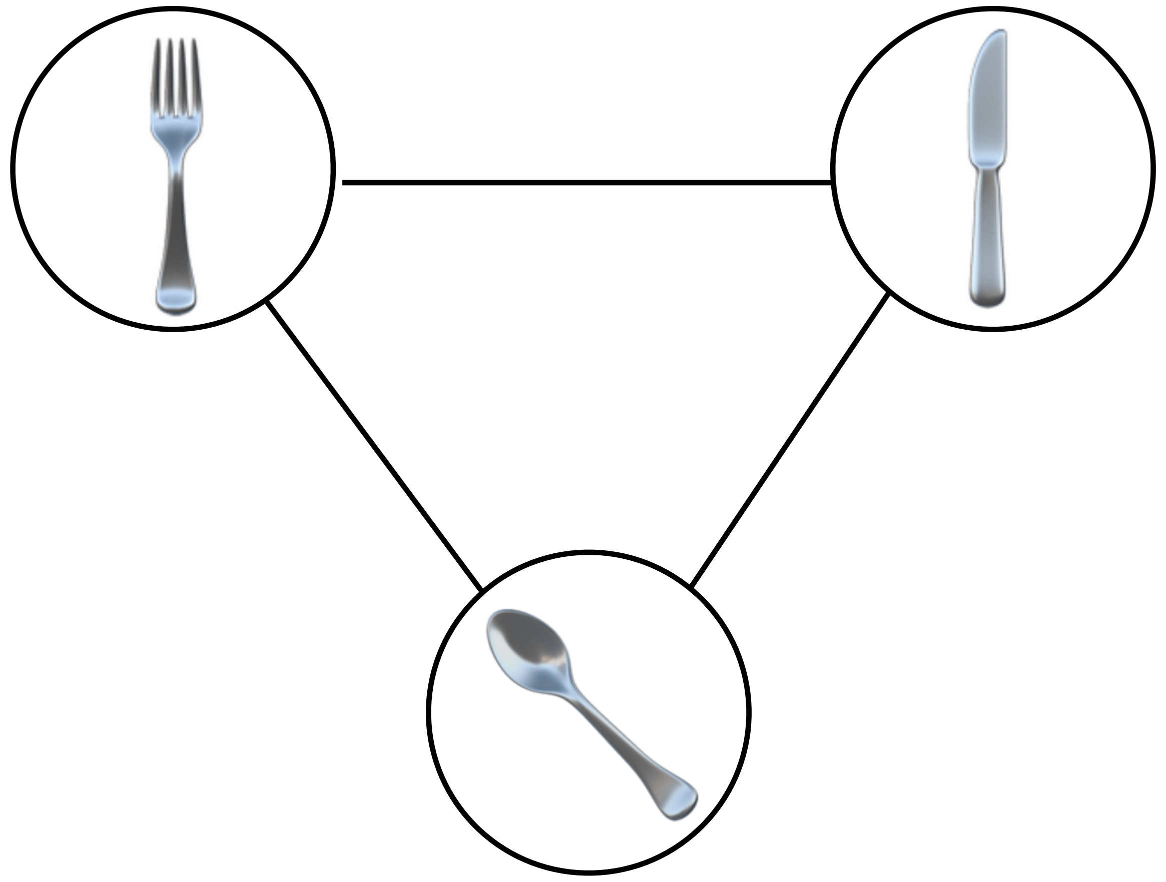

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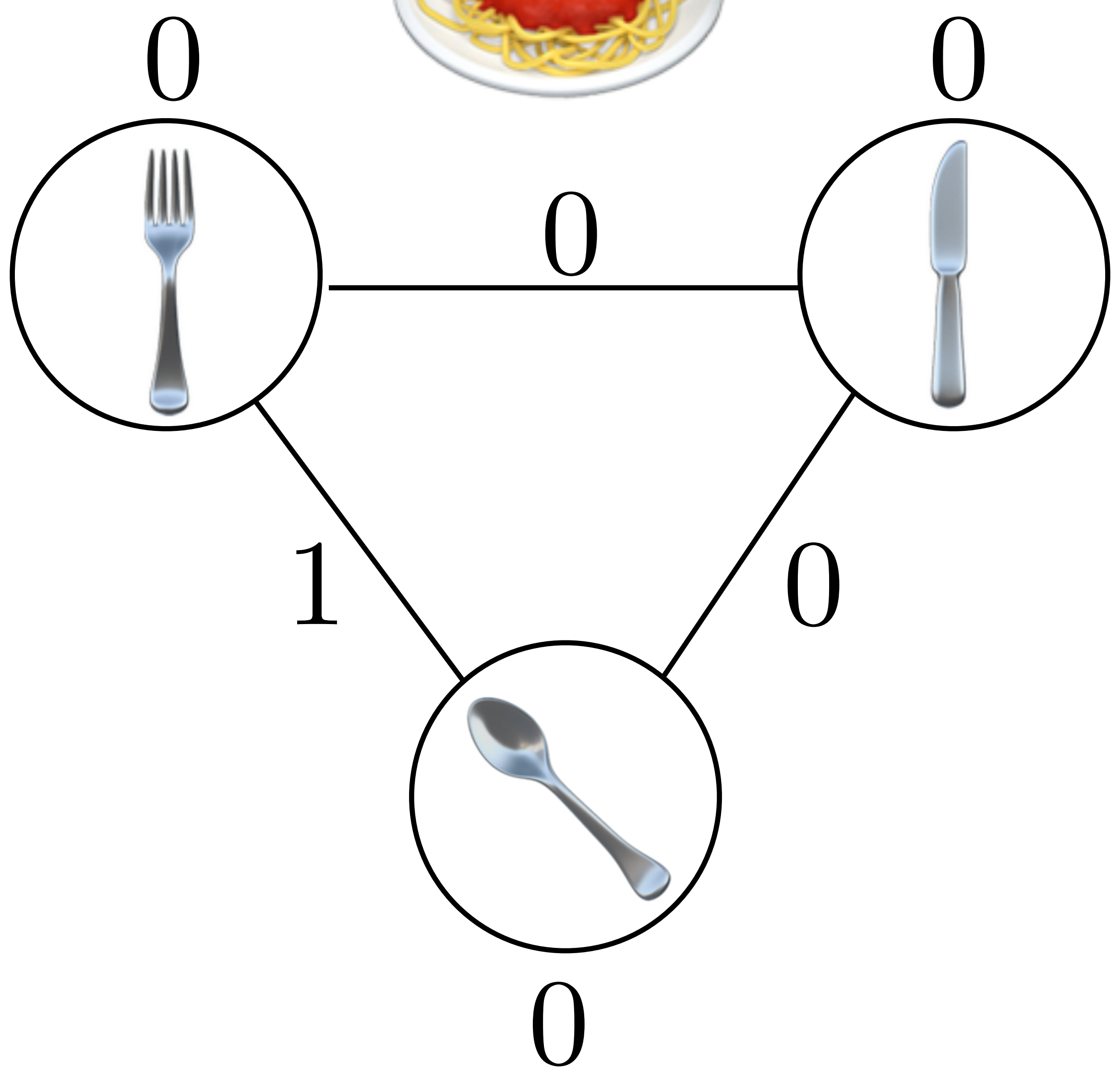

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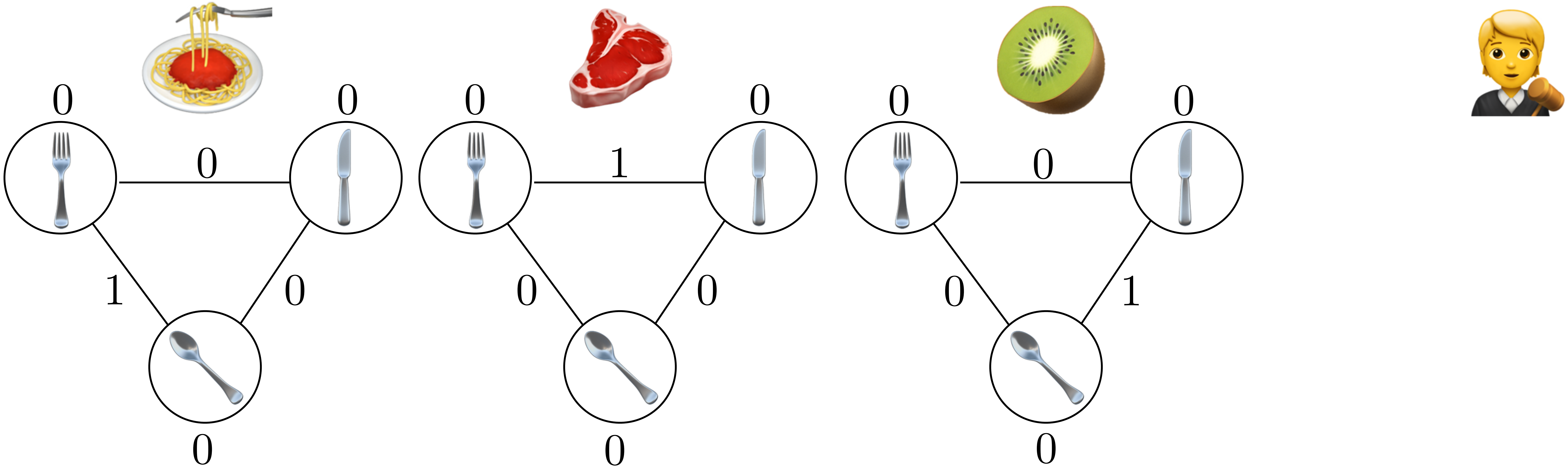

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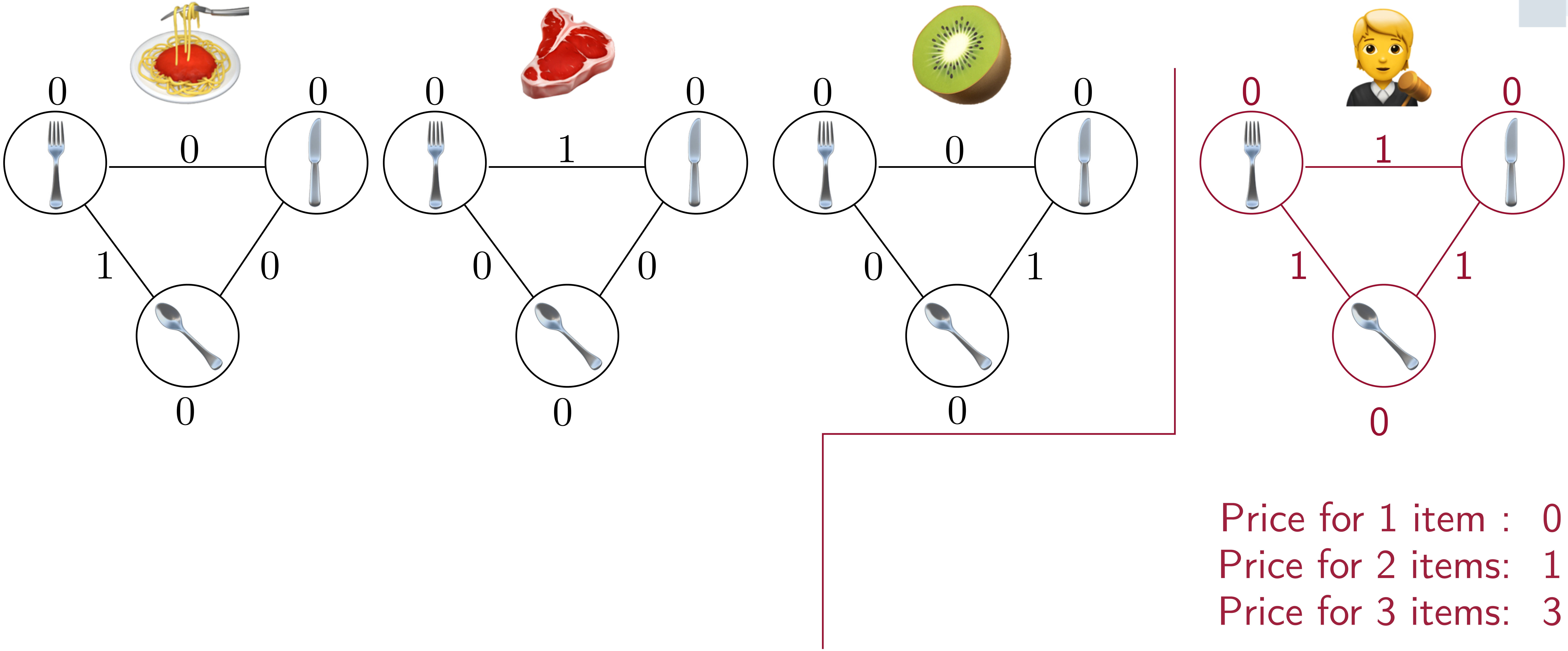

Price for 1 item : 0
Price for 2 items: 1
Price for 3 items: 3

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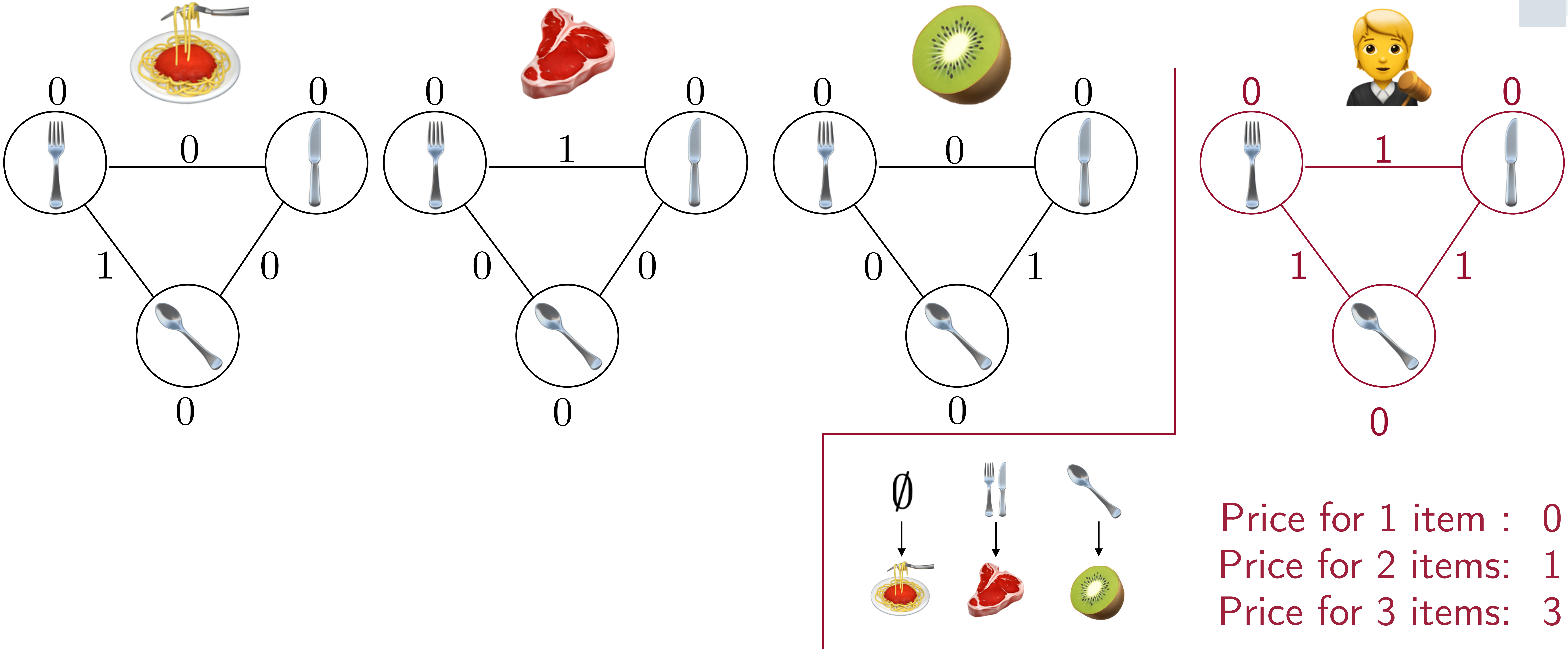

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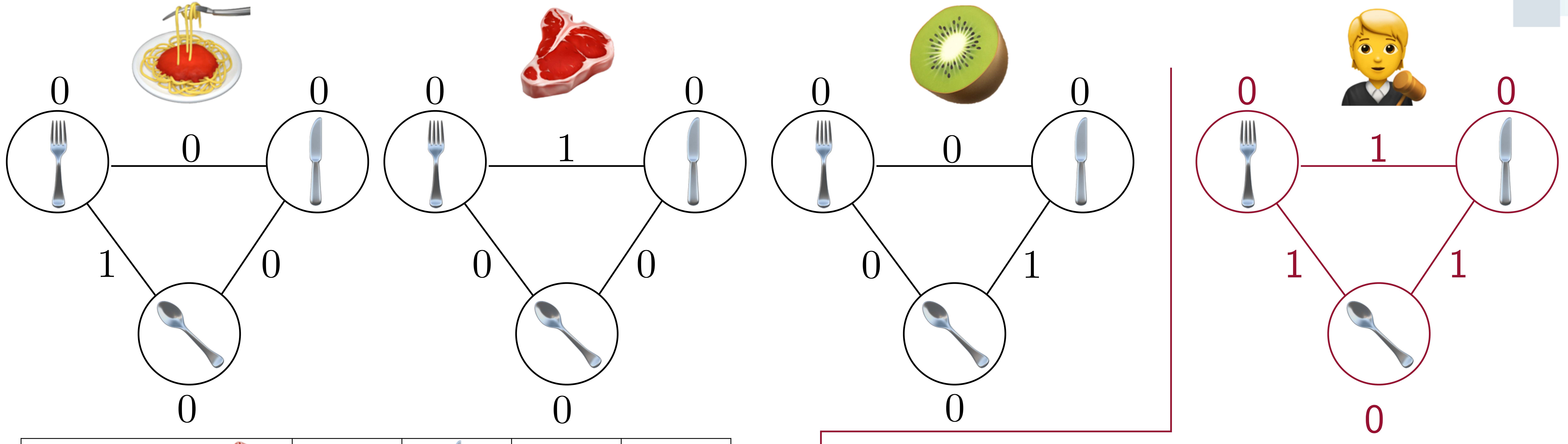
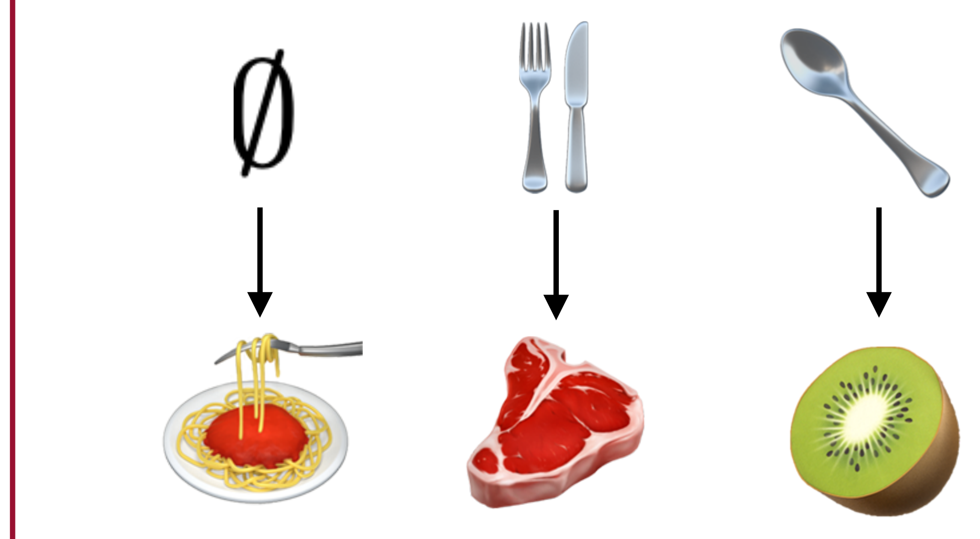


Opinion of 🥩	∅			
Willing to pay	0	0	1	1
Price charged	0	0	1	3
Profit	0	0	0	-2



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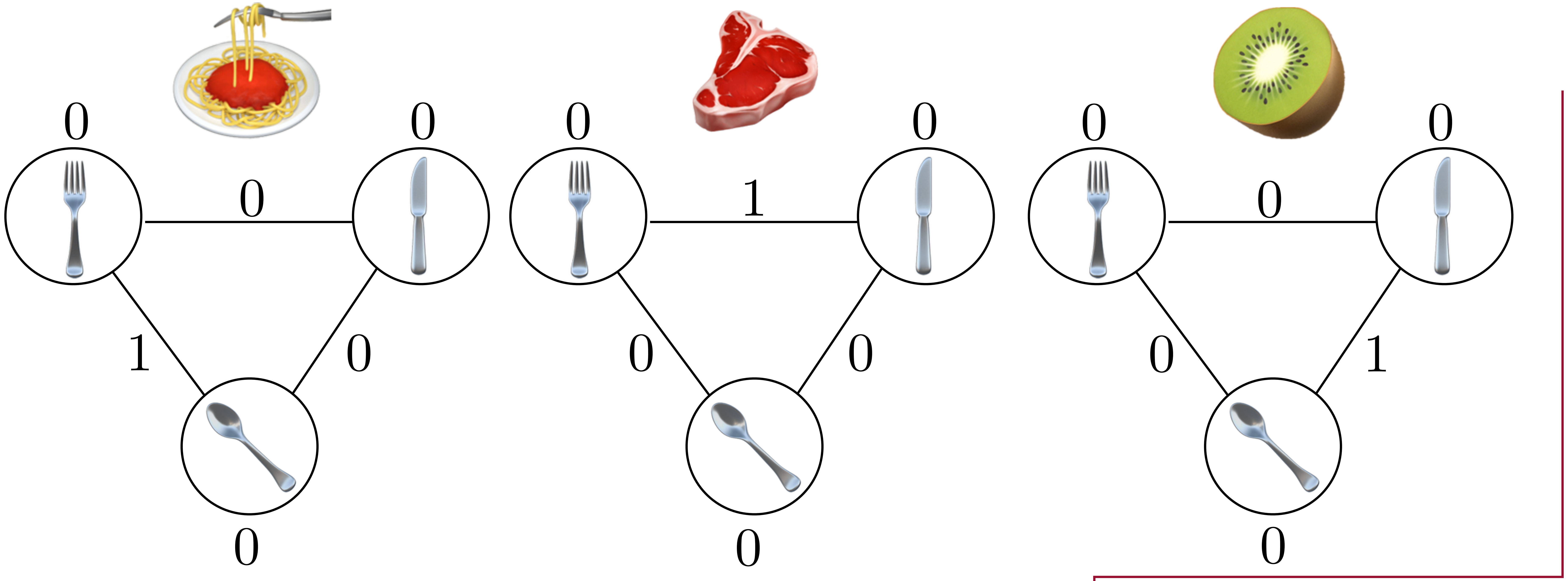

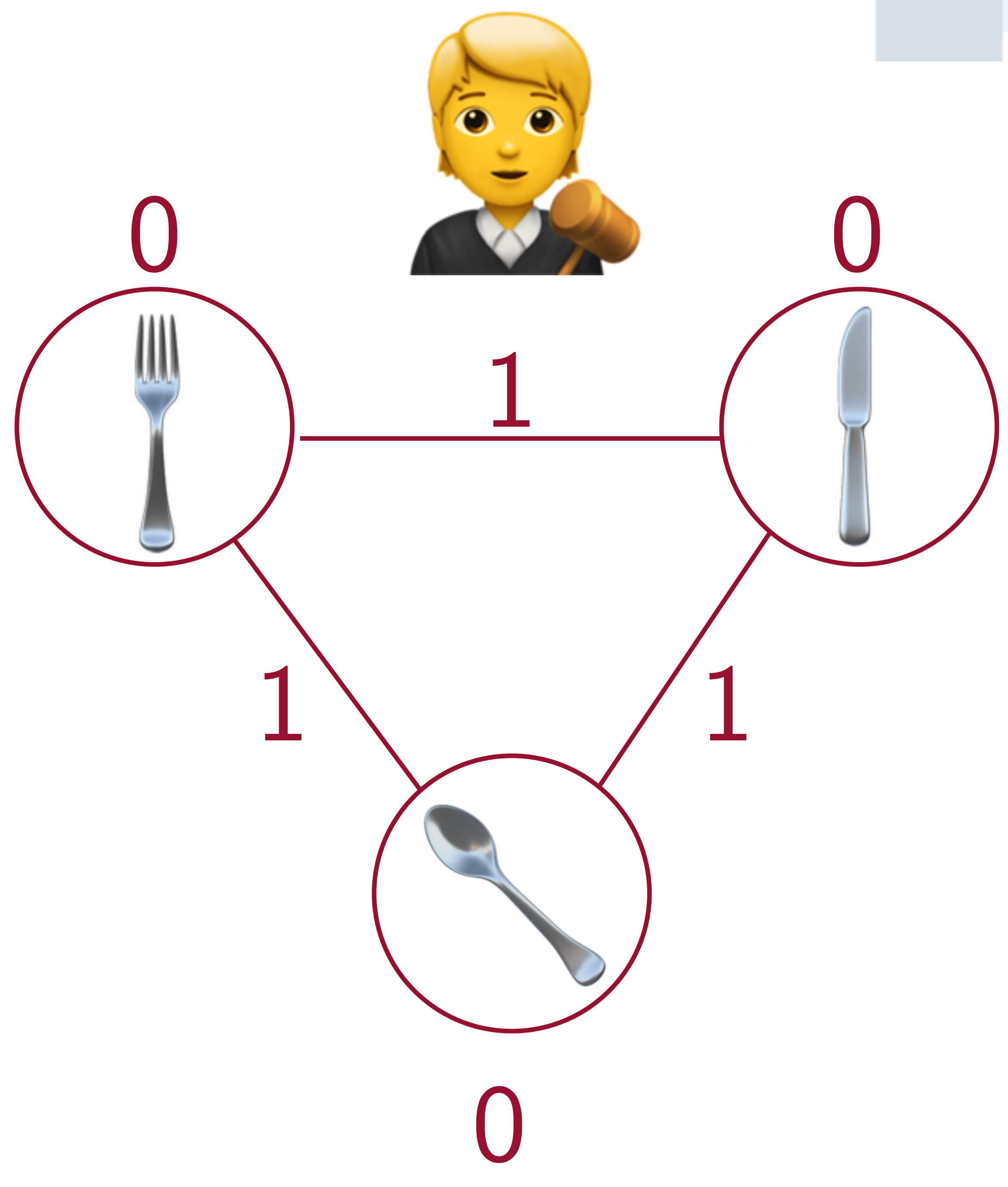

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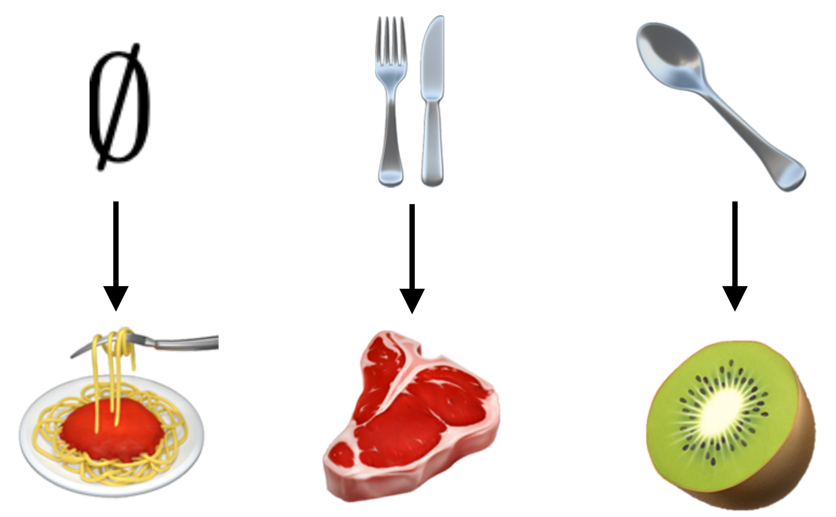

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  → bad properties from game-theoretic perspective

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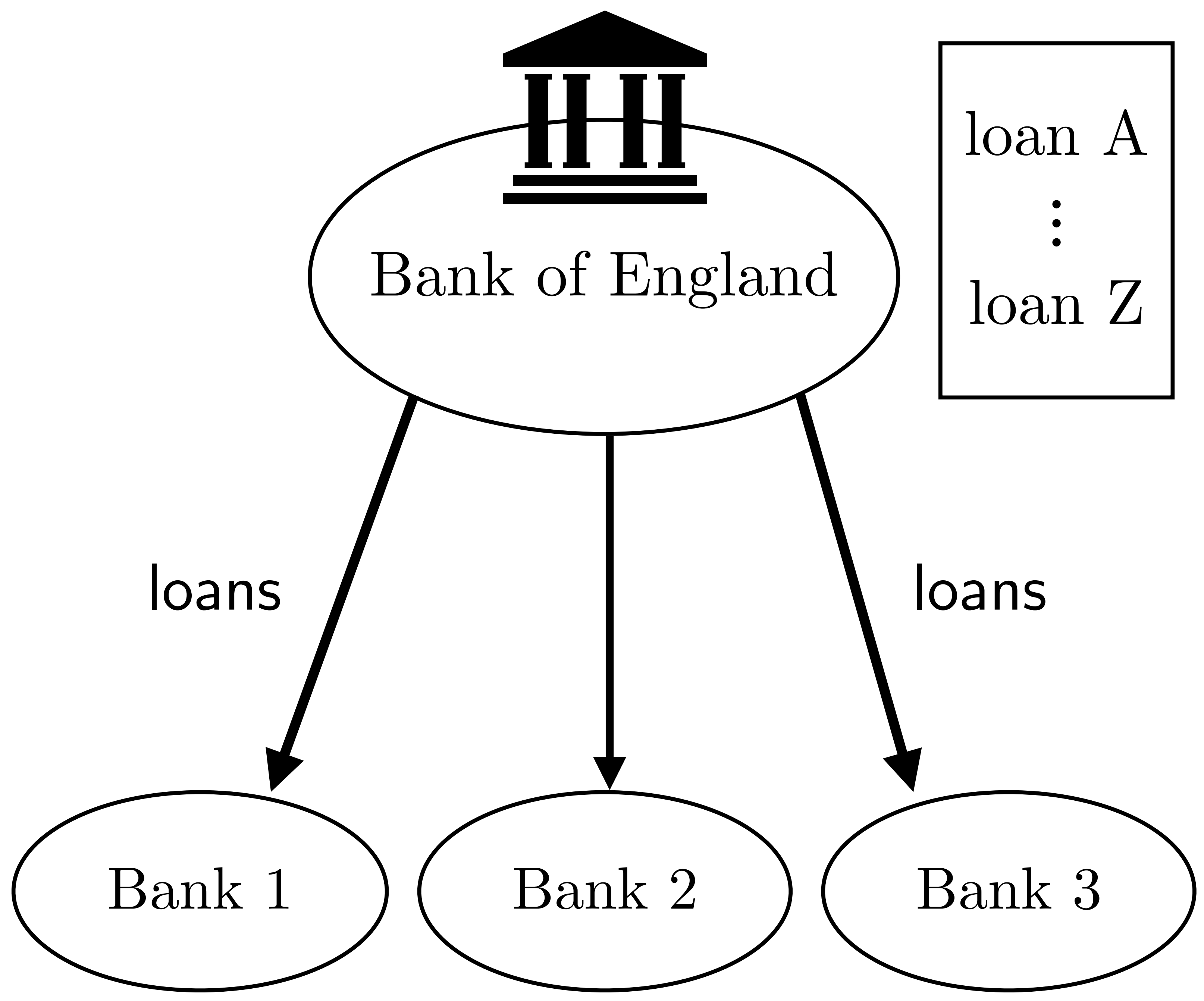

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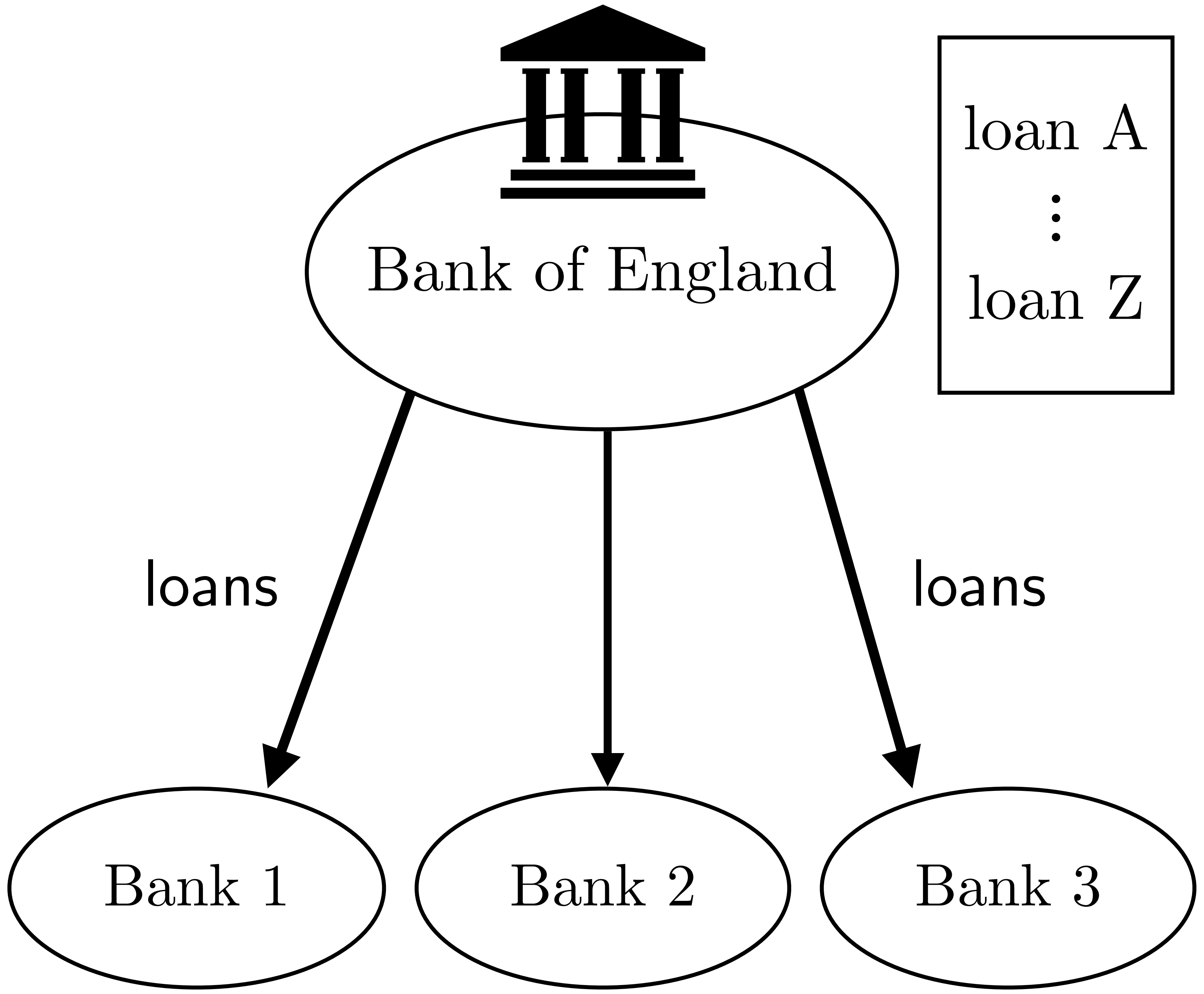
Question:
Which distribution of loans is best for the general economy of England? Fast way to decide?



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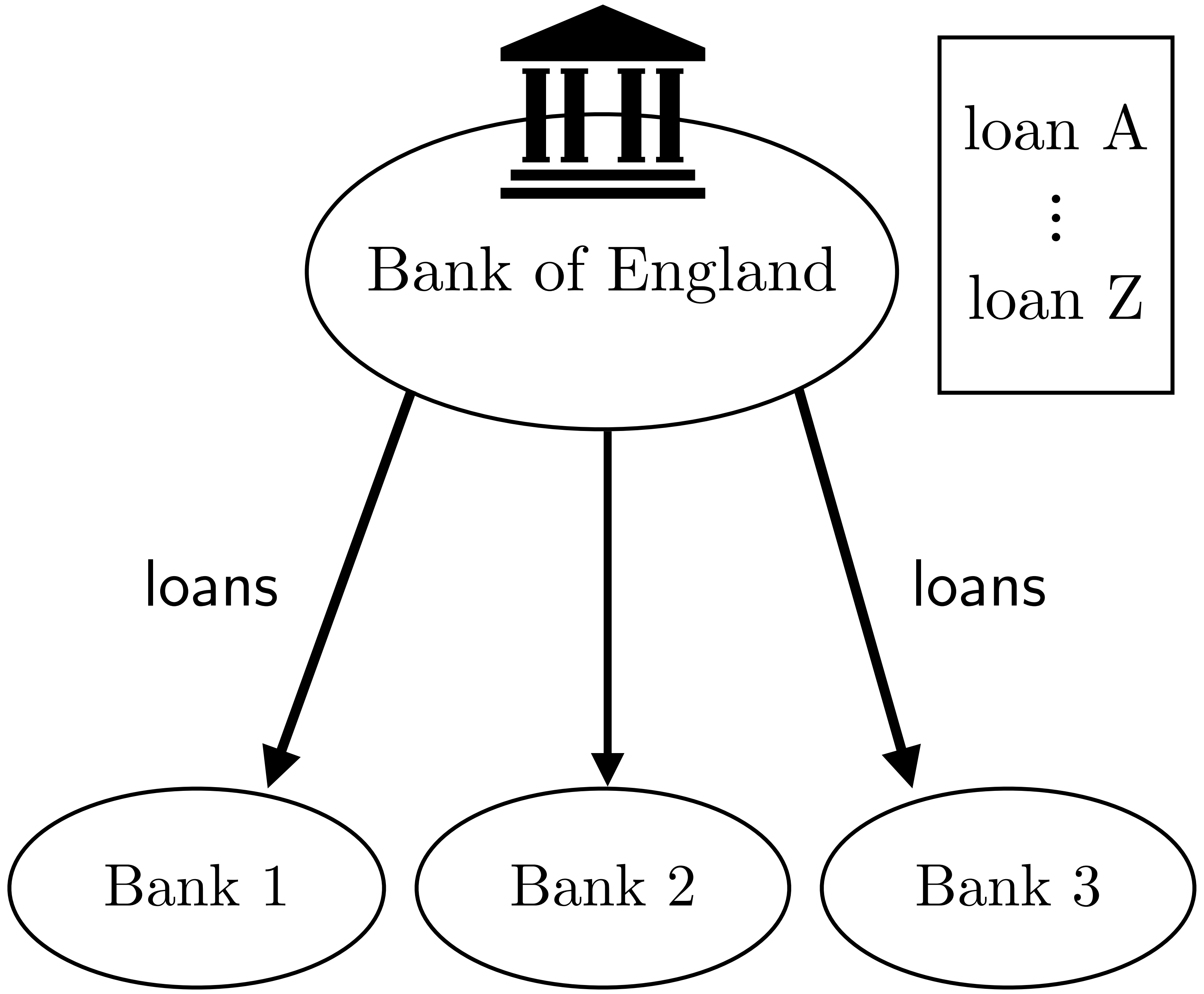

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[Baldwin-Klemperer, 2011]

1. Bidding round:
   Bidders tell the auctioneer (secretly, honestly) about their preferences.
2. Auctioneer sets price and decides a distribution of goods.

The graphical model and its polytope

[Candogan-Ozdaglar-Parillo ’18]

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$$(\chi_{G'})_i = \begin{cases} 1 & \text{if } i \in V(G') \\ 0 & \text{if } i \notin V(G') \end{cases} \qquad (\chi_{G'})_{ij} = \begin{cases} 1 & \text{if } ij \in E(G') \\ 0 & \text{if } ij \notin E(G') \end{cases}$$

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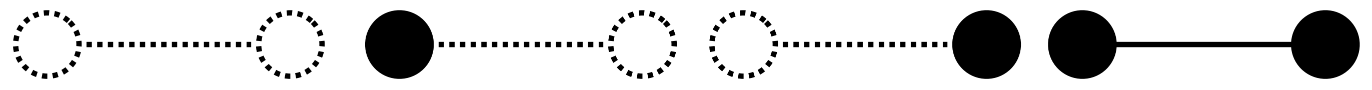

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$$P(G) = \mathrm{conv}(\chi_{G'} \mid G' \subseteq G \text{ induced})$$

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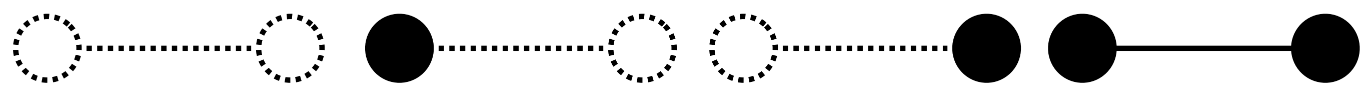

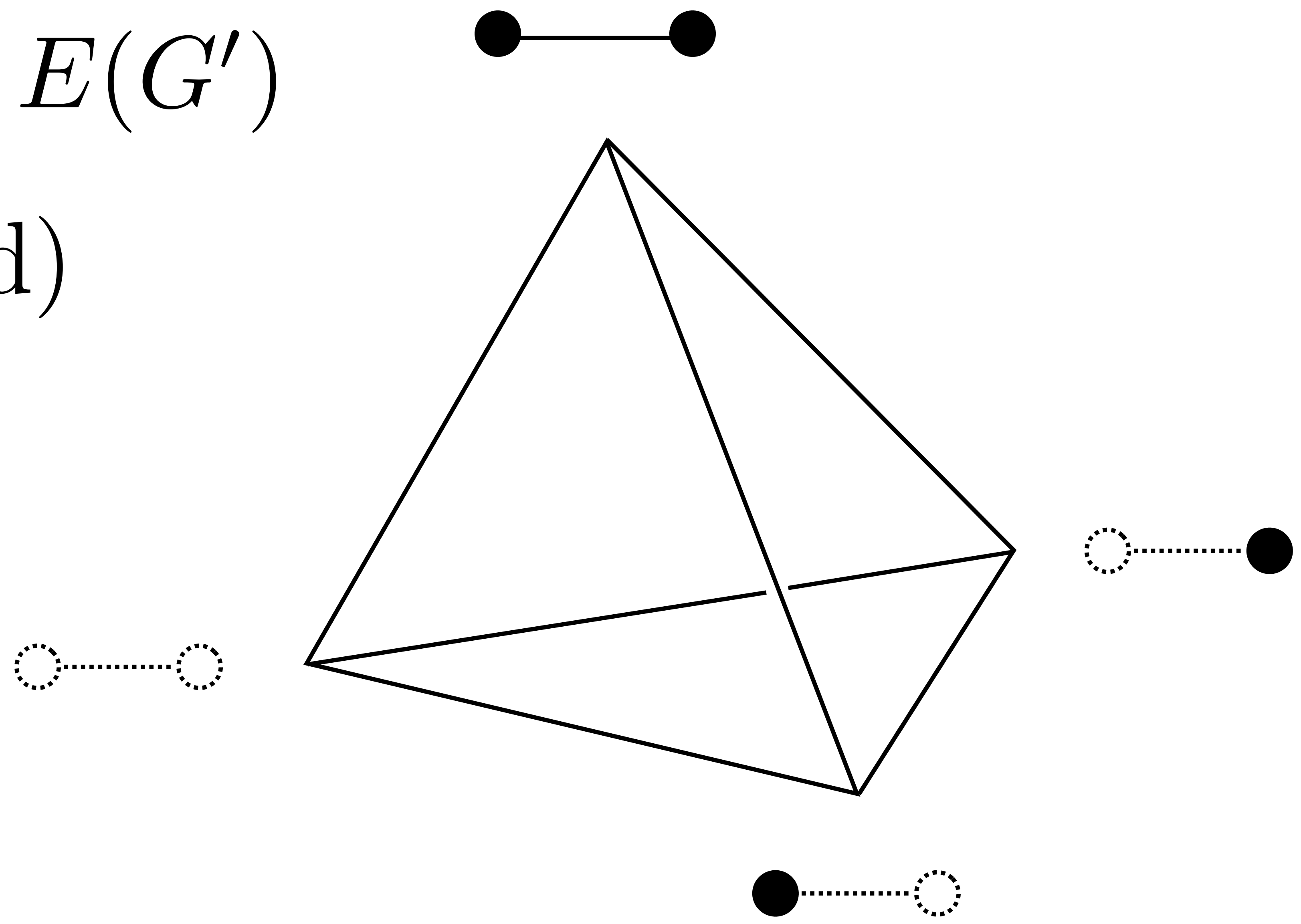

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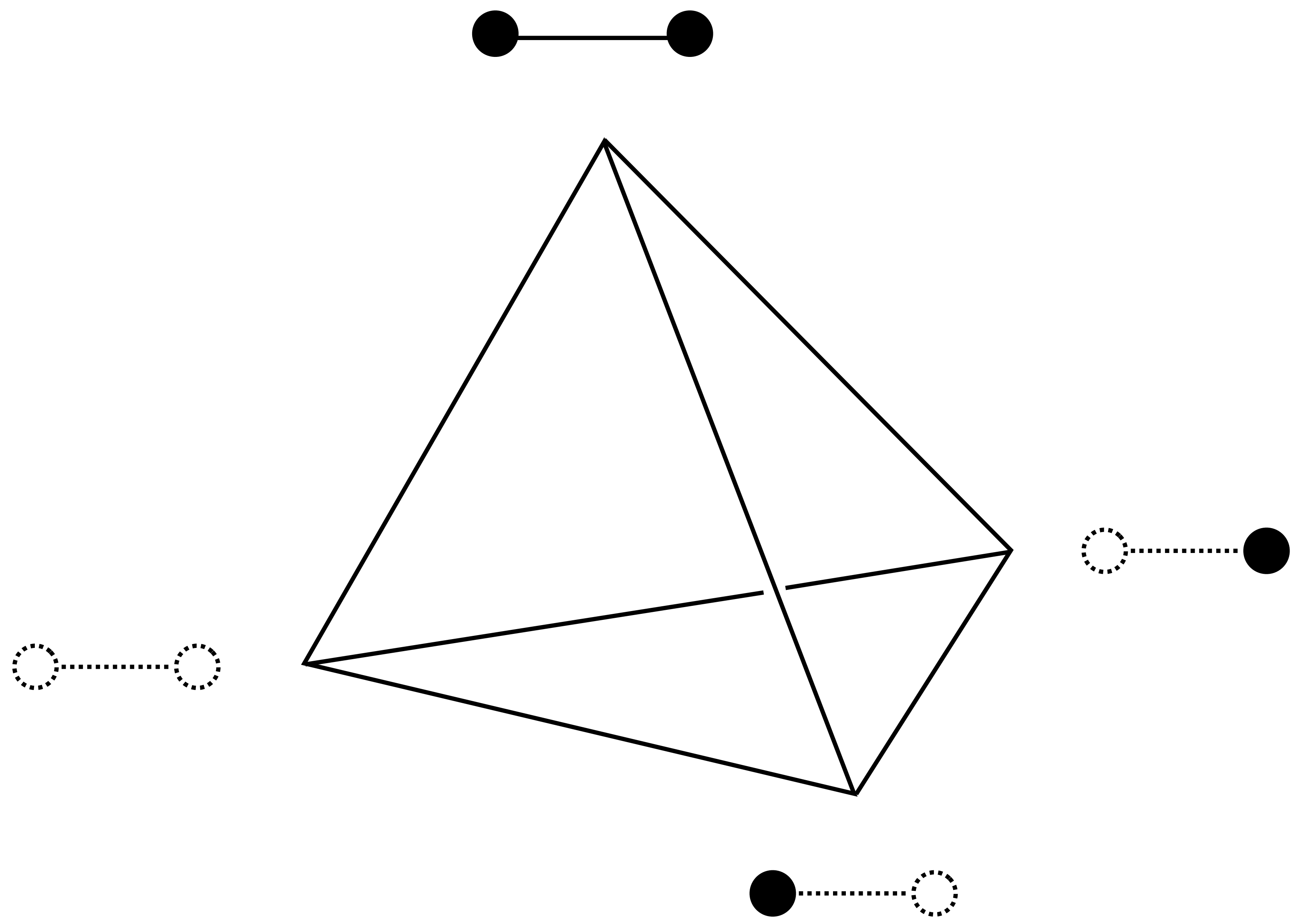

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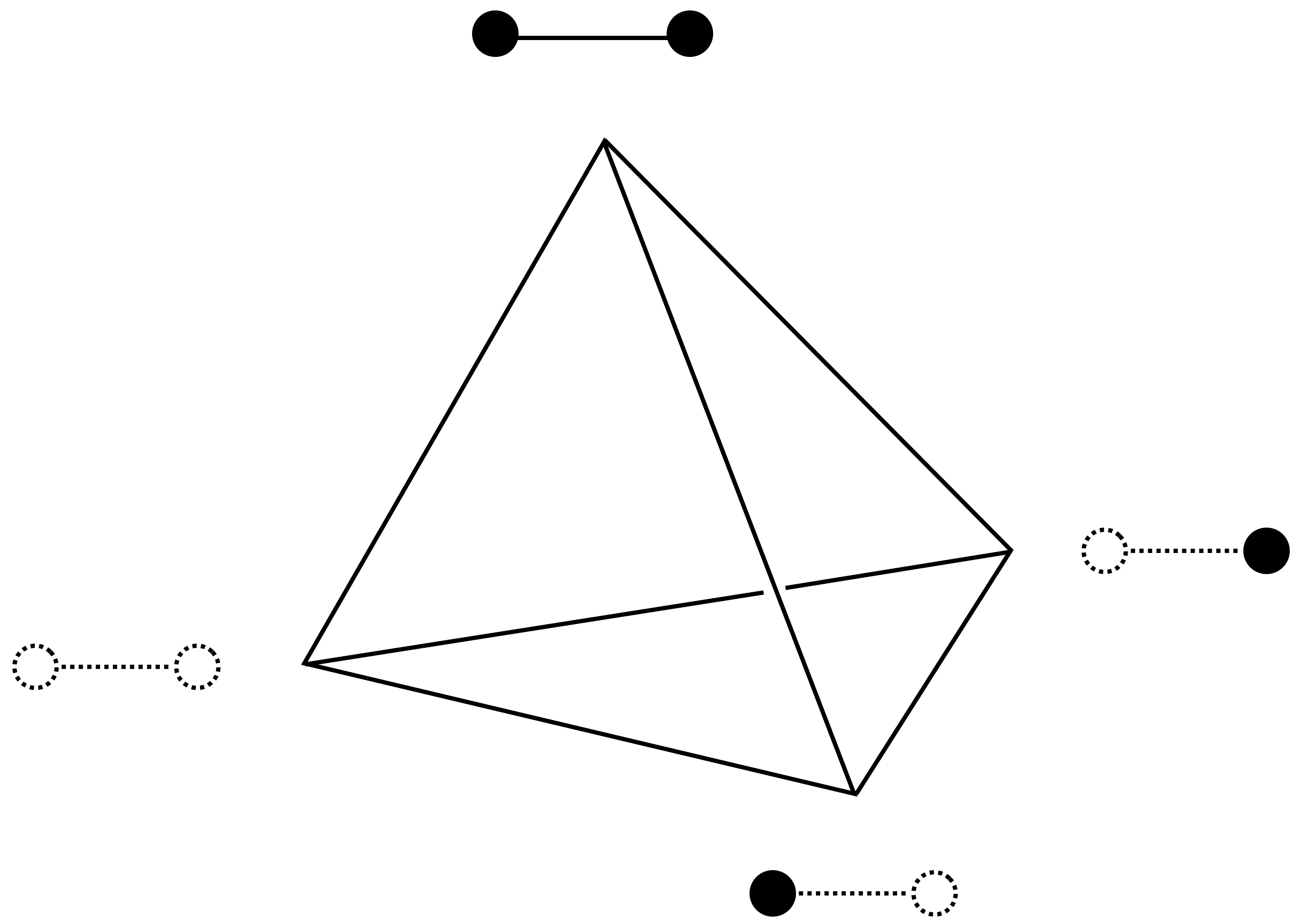
Bidder $b \in [m]$ communicates preferences to auctioneer



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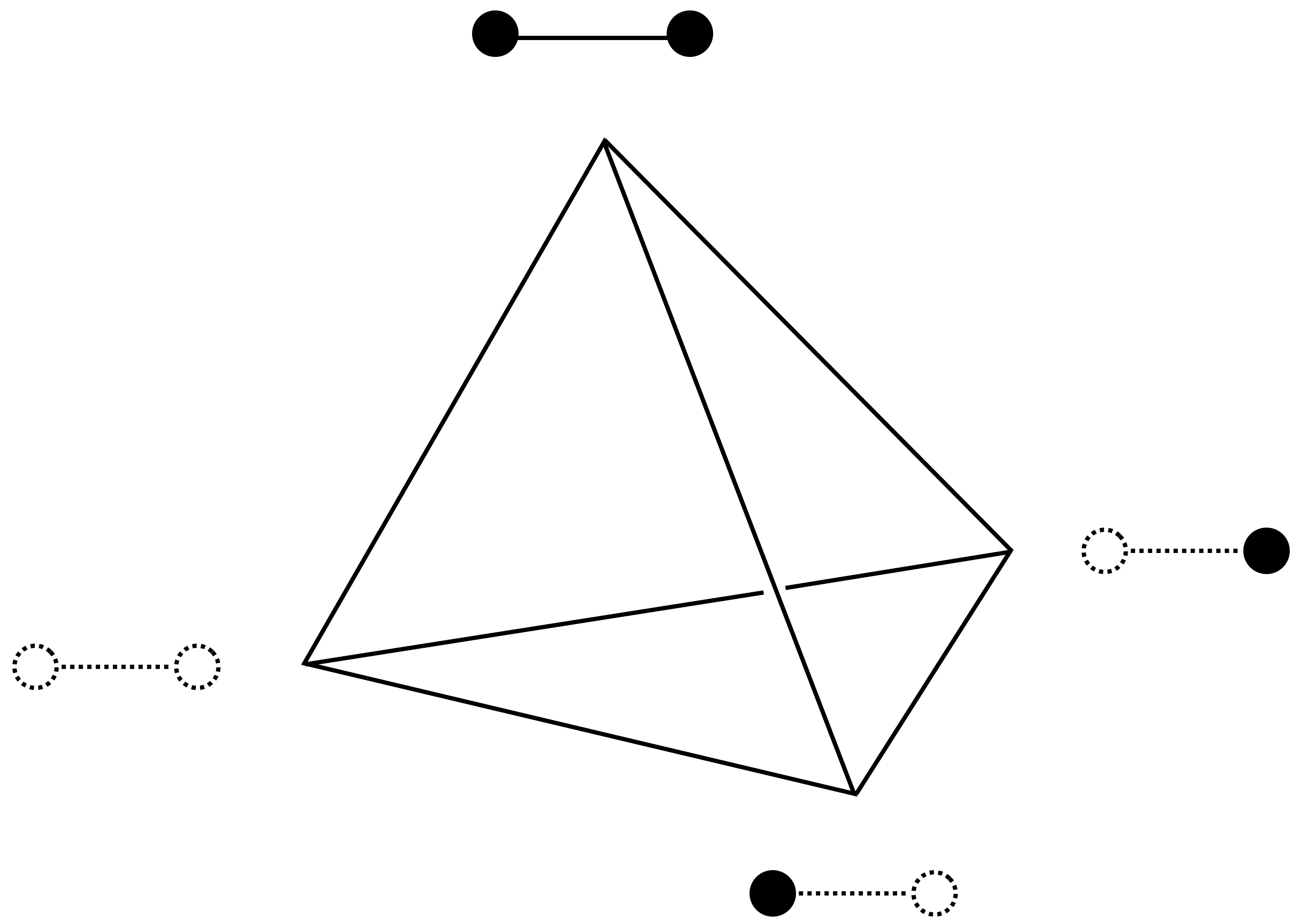
Valuation function $v^b : P \cap \mathbb{Z}^{n+|E|} \to \mathbb{R}$, $v^b(a) = \langle w^b, a \rangle$ for some $w^b \in \mathbb{R}^{n+|E|}$



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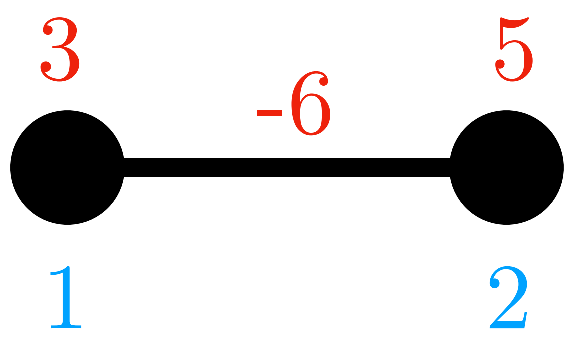

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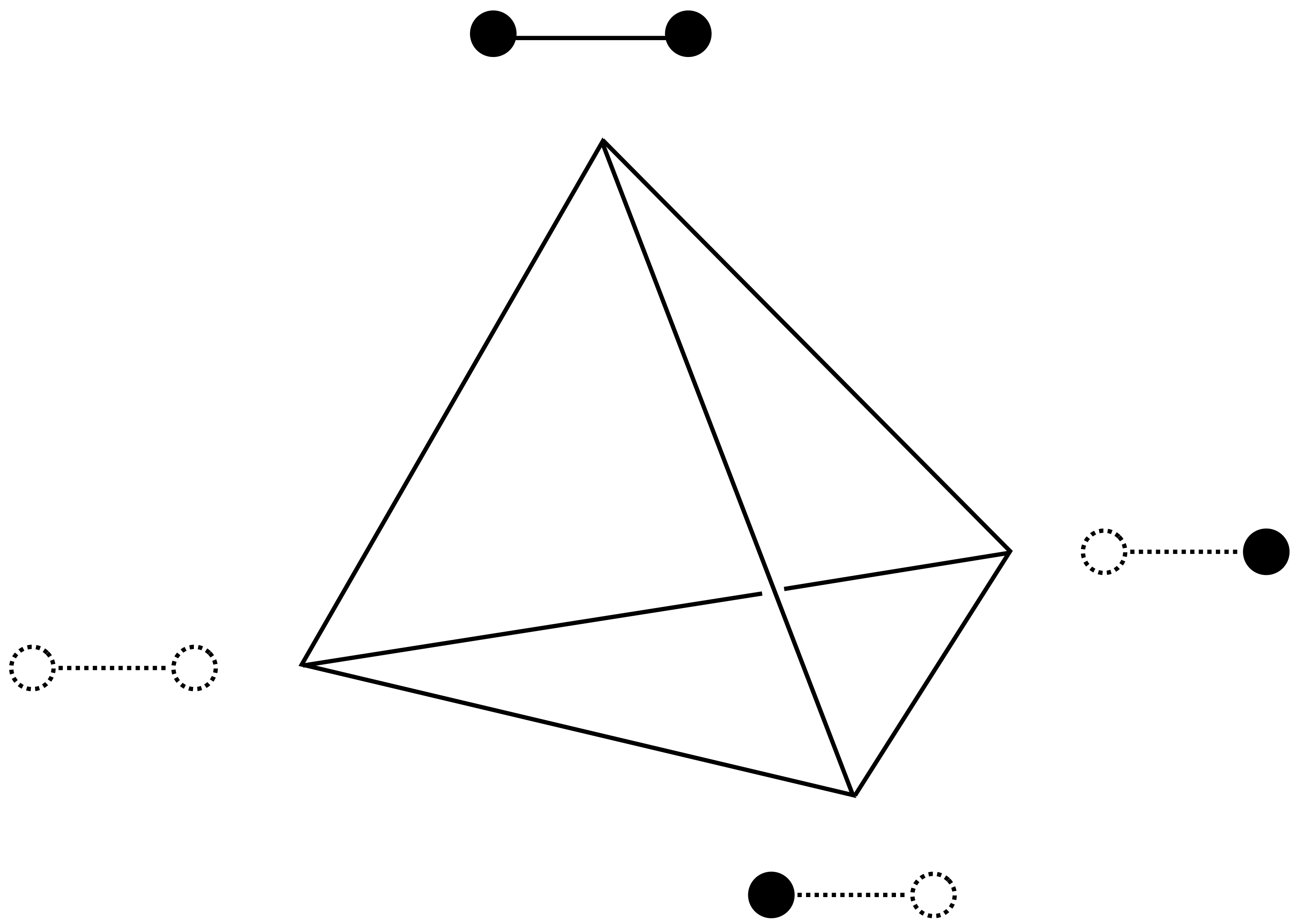

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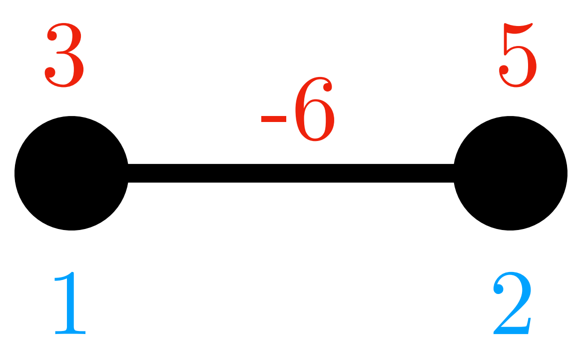

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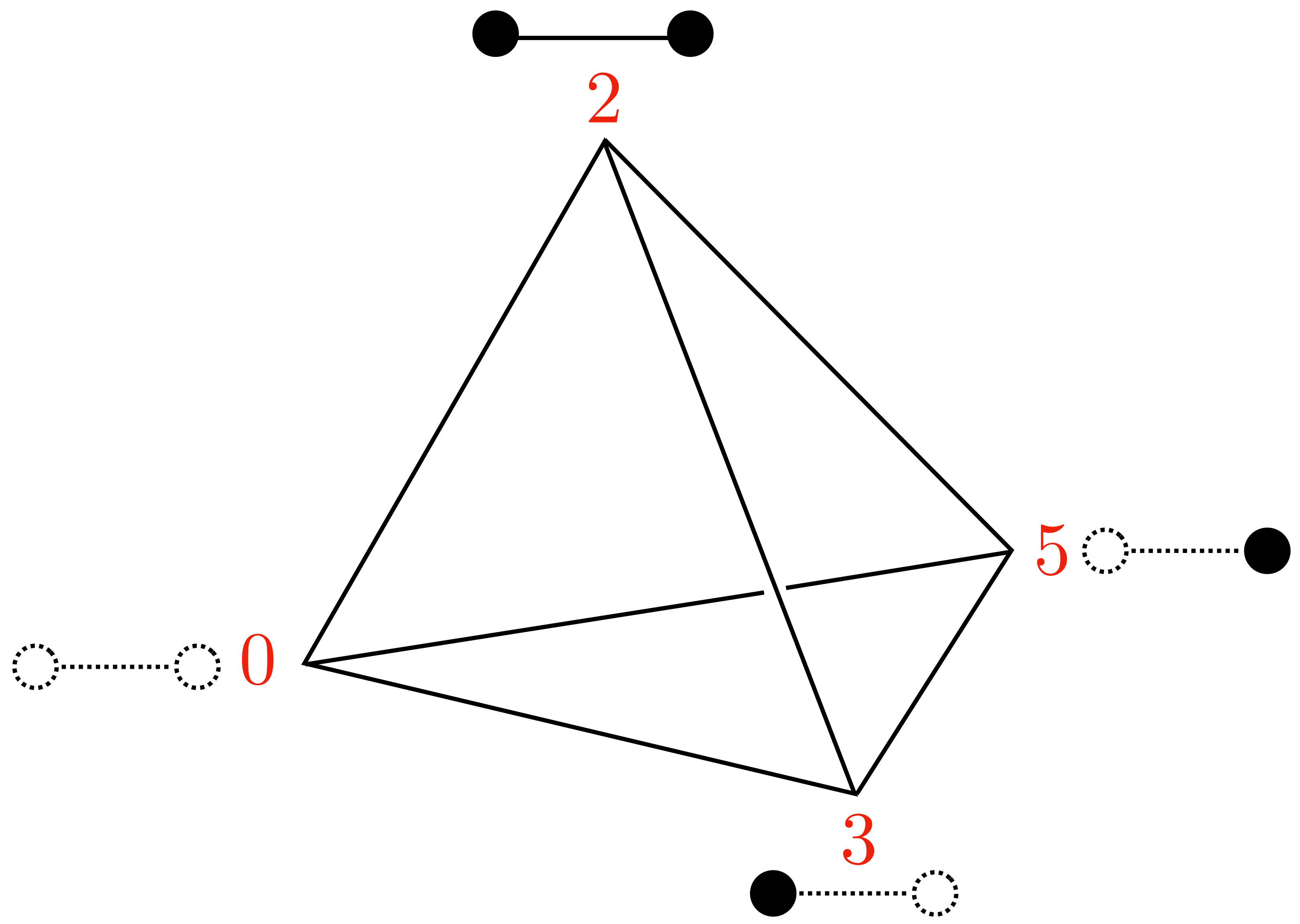

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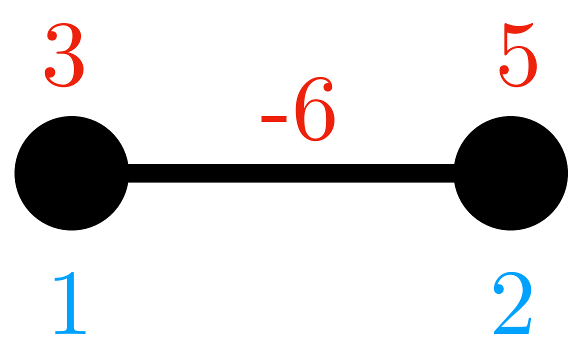

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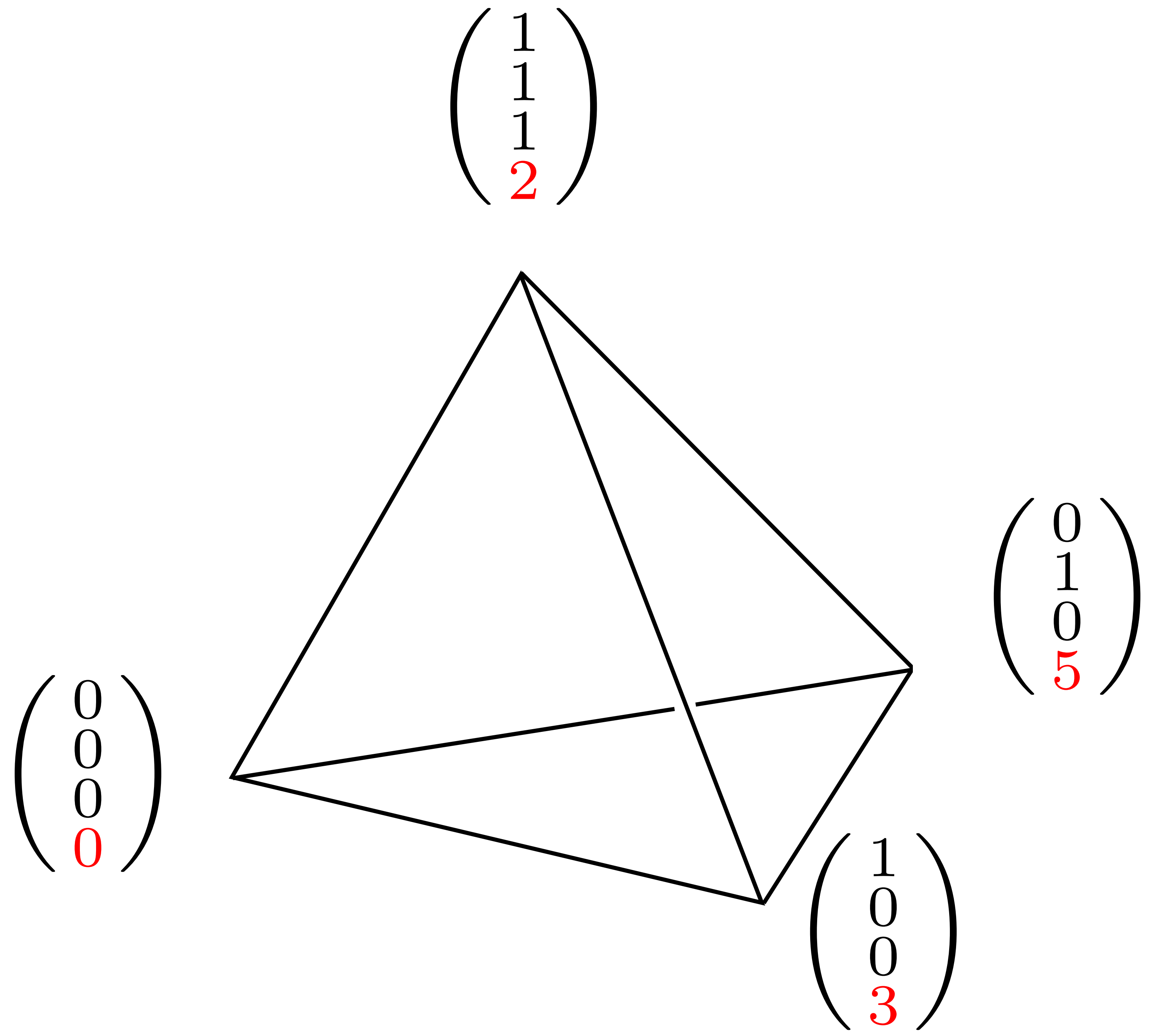
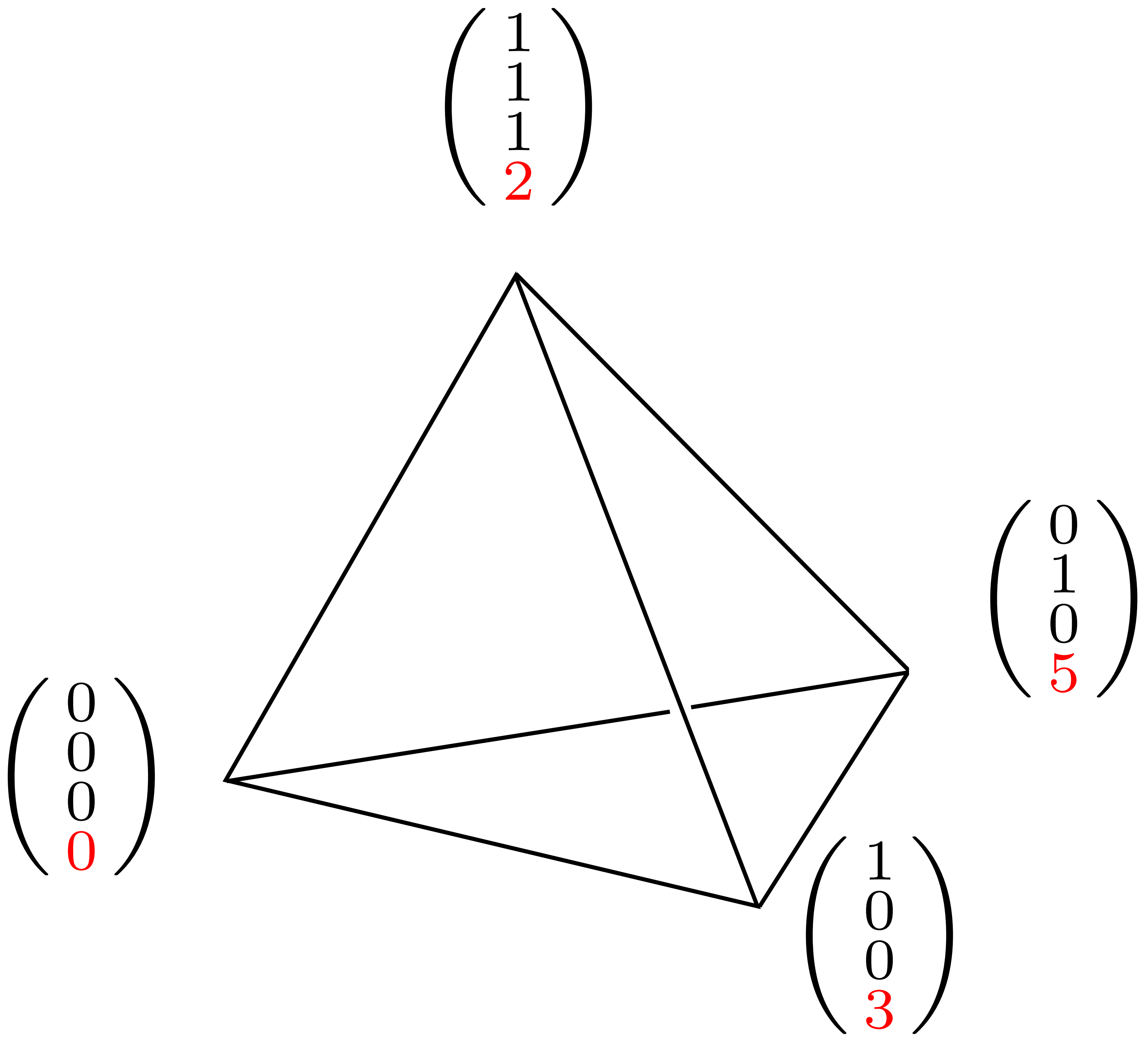


2. Auctioneer's decision

Auctioneer sets a price



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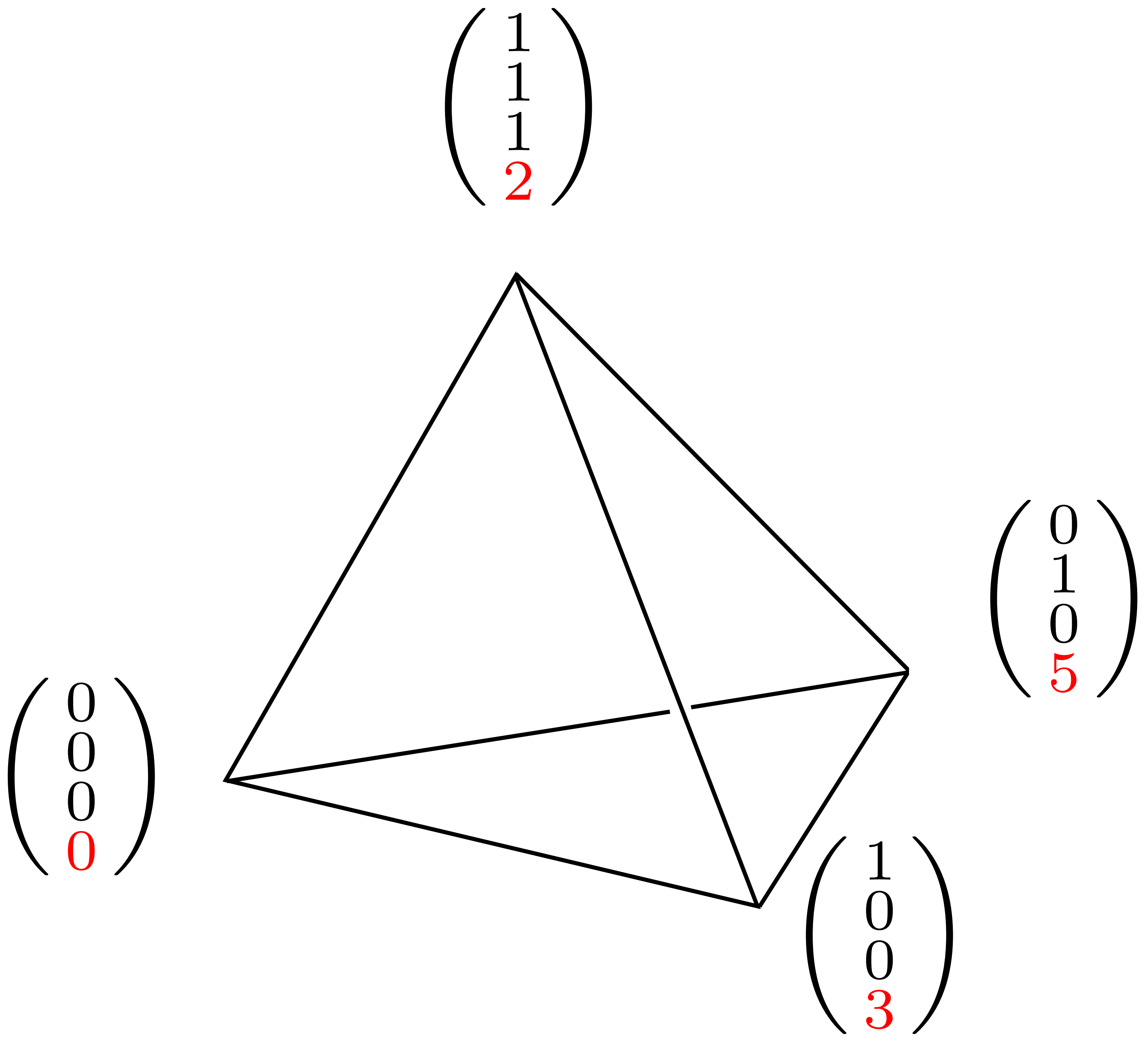

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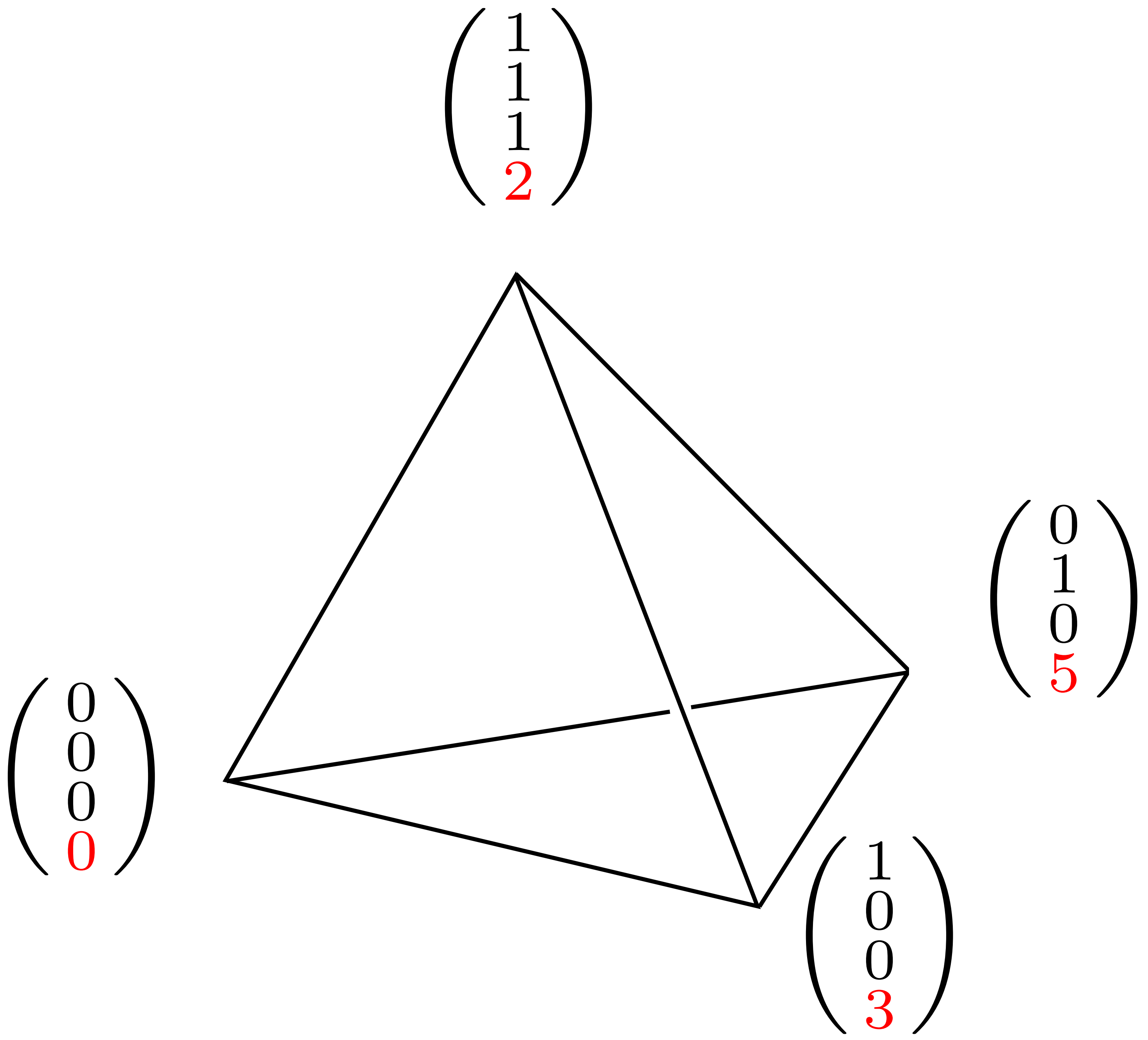

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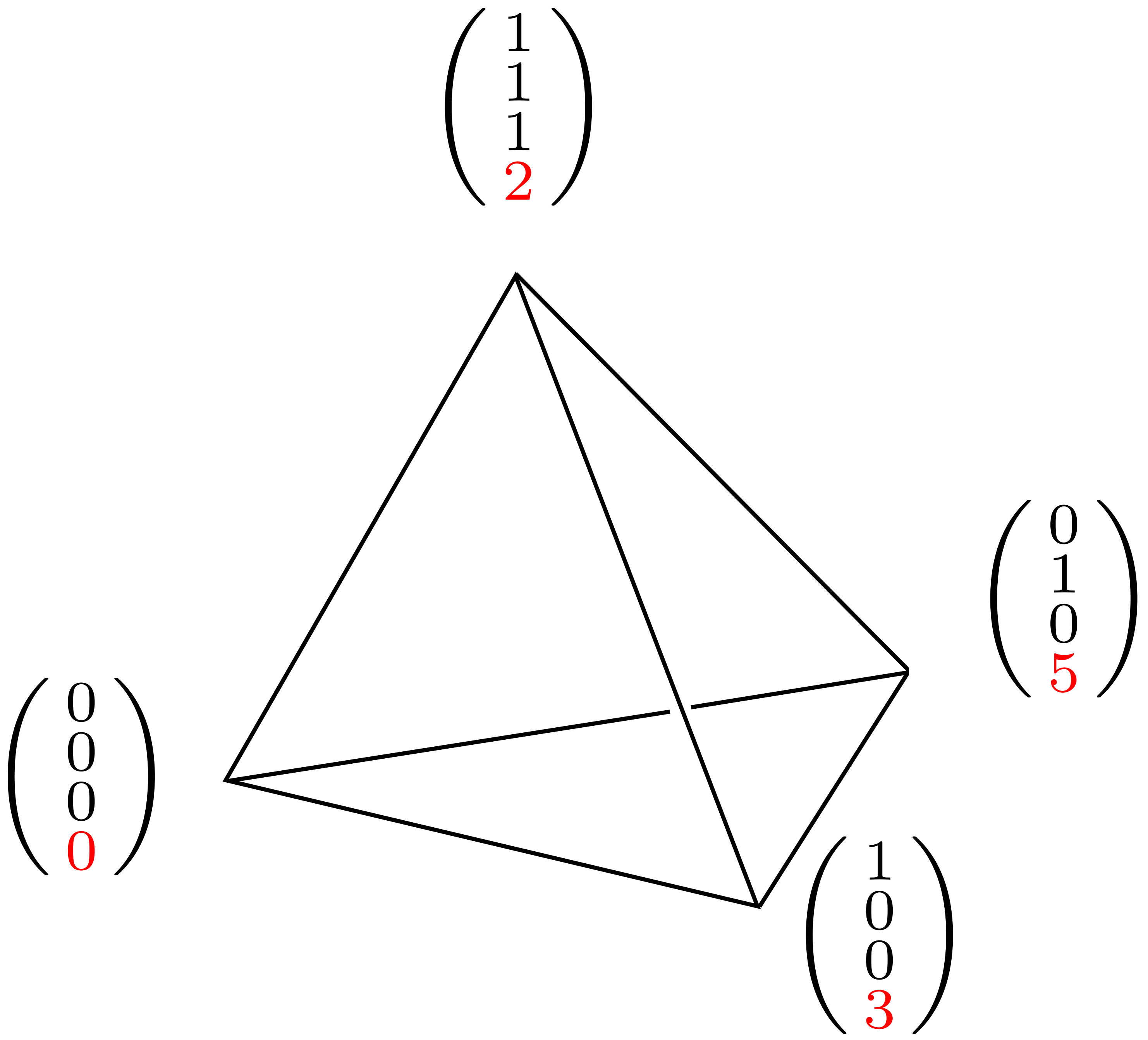
$v^b(a) = \langle w^b, a \rangle$	0	3	5	2



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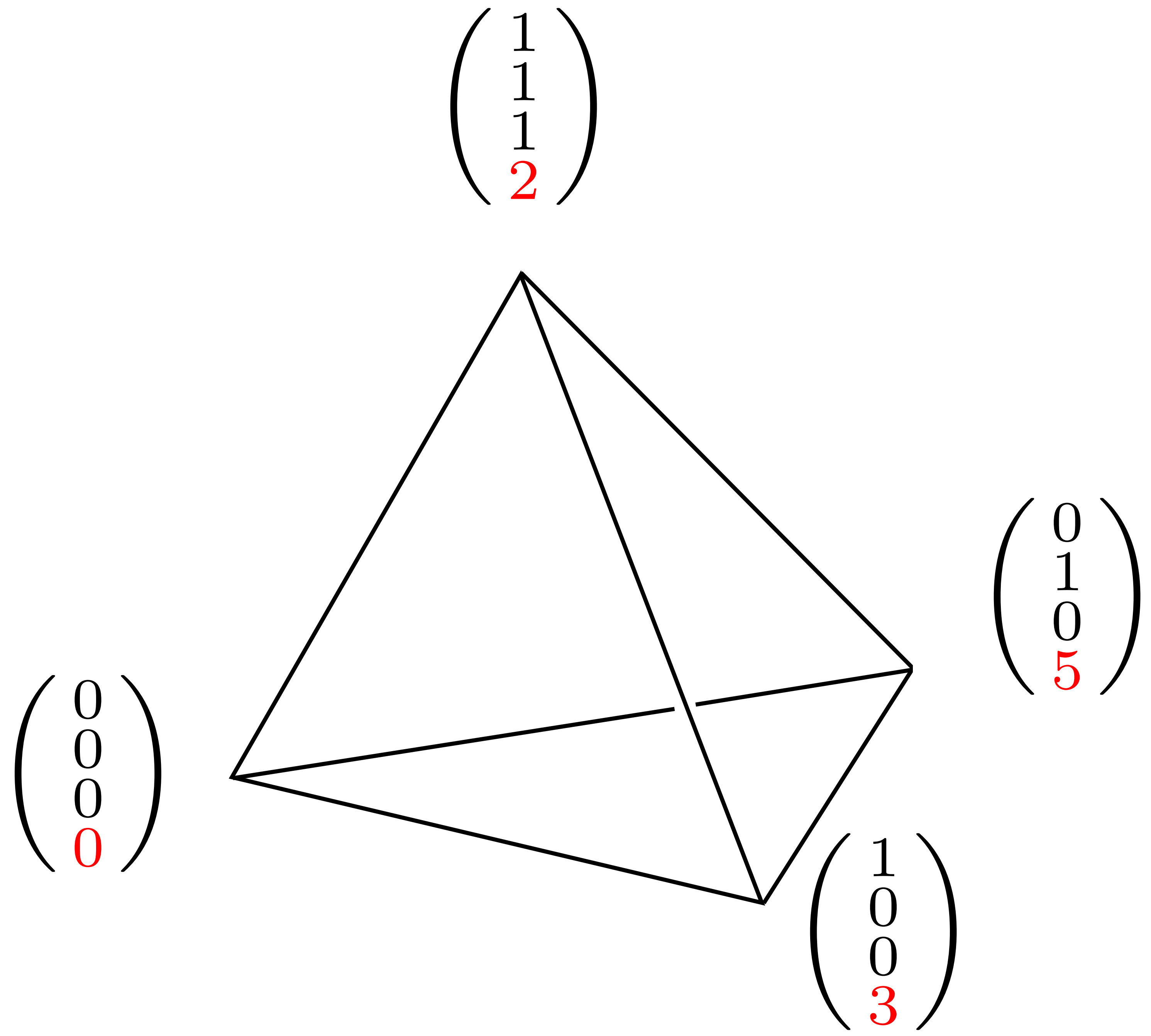

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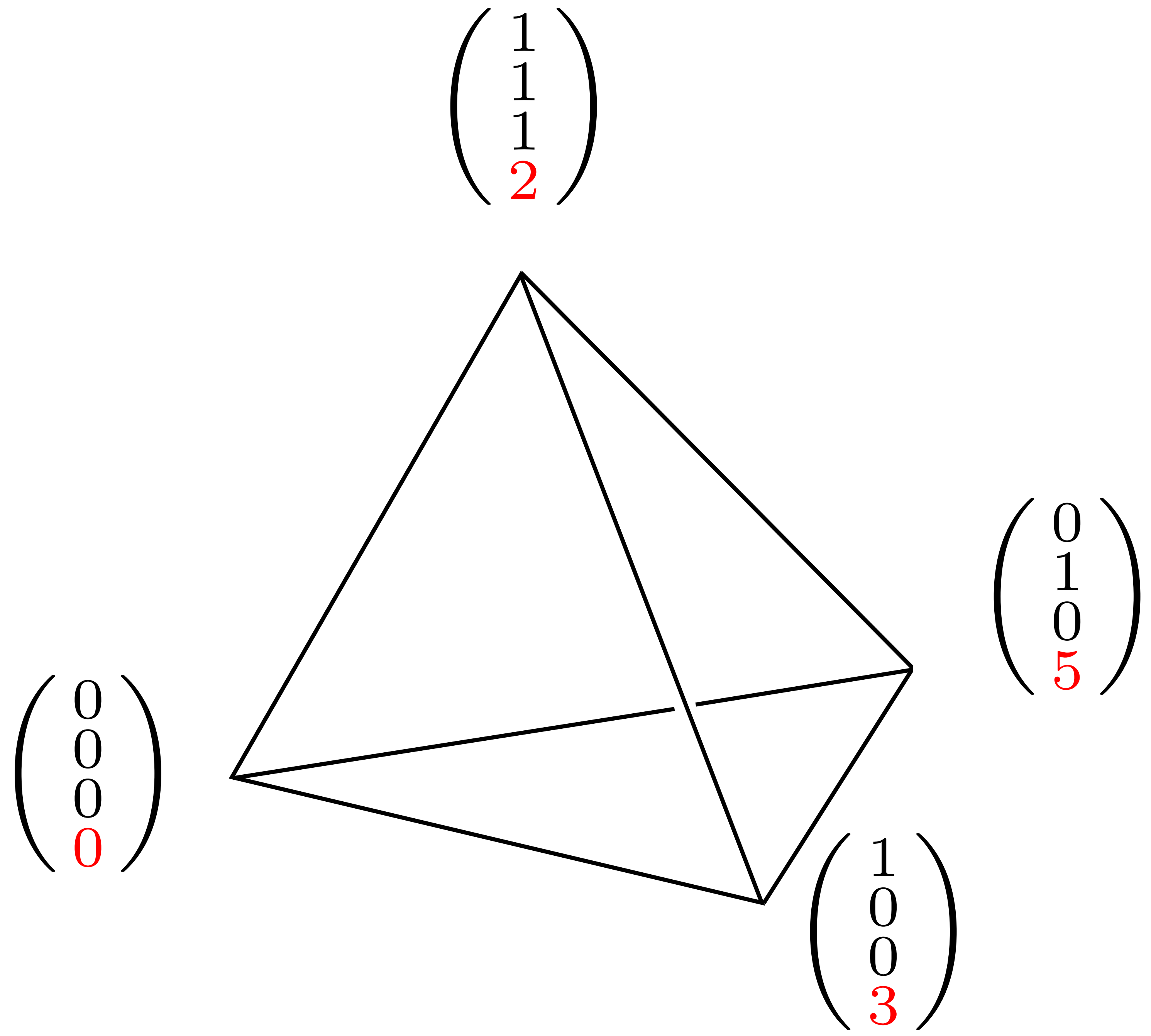

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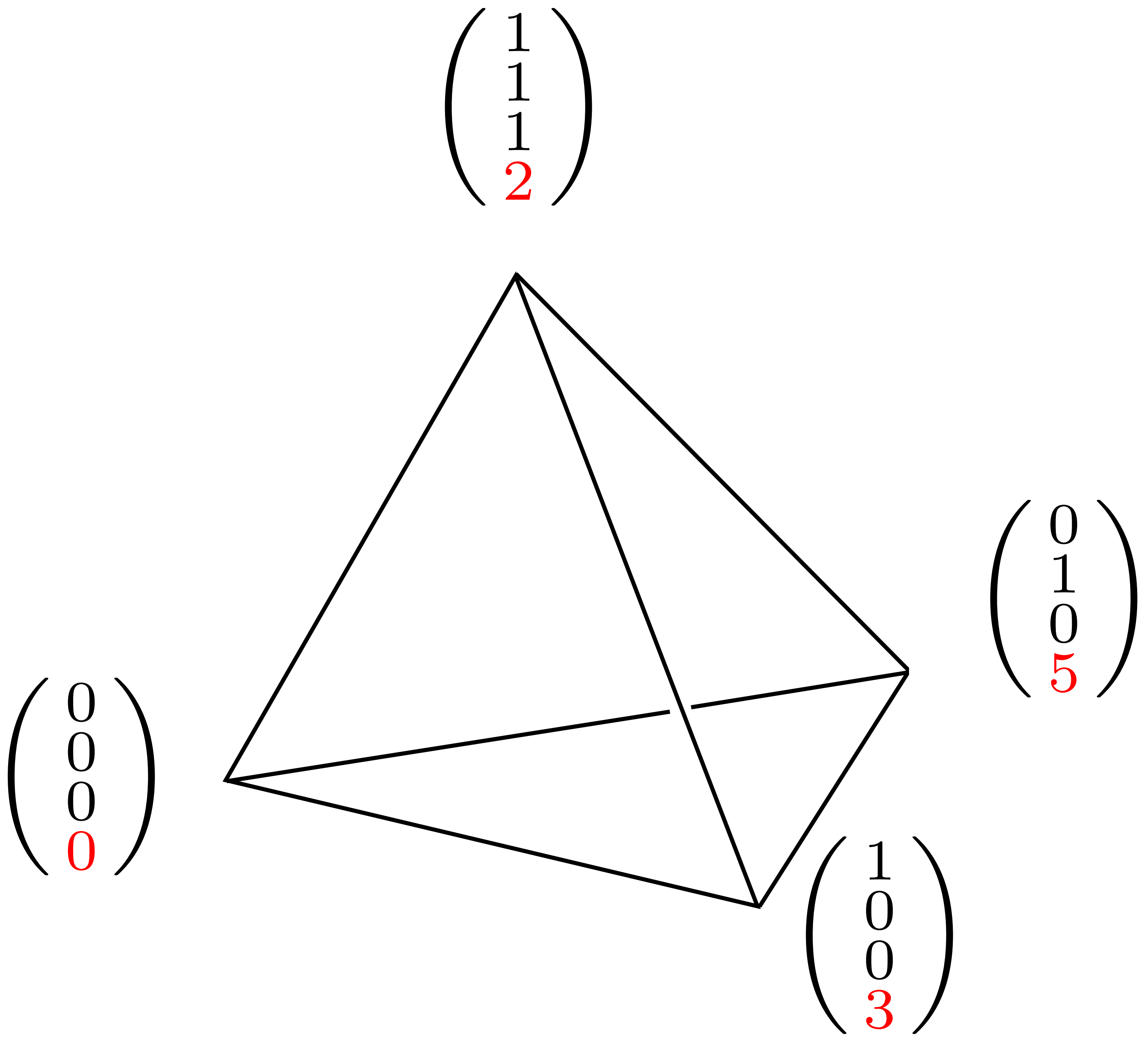

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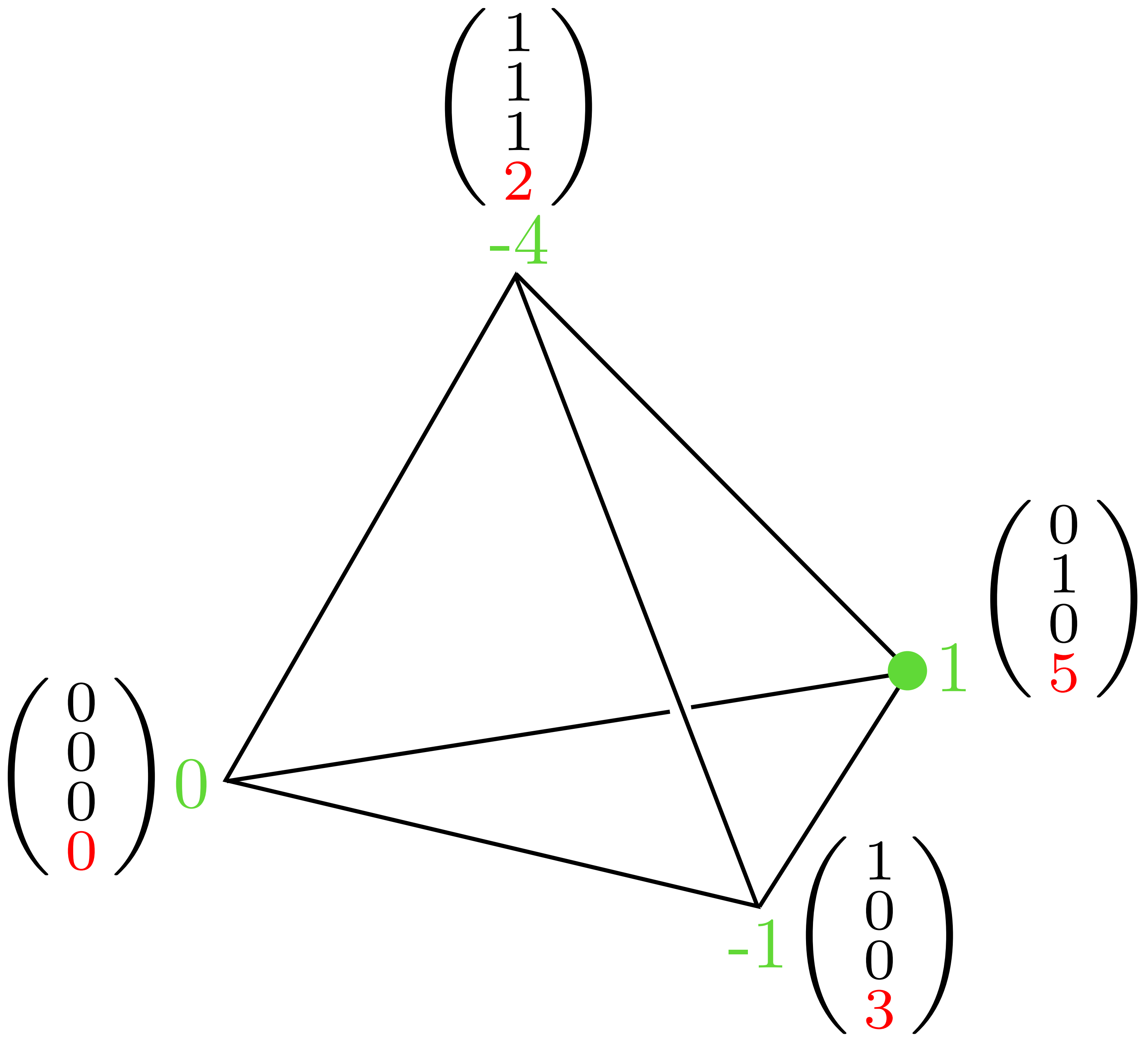

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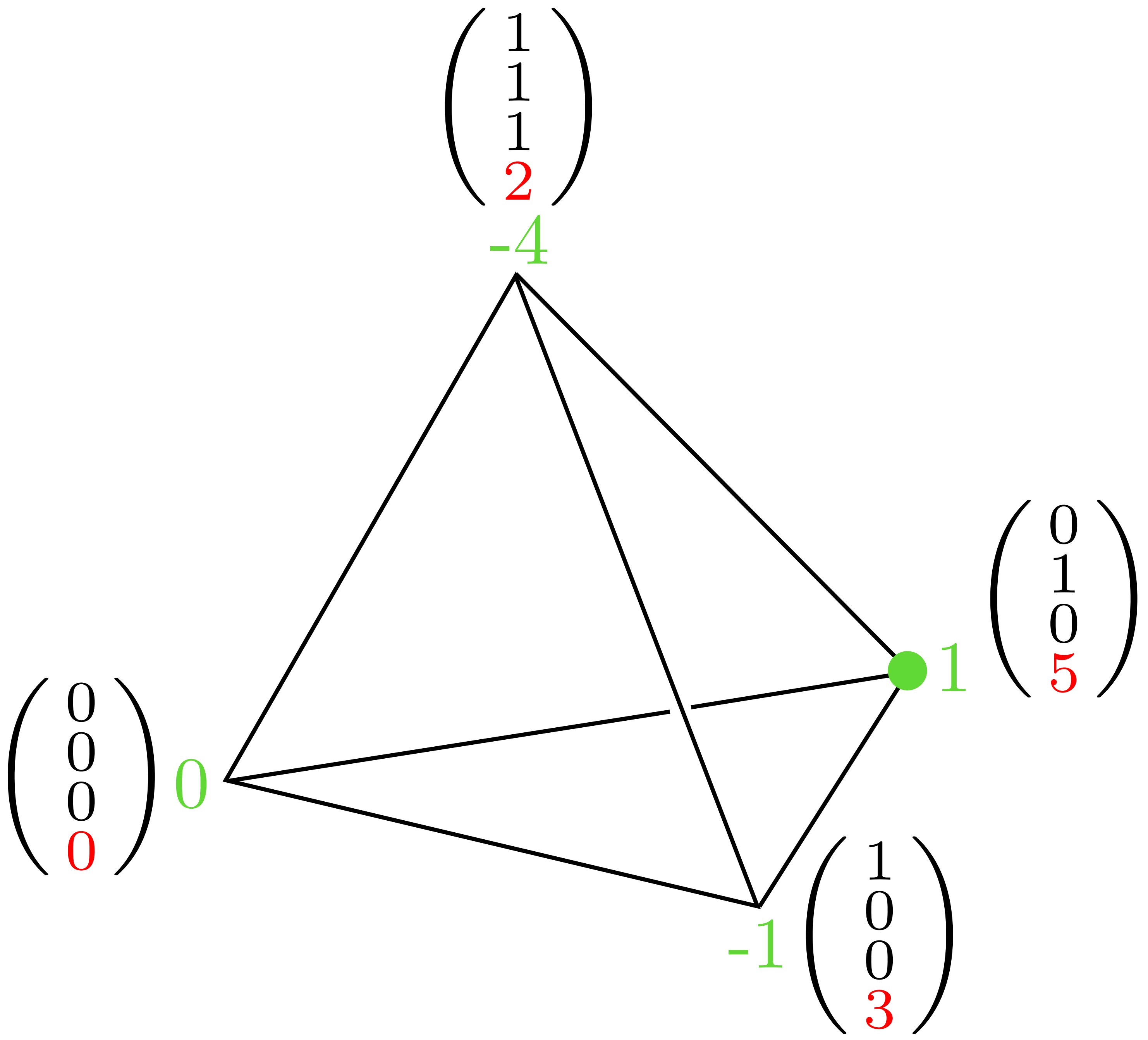

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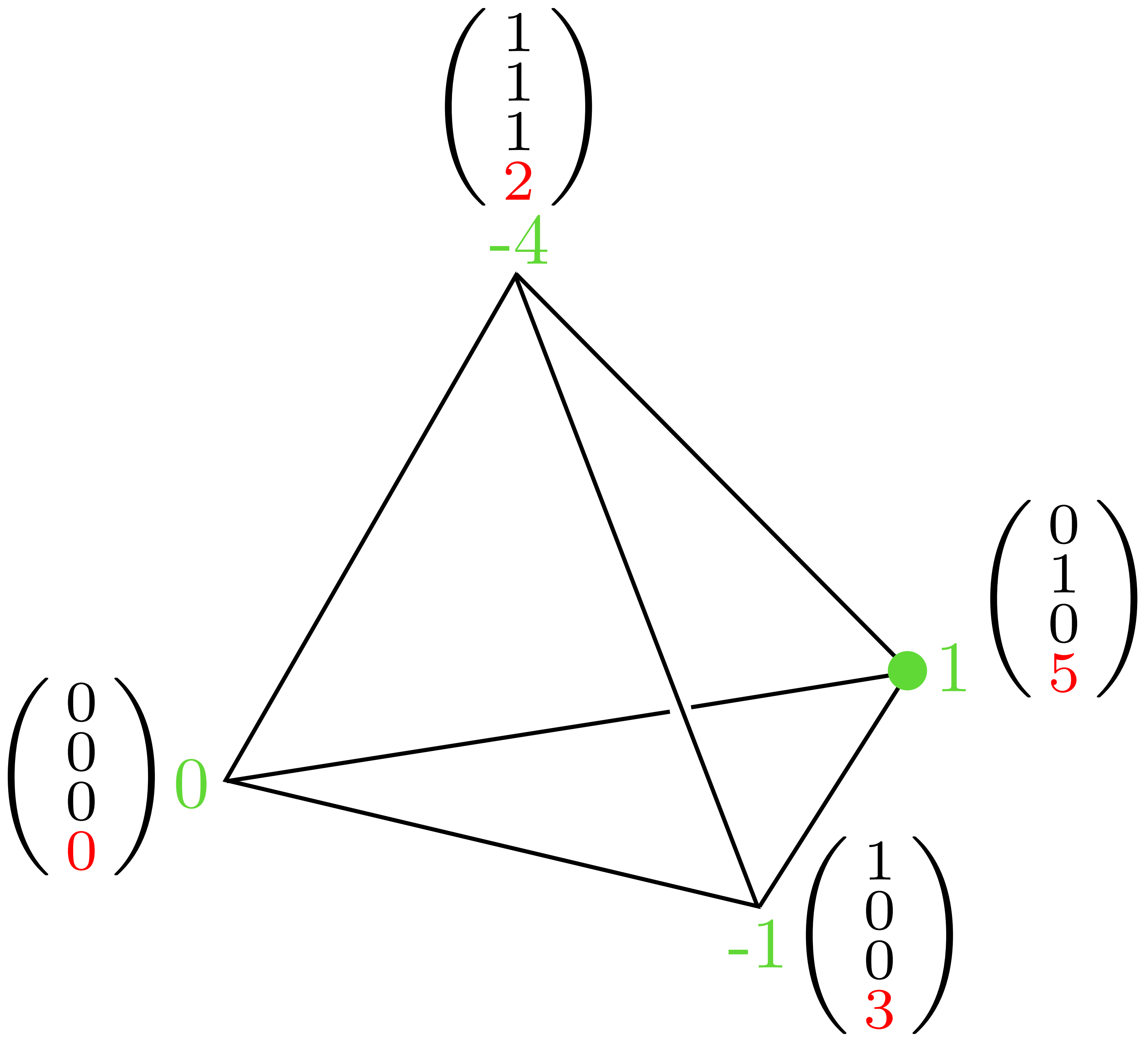

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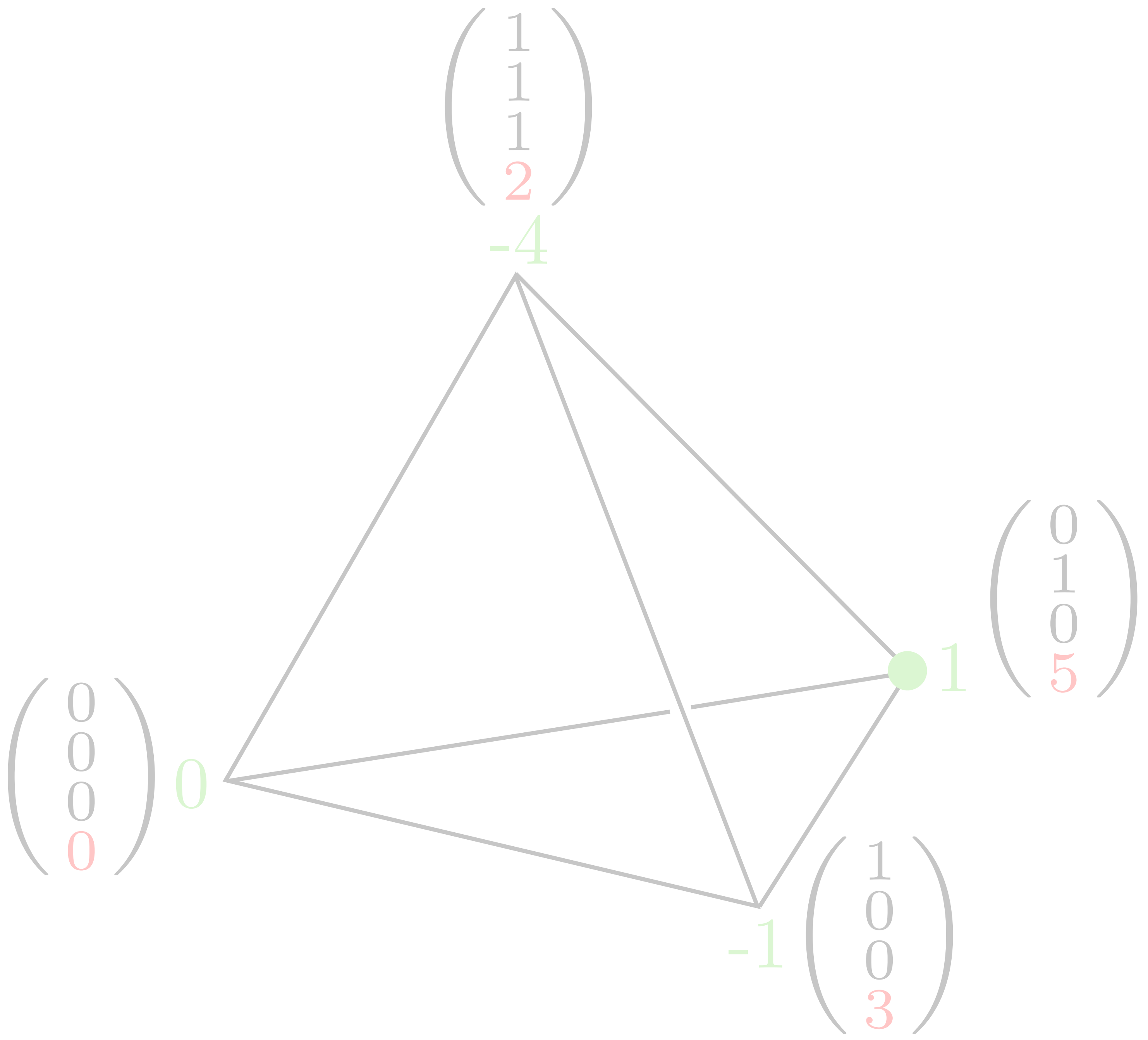

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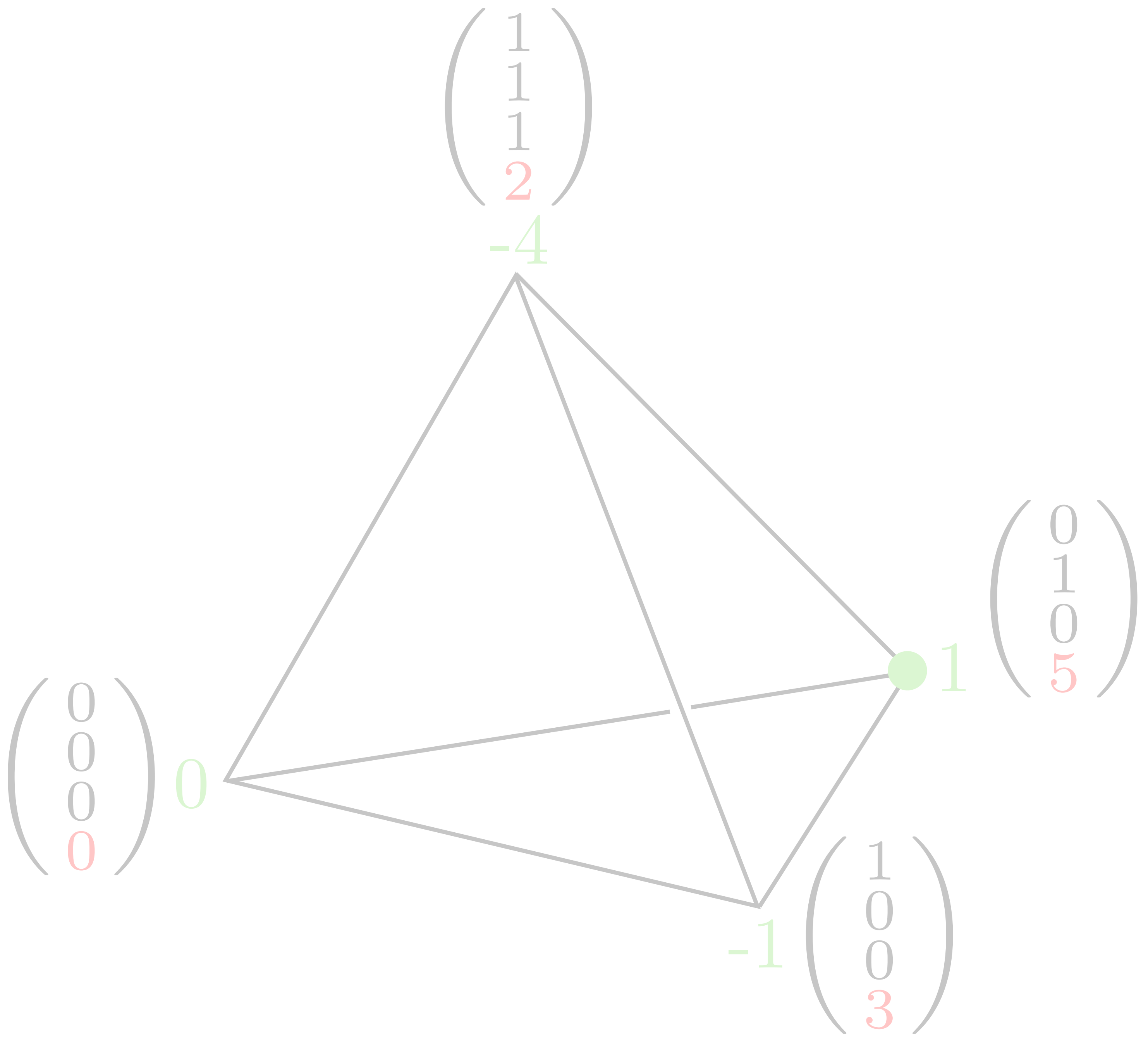

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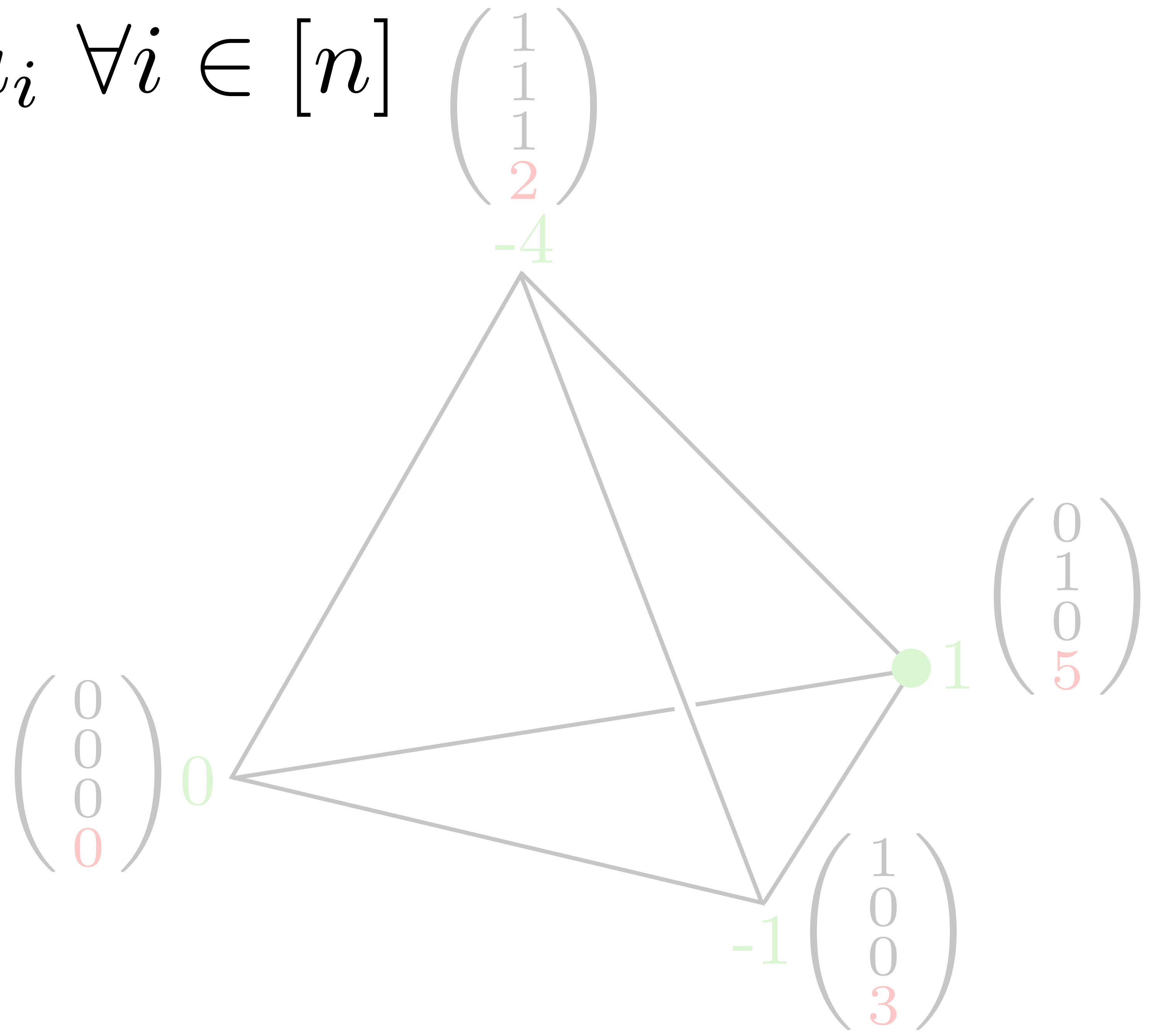

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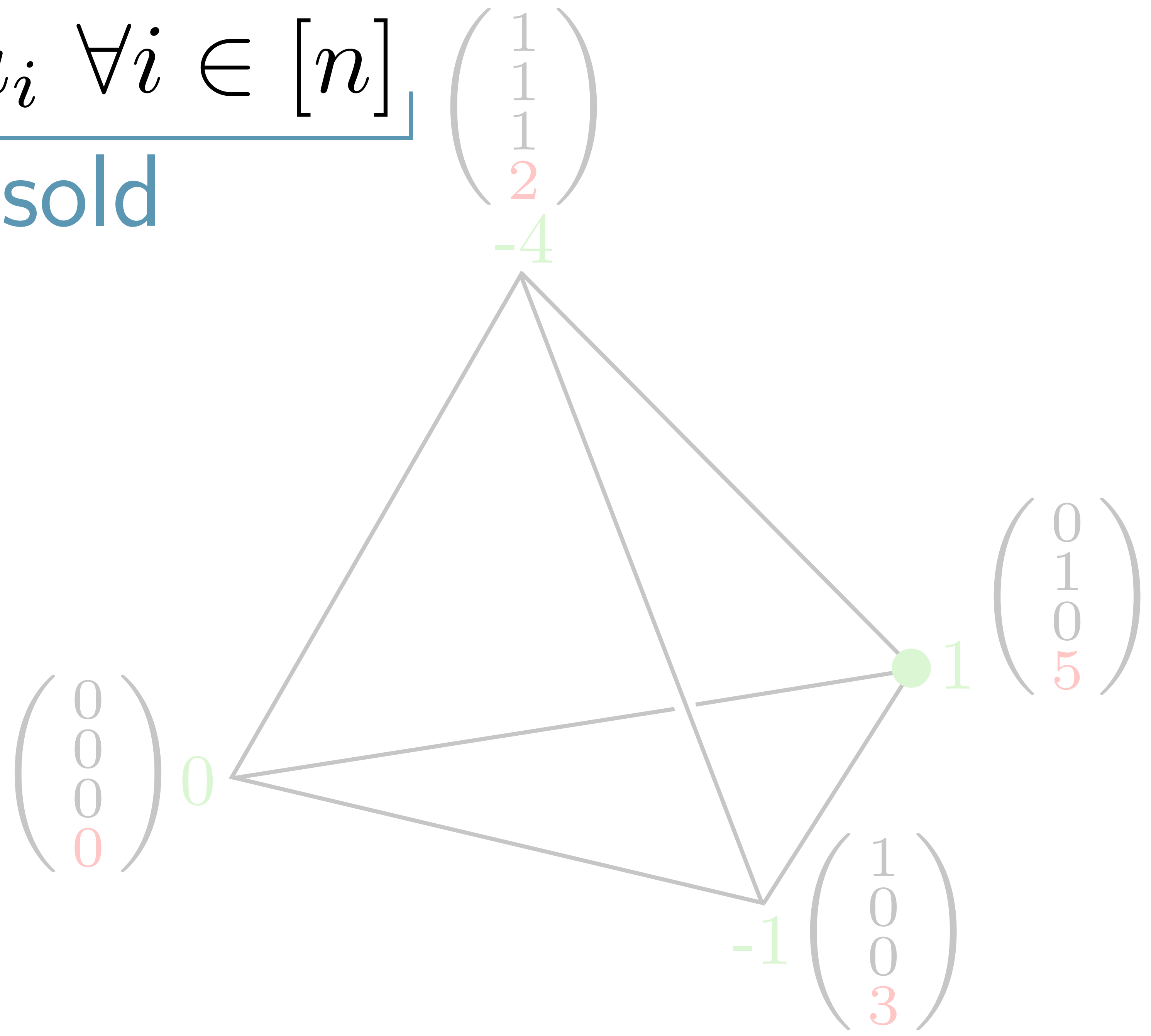

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A competitive equilibrium is *guaranteed to exist* if for any set of valuations $\{v^b \mid b \in [m]\}$ there exists $p \in \mathbb{R}^{n+|E|}, a \in \sum_{b\in[m]} D(v^b, p)$ such that $a \in \pi^{-1}(a^*)$.

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In particular, then a CE is guaranteed to exist.

Mixed regular subdivisions

Aggregate valuation function:

$$V(a) = \max\{\sum_{b\in[m]} v^b(a^b) \mid a^b \in P(G)\cap\mathbb{Z}^d, \sum_{b\in[m]} a^b = a\}$$

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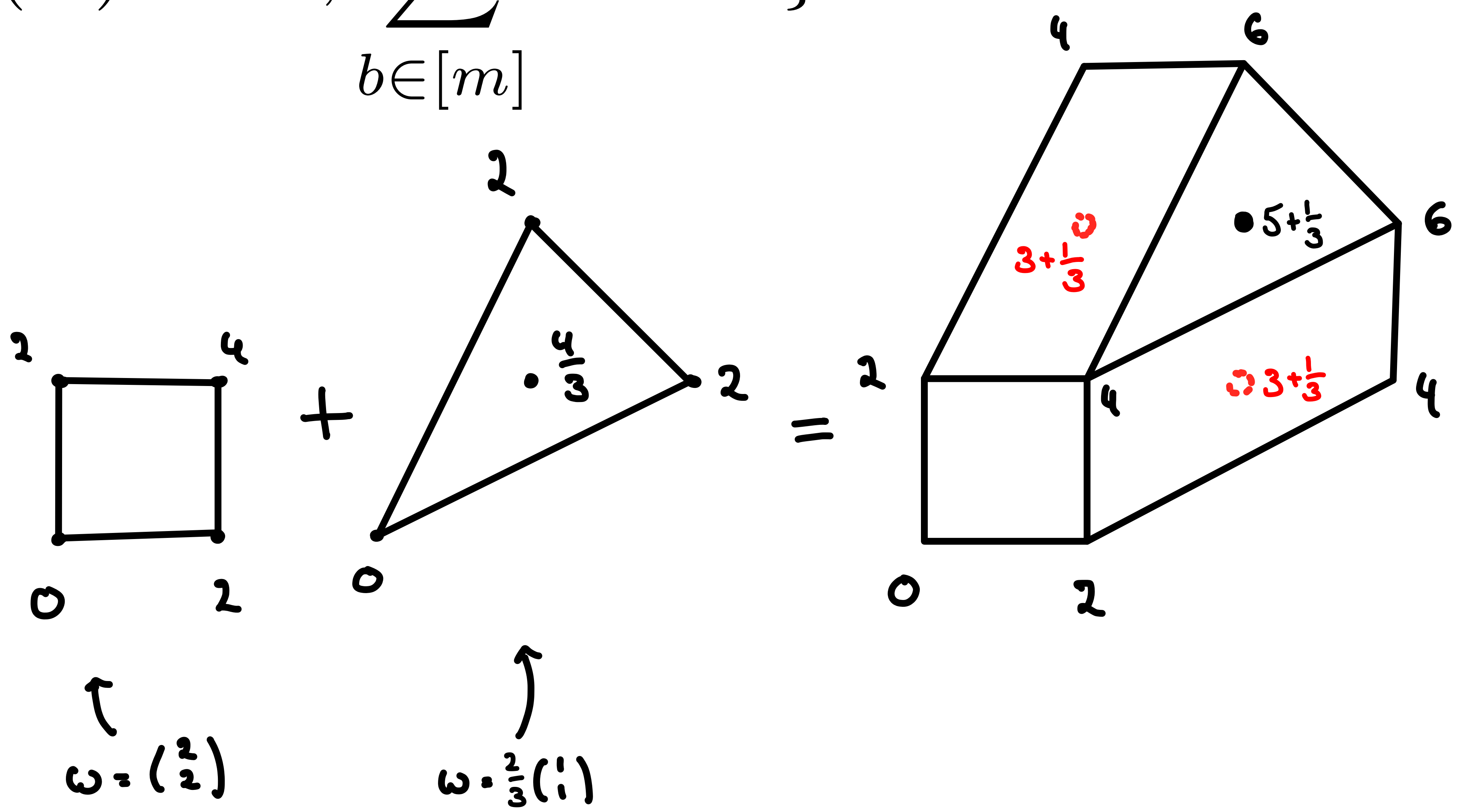

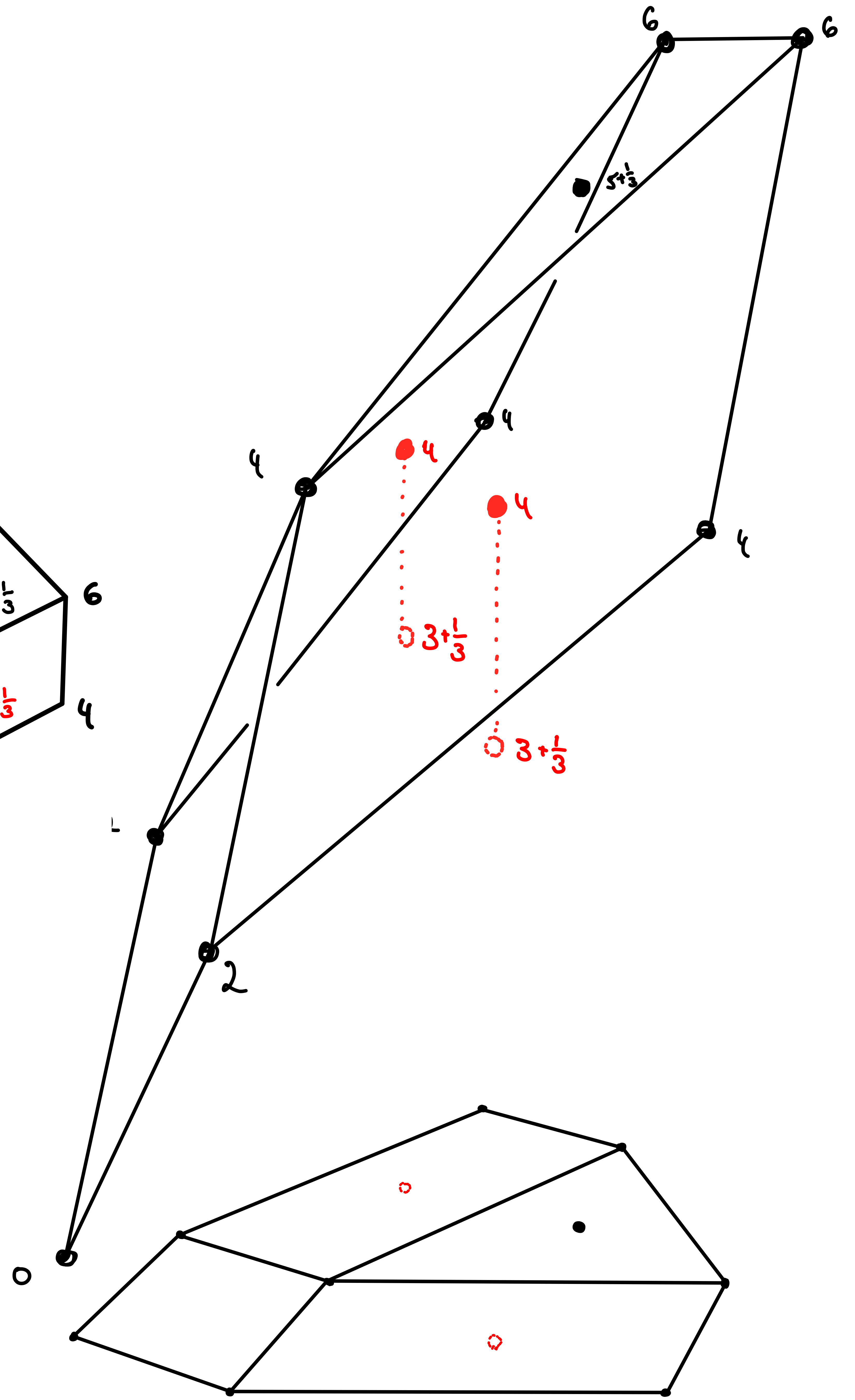

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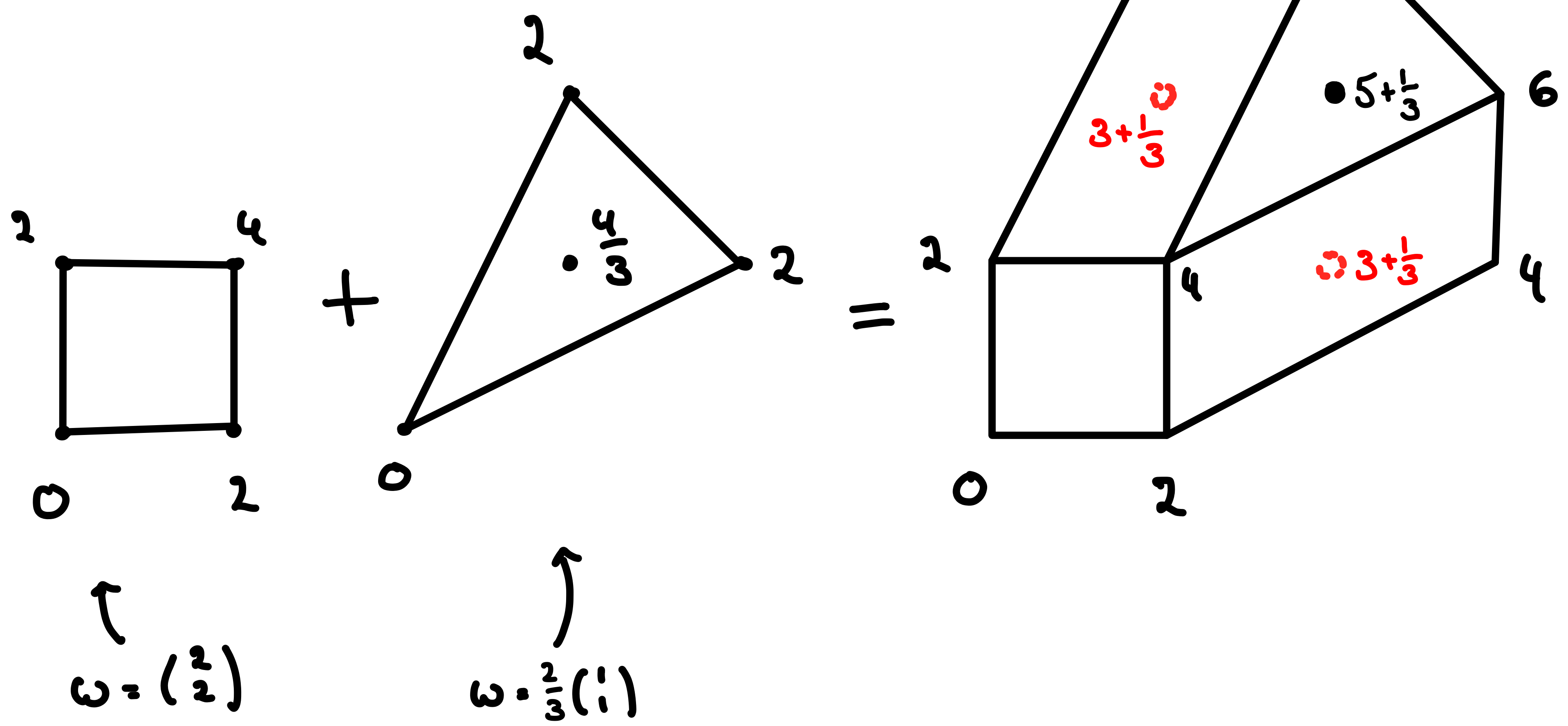

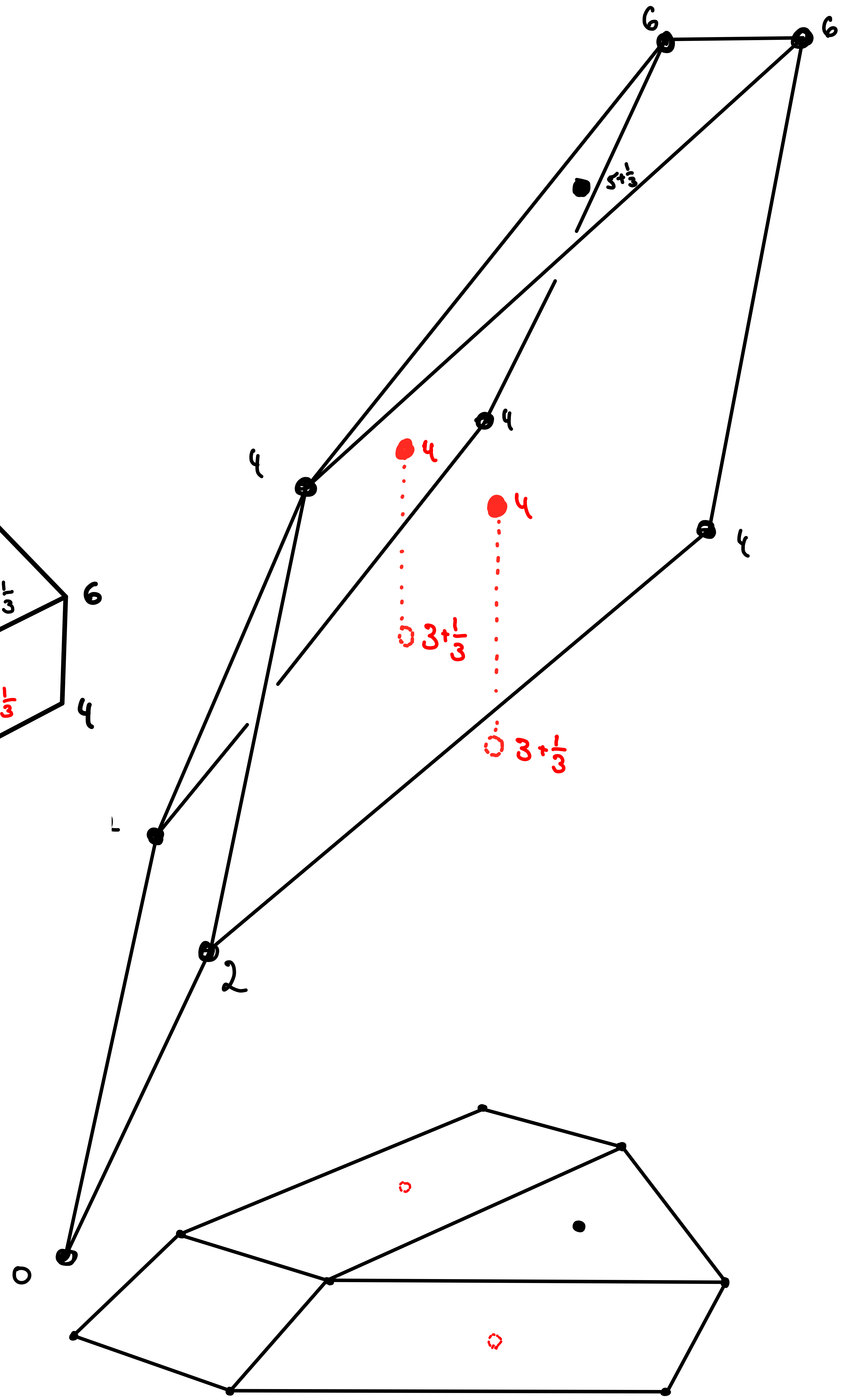

$$\forall \, F^1, \ldots, F^m \preceq P(G) :$$

$$\text{if } a \in \underbrace{\sum_{b \in [m]} F^b}_{\text{face in mixed regular subdivision}} \text{ then } a \in \underbrace{\sum_{b \in [m]} \text{vert}(F^b)}_{\text{Points that are always in the upper convex hull of the lifted } mP(G)}$$

face in mixed regular subdivision

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The complete graph

and 0/1-bundles

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Let $G = K_n$. The polytope $P(K_n)$ is the *correlation polytope (boolean quadric polytope).* $P(K_n) \cong$ cut polytope, but not lattice isomorphic!

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Let $a^* \in \{0,1\}^n$. Then $\forall a \in \pi^{-1}(a^*)$ such that

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Reminder

Let $a^* \in \mathbb{Z}^n_{\geq 0}$. A CE is guaranteed to exist if $\exists\ a \in \pi^{-1}(a^*)$ such that

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$G = K_4, a^* = (2,2,2,2)$. There are edges e_1, e_2, e_3, e_4 of $P(K_4)$ s.t.
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Other graphs

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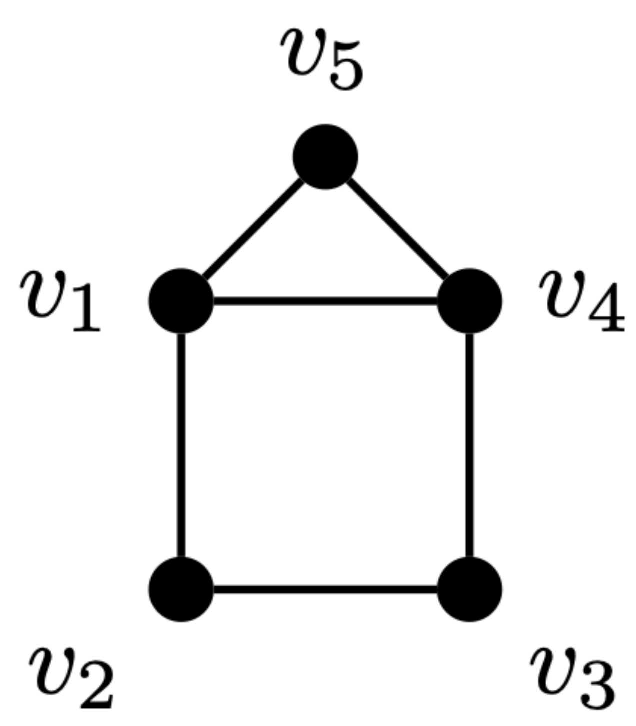

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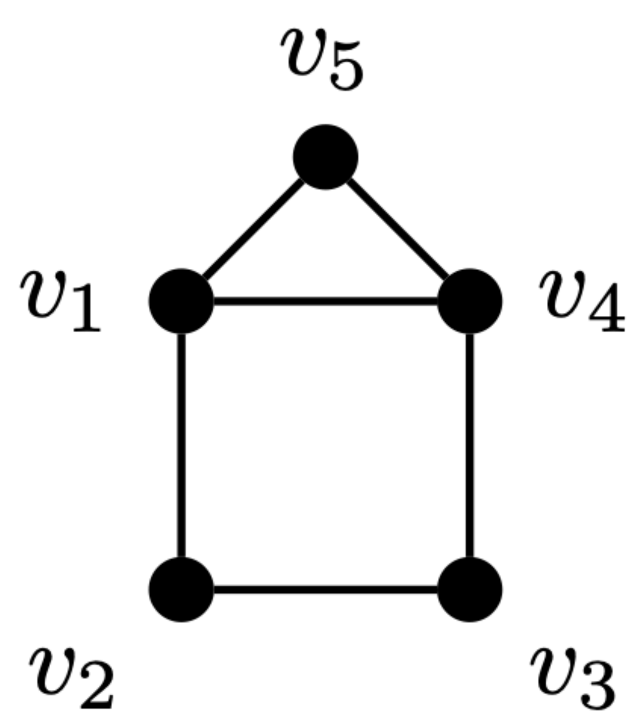


$a^* = (1,1,1,1,1)$. There are edges e_1, e_2, e_3, e_4 of $P(G)$ s.t.

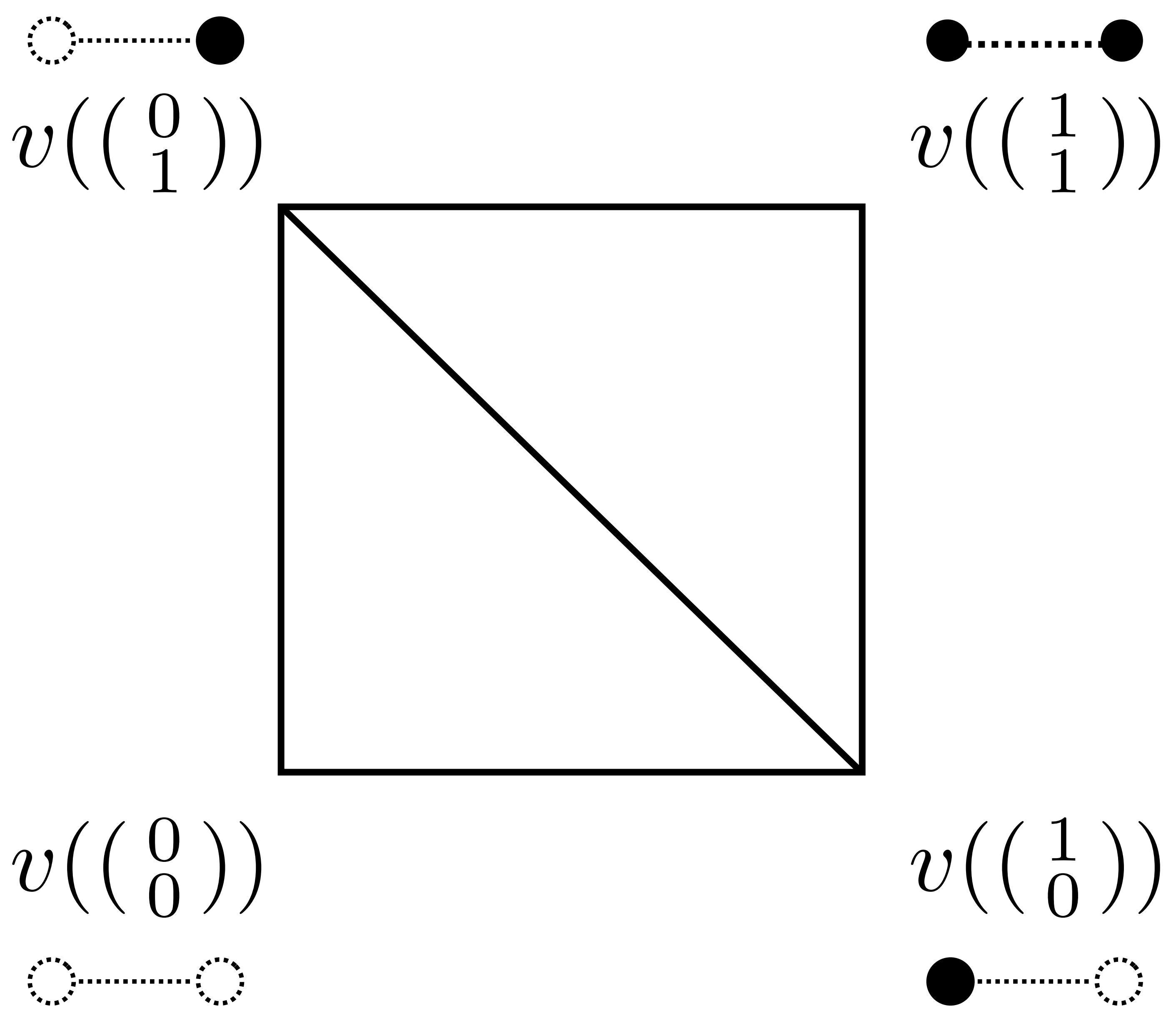
$$\pi^{-1}(a^*) \cap \sum_{i=1}^{4} e_i = \{(1,1,1,1,\ 0,0,0,0,0,0)\}$$

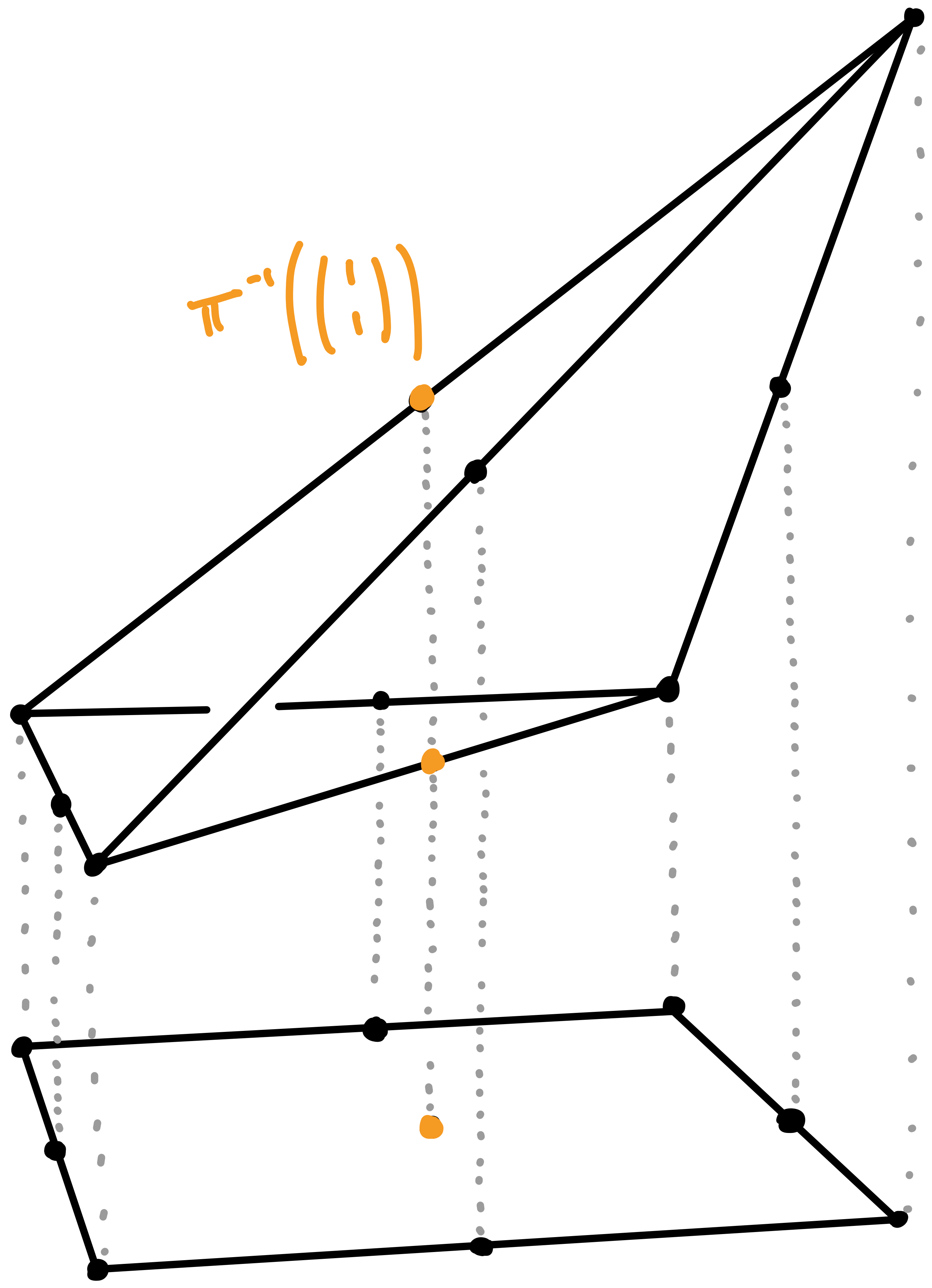
and

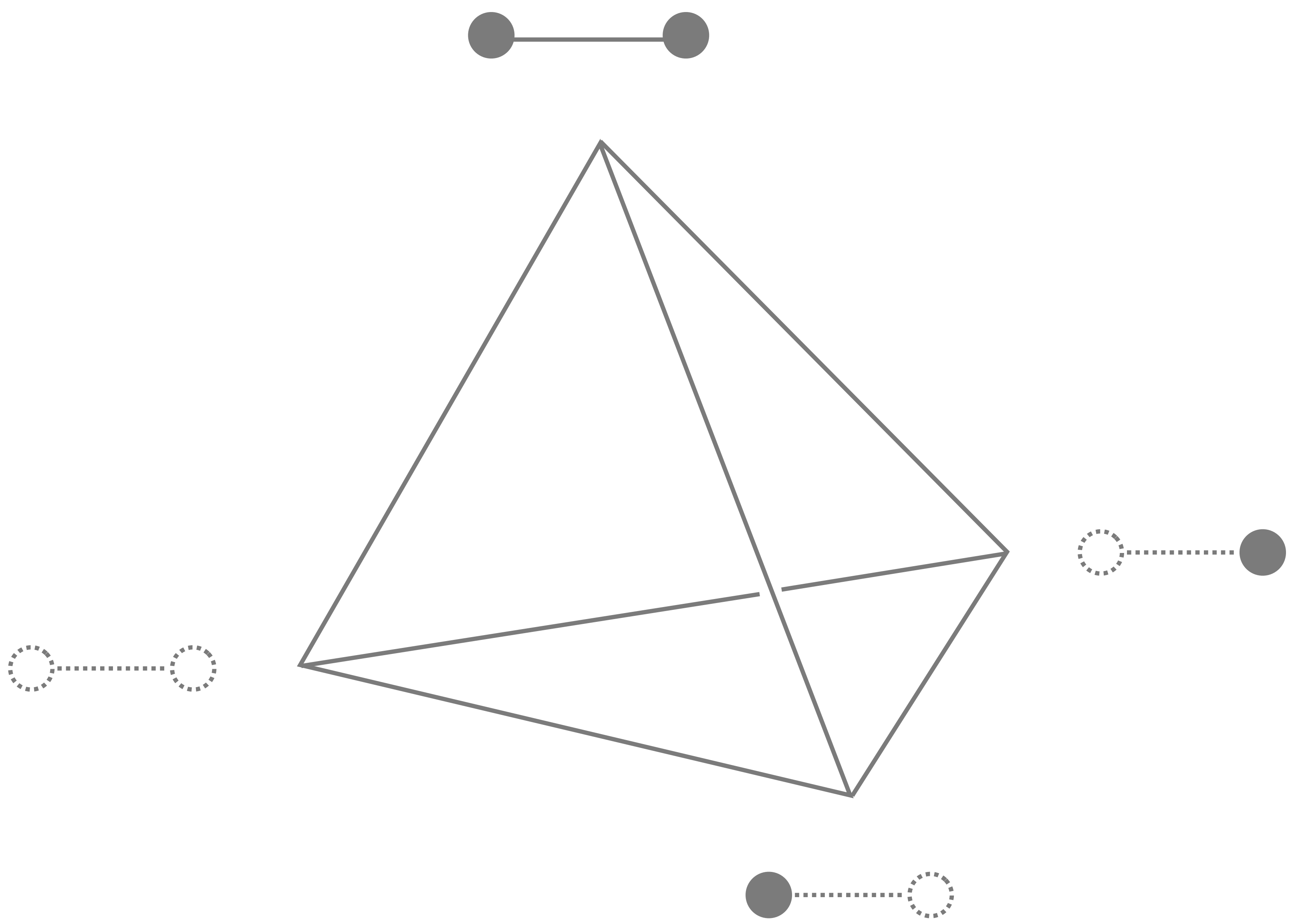
$$\pi^{-1}(a^*) \cap \sum_{i=1}^{4} \mathrm{vert}(e_i) = \emptyset.$$

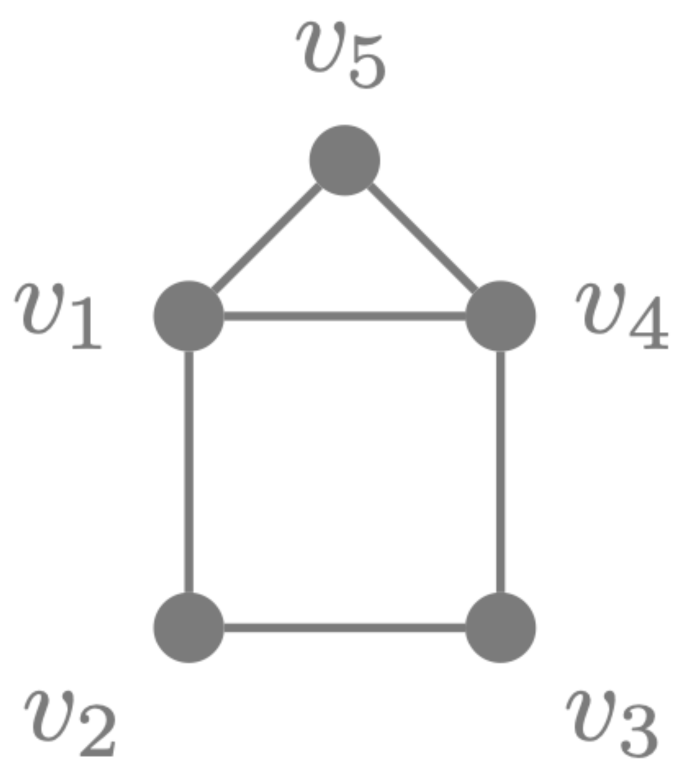
Comparison: classical approach

Non-linear valuations on the cube









Thank you!