Competitive Equilibrium and Lattice Polytopes

DM/G Seminar

16 March 2022

Marie-Charlotte Brandenburg
based on joint work with Christian Haase and Ngoc Mai Tran

Max-Planck-Institut für
Mathematik
in den Naturwissenschaften

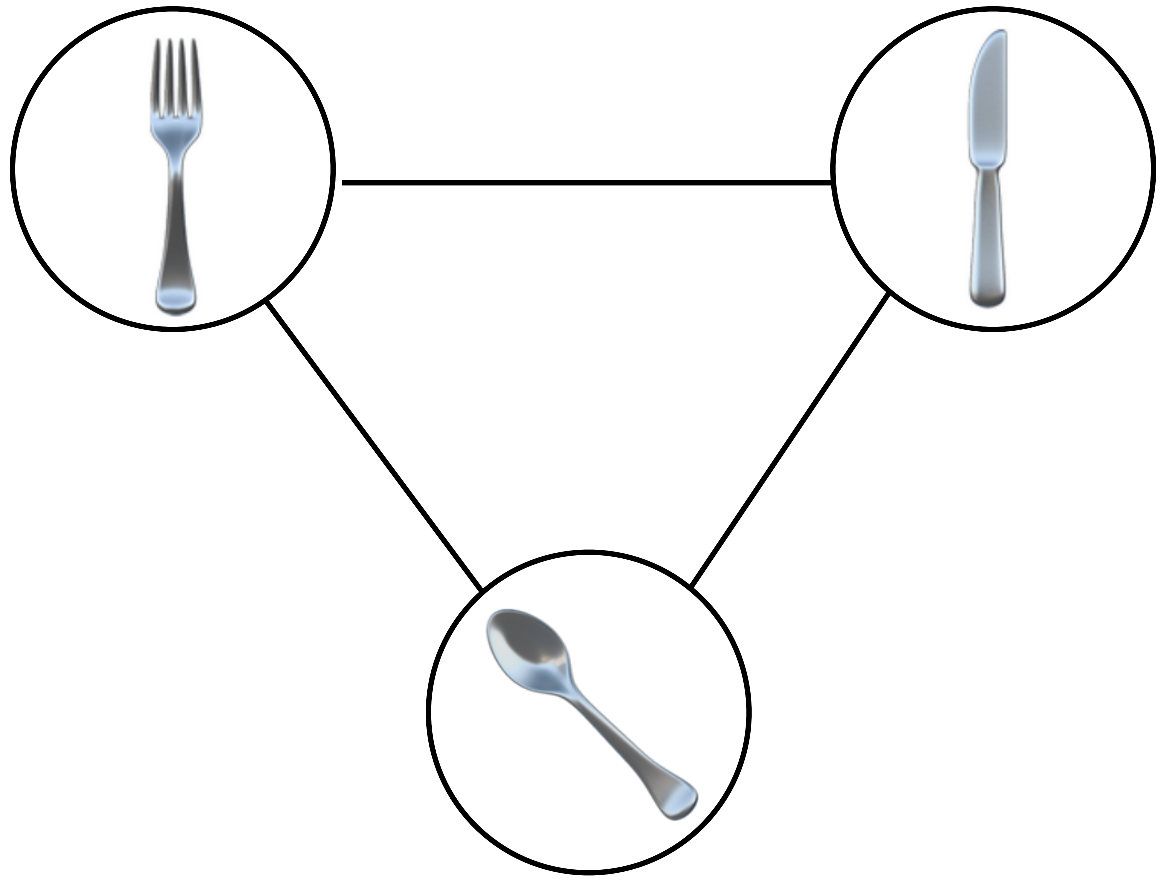
MAX-PLANCK-GESELLSCHAFT

Overview

1. **First Example**
2. **History | Motivation**
3. **Mathematical Model | Connections to Polytopes**
4. **Can we guarantee the existence of a competitive equilibrium?**
   (Answer: yes, if $G = K_n$)

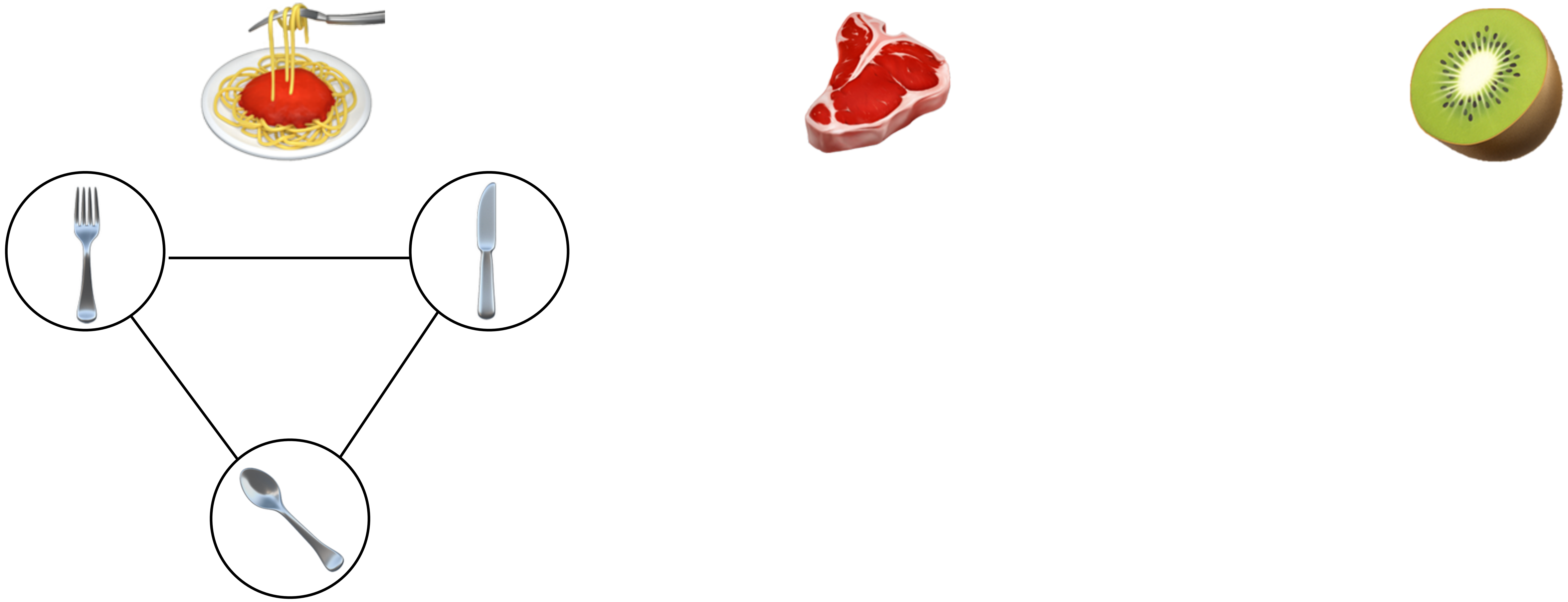
First Example

The cutlery auction at dinner time



First Example

The cutlery auction at dinner time

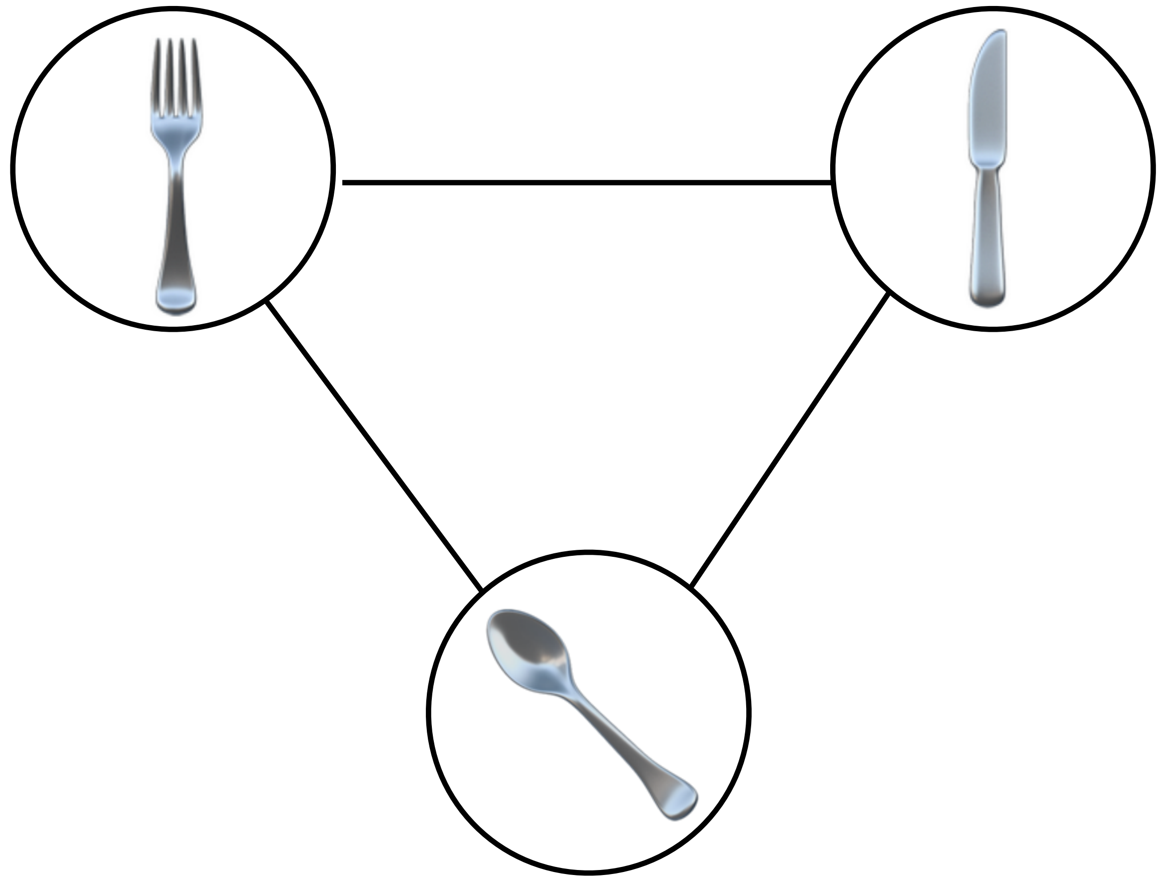


First Example

The cutlery auction at dinner time

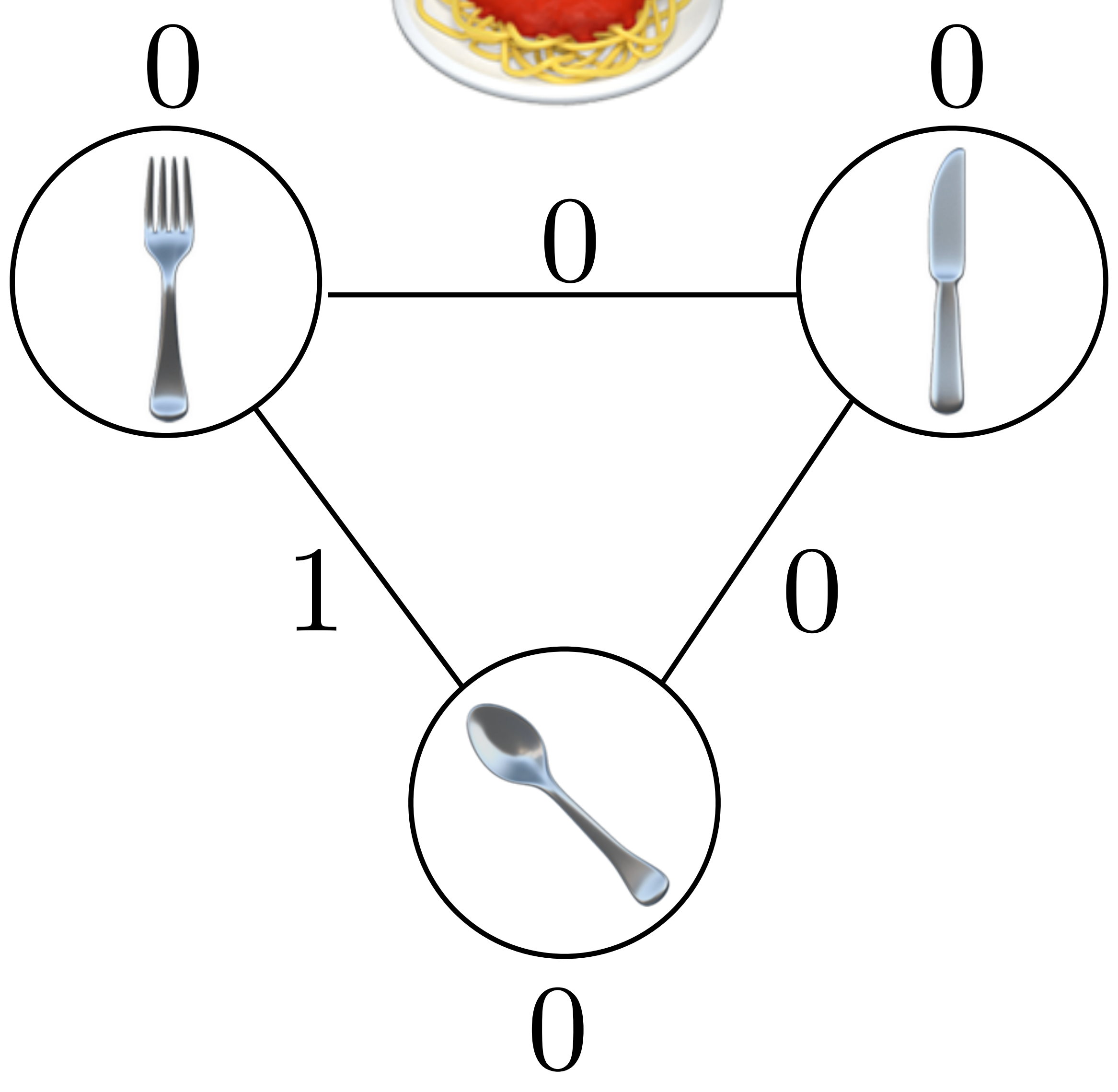






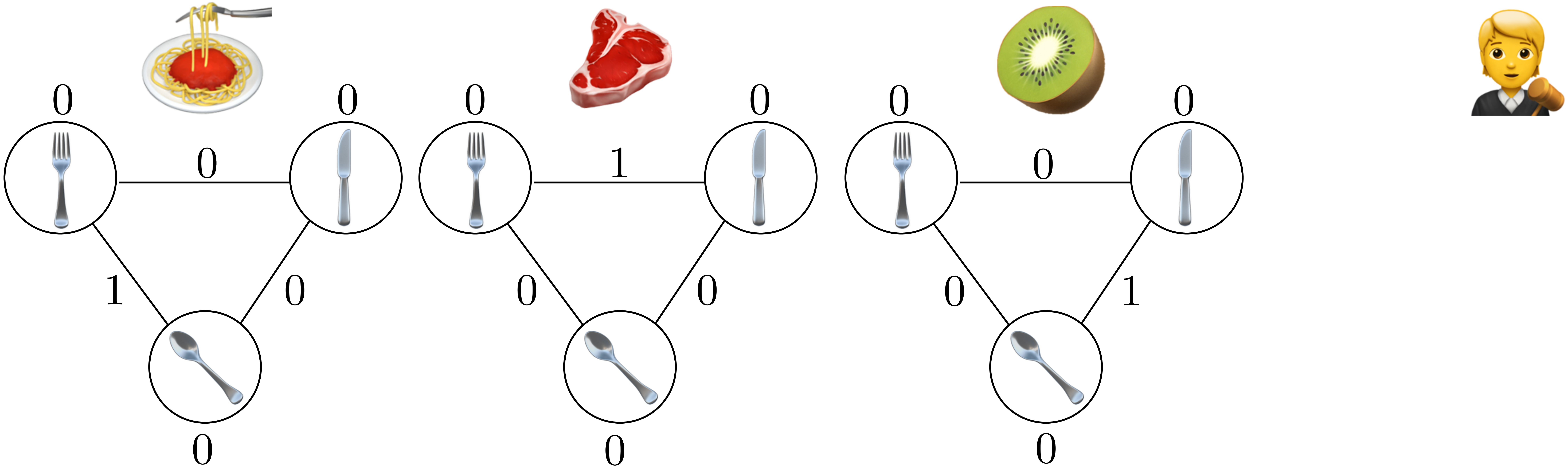
First Example

The cutlery auction at dinner time



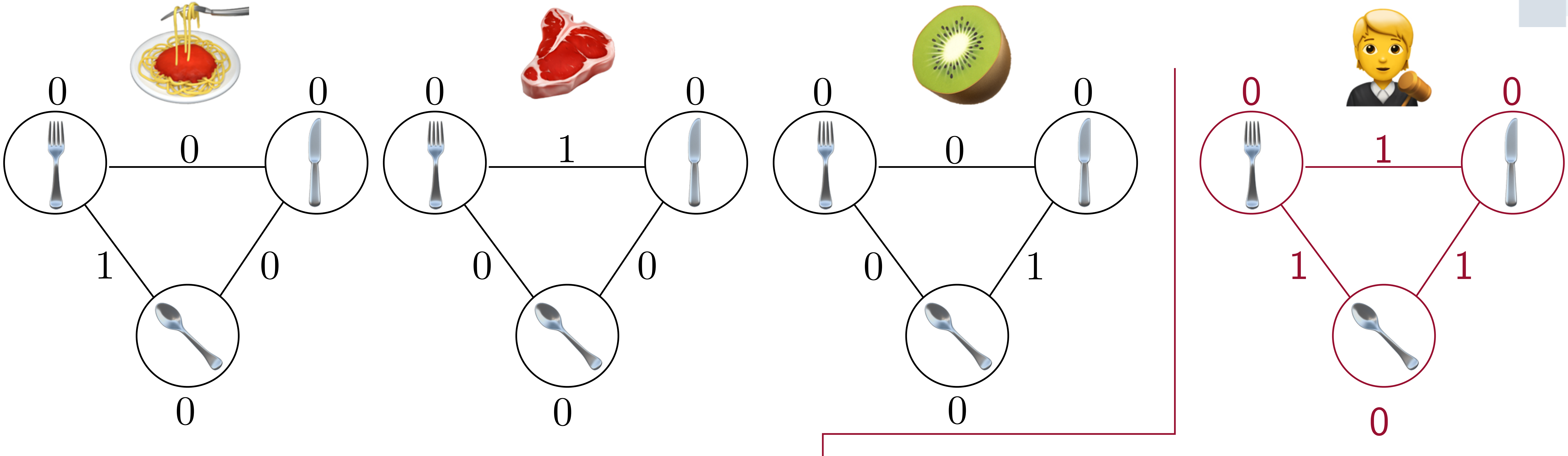
First Example

The cutlery auction at dinner time



First Example

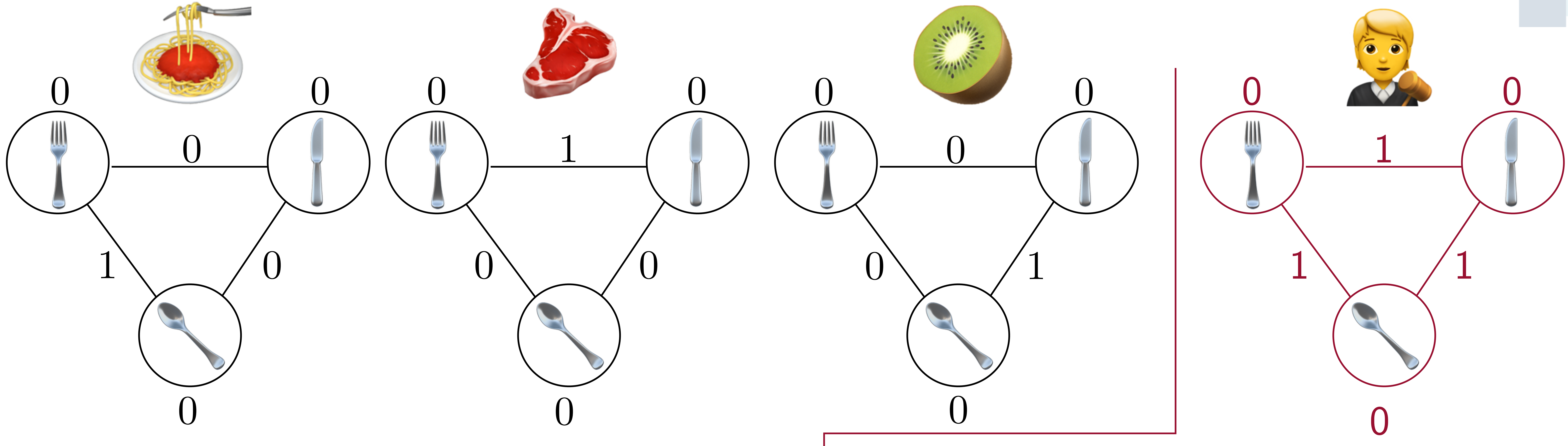
The cutlery auction at dinner time

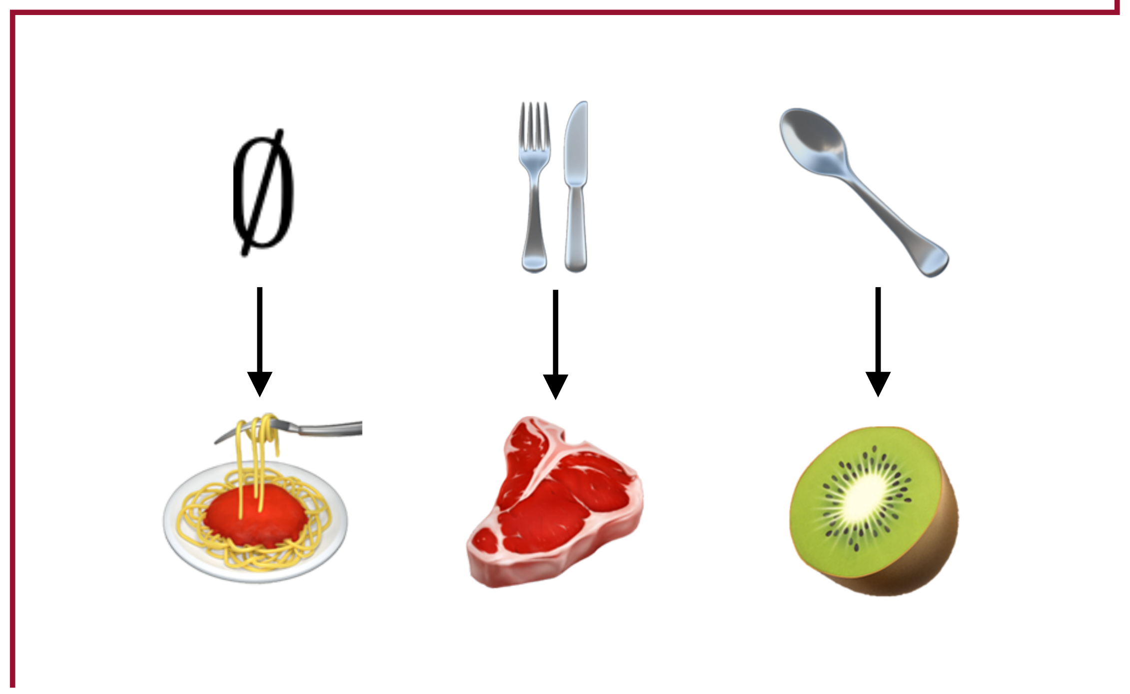


Price for 1 item : 0
Price for 2 items: 1
Price for 3 items: 3

First Example

The cutlery auction at dinner time

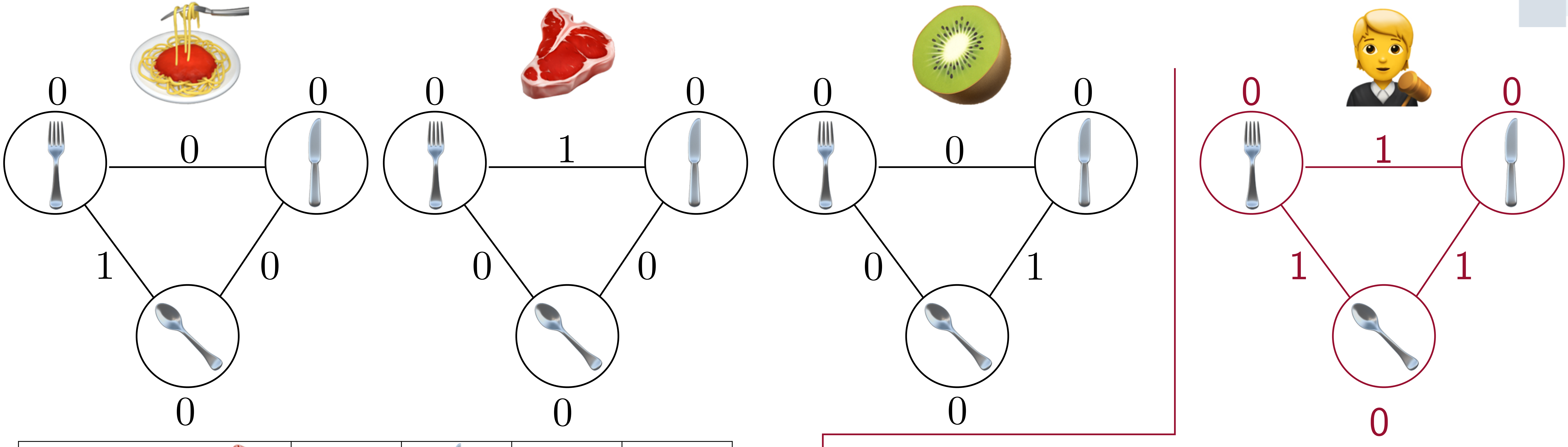




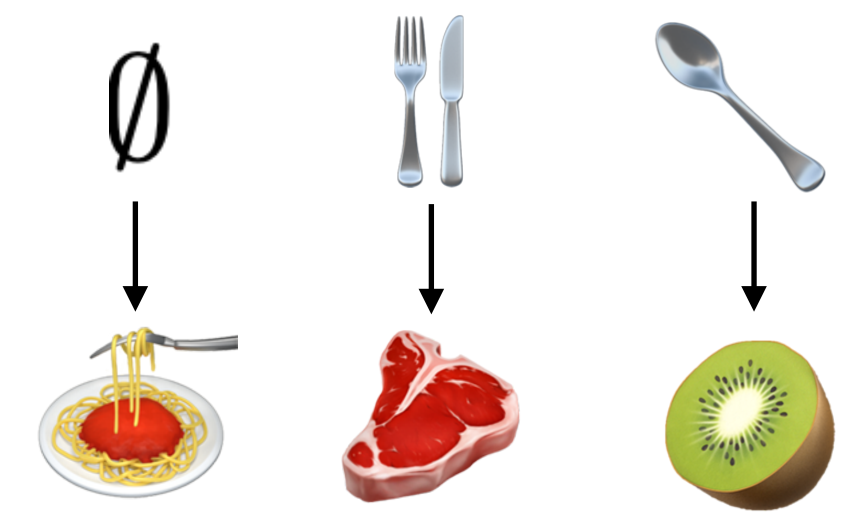
Price for 1 item : 0
Price for 2 items: 1
Price for 3 items: 3

First Example

The cutlery auction at dinner time



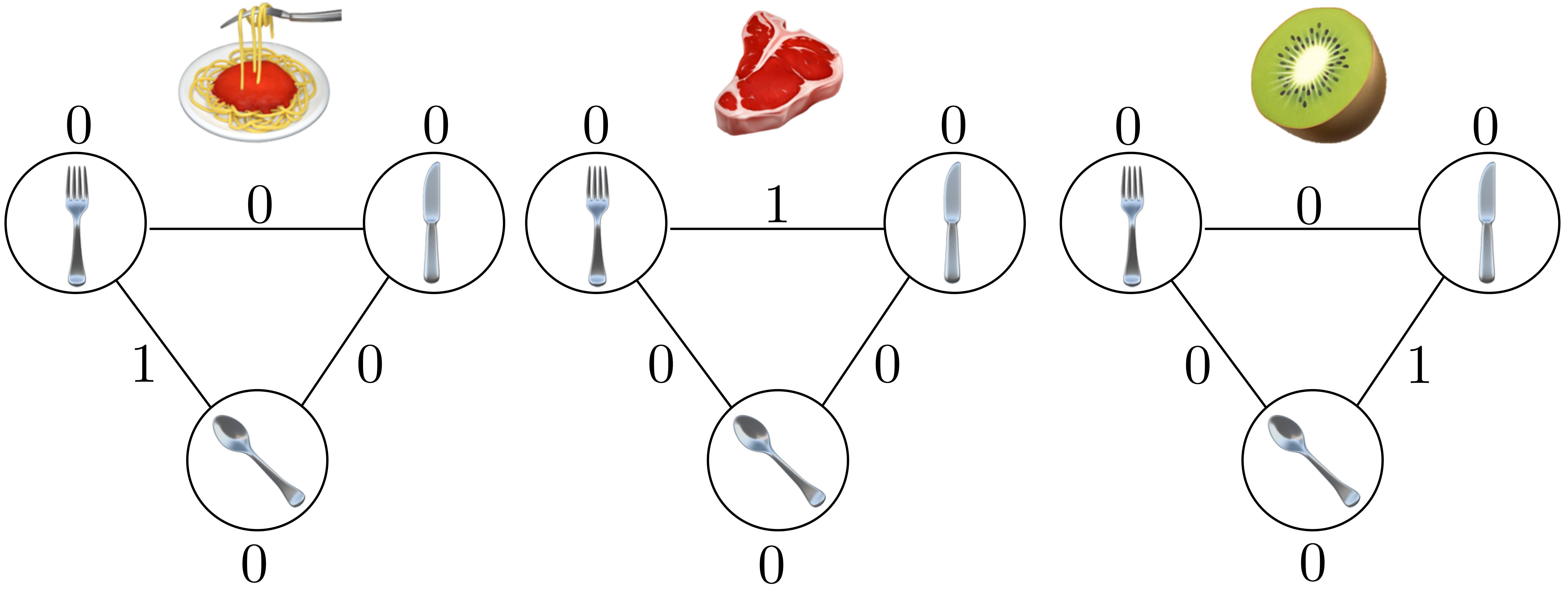
Opinion of 🥩	∅	🔪	🍴🔪	🍴🔪🥄
Willing to pay	0	0	1	1
Price charged	0	0	1	3
Profit	0	0	0	-2

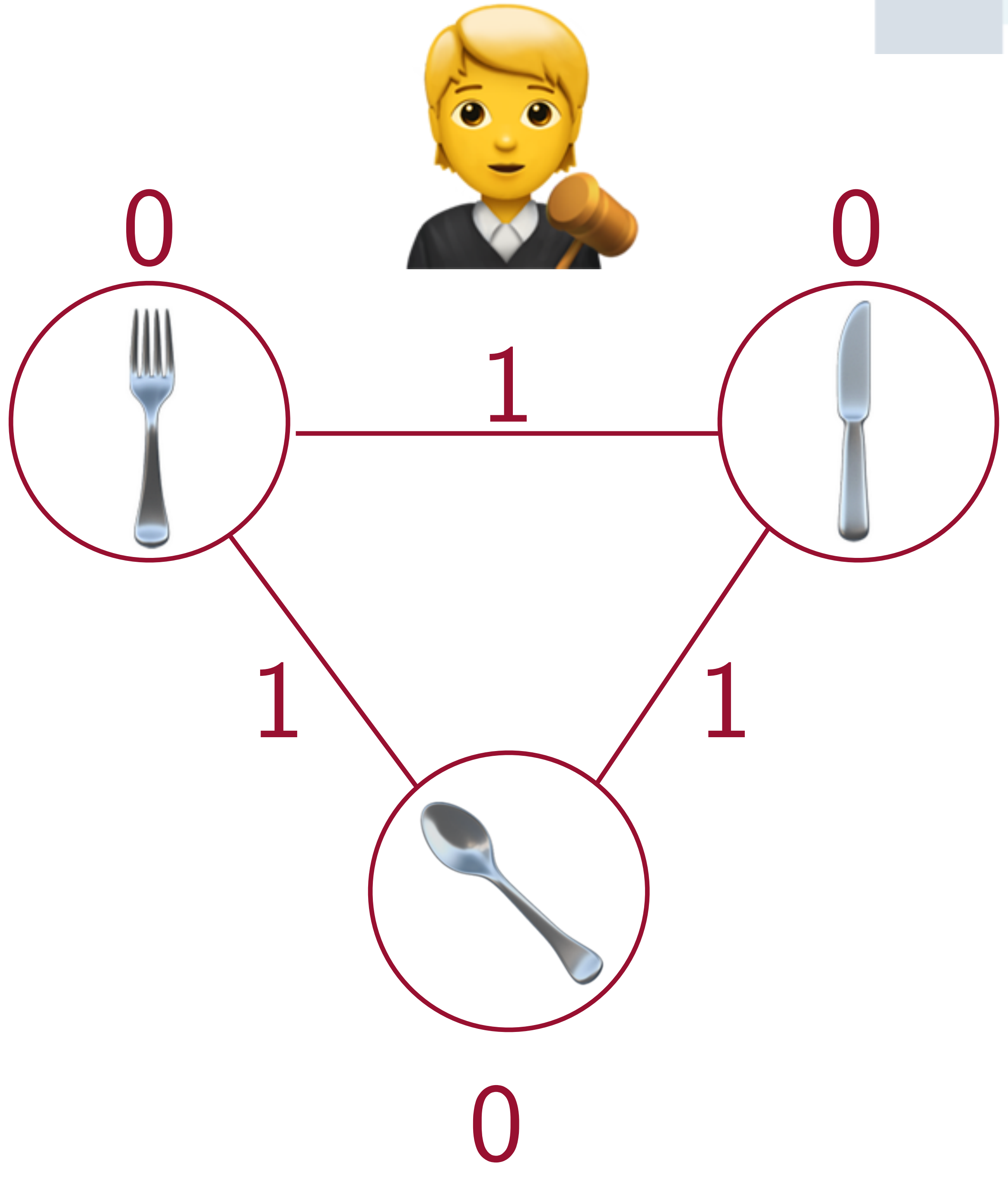


Price for 1 item : 0
Price for 2 items: 1
Price for 3 items: 3

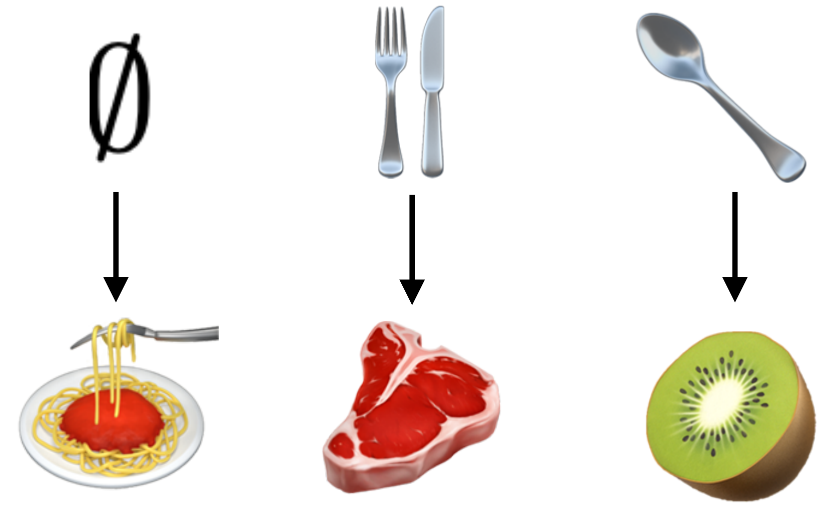
First Example

The cutlery auction at dinner time





Opinion of	∅			
Willing to pay	0	0	1	1
Price charged	0	0	1	3
Profit	0	0	0	-2



Price for 1 item : 0
Price for 2 items: 1
Price for 3 items: 3

Background: Product-Mix Auctions

Background: Product-Mix Auctions

- Round-based auction
  → bad properties from game-theoretic perspective

Background: Product-Mix Auctions

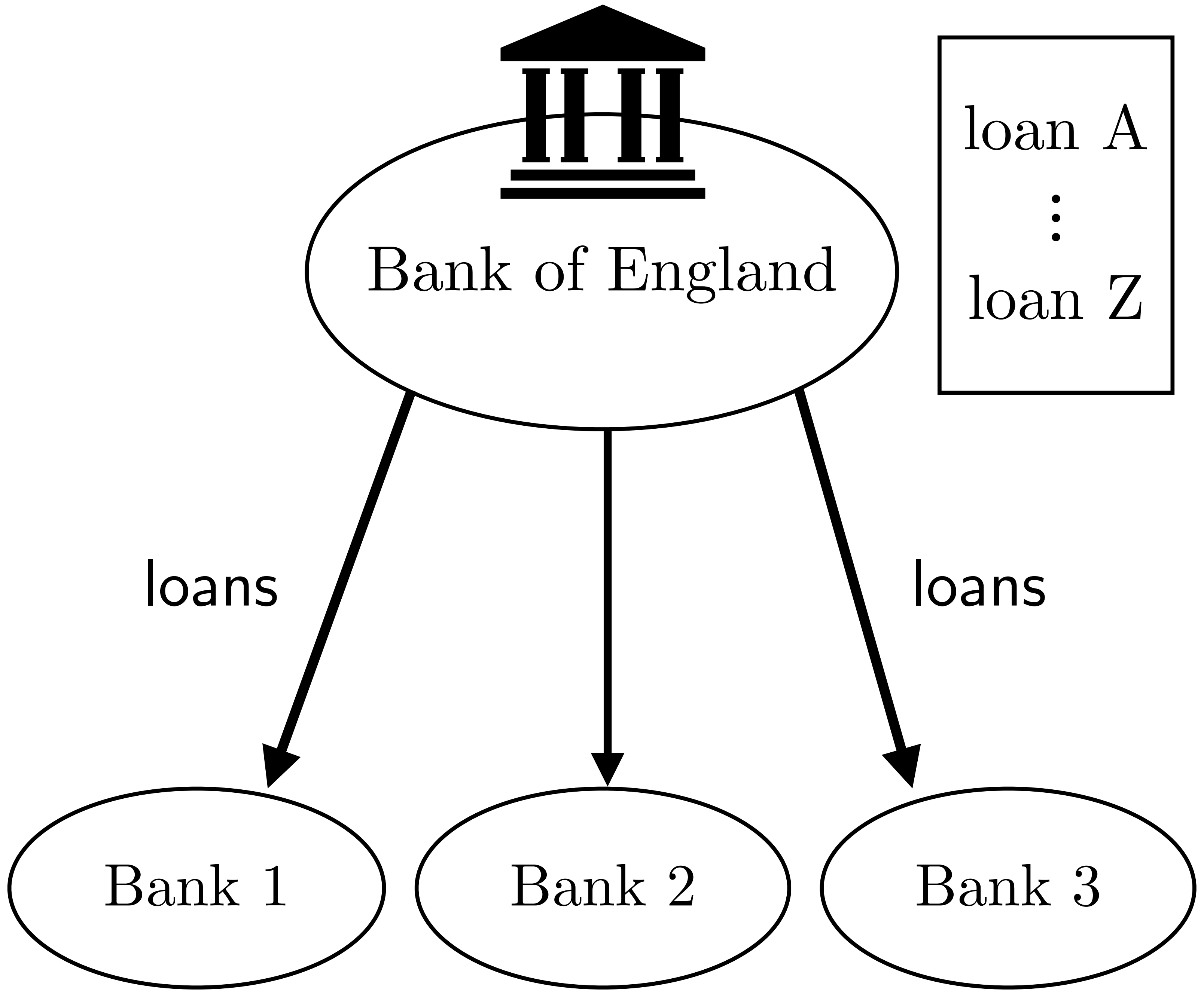
- Round-based auction
  → bad properties from game-theoretic perspective
- Product-Mix auctions
  bank crisis 2011, England

Background: Product-Mix Auctions

- Round-based auction
  → bad properties from game-theoretic perspective
- Product-Mix auctions
  bank crisis 2011, England

Background: Product-Mix Auctions

- Round-based auction
  → bad properties from game-theoretic perspective
- Product-Mix auctions
  bank crisis 2011, England

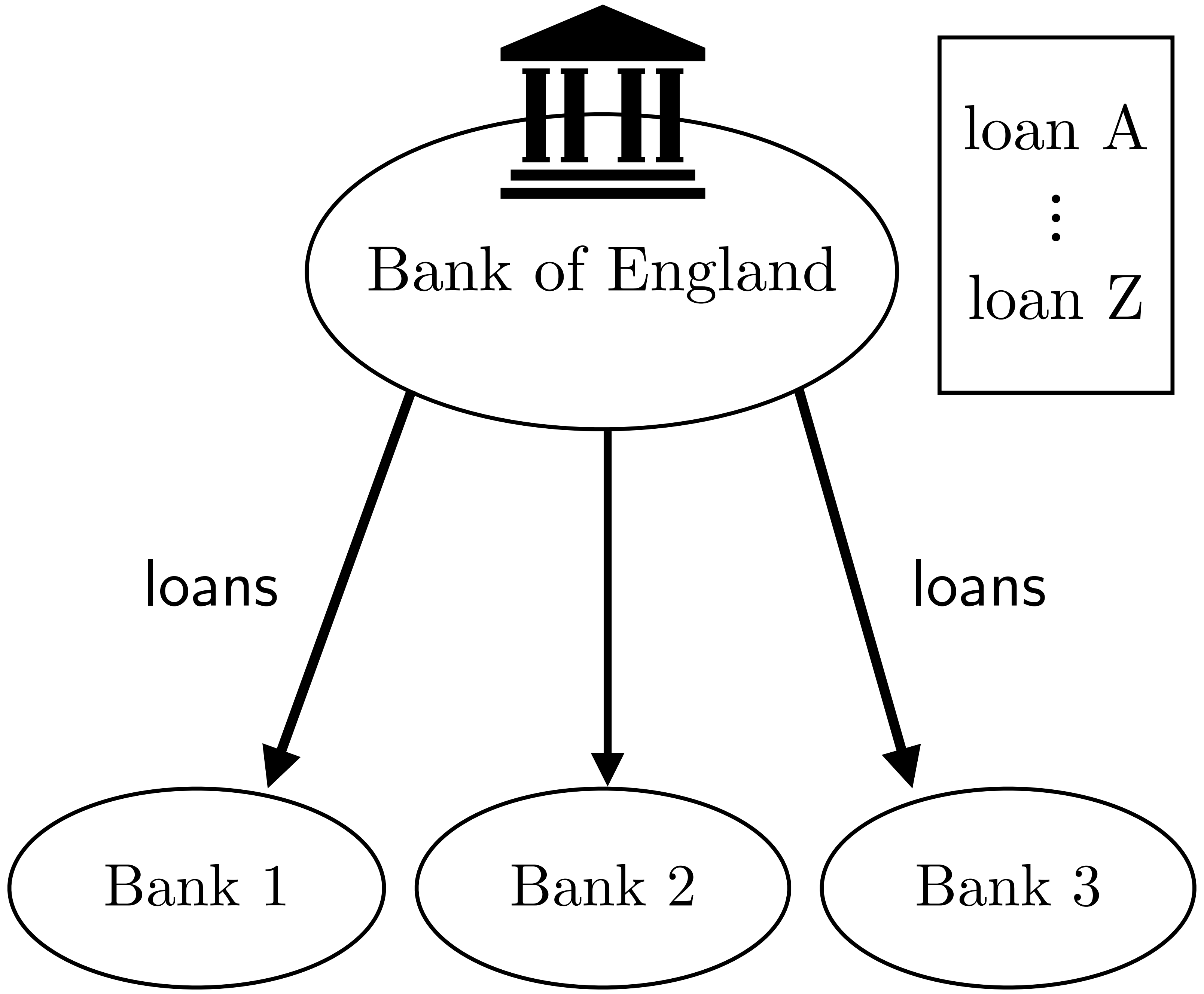


Background: Product-Mix Auctions

- Round-based auction
  → bad properties from game-theoretic perspective

- Product-Mix auctions
  bank crisis 2011, England

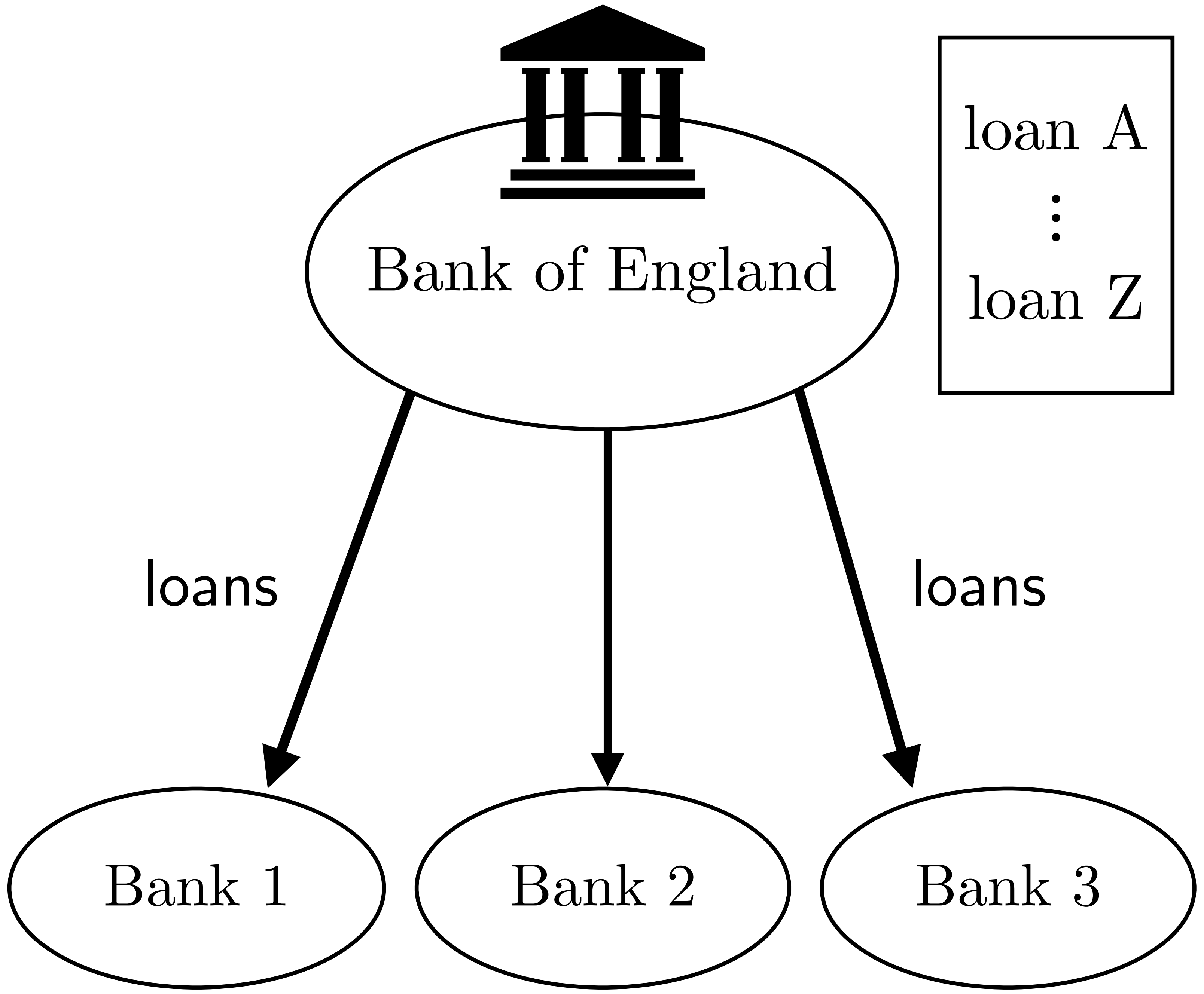
Question:
Which distribution of loans is best for the general economy of England? Fast way to decide?



Background: Product-Mix Auctions

- Round-based auction
  → bad properties from game-theoretic perspective

- Product-Mix auctions
  bank crisis 2011, England



Question:
Which distribution of loans is best for the general economy of England? Fast way to decide?

[Baldwin-Klemperer, 2011]

1. Bidding round:
   Bidders tell the auctioneer (secretly, honestly) about their preferences.
2. Auctioneer sets price and decides a distribution of goods.

The graphical model and its polytope

[Candogan-Ozdaglar-Parillo ’18]

The graphical model and its polytope

[Candogan-Ozdaglar-Parillo ’18]

$n = \#$ types of goods, $a_i^* = \#$ items of type i, $a^* \in \mathbb{Z}^n_{\geq 0}$

The graphical model and its polytope

[Candogan-Ozdaglar-Parillo '18]

$n = \#$ types of goods, $a_i^* = \#$ items of type i, $a^* \in \mathbb{Z}^n_{\geq 0}$

General assumptions:

1. Each bidder wants to buy ≤ 1 item per type.
2. Auctioneer wants to sell everything.

The graphical model and its polytope

[Candogan-Ozdaglar-Parillo '18]

$n = \#$ types of goods, $a_i^* = \#$ items of type i, $a^* \in \mathbb{Z}^n_{\geq 0}$

General assumptions:

1. Each bidder wants to buy ≤ 1 item per type.
2. Auctioneer wants to sell everything.

$G = ([n], E)$ graph,

The graphical model and its polytope

[Candogan-Ozdaglar-Parillo '18]

$n = \#$ types of goods, $a_i^* = \#$ items of type i, $a^* \in \mathbb{Z}_{\geq 0}^n$

General assumptions:

1. Each bidder wants to buy ≤ 1 item per type.
2. Auctioneer wants to sell everything.

$G = ([n], E)$ graph, $G' \subseteq G$ induced subgraph. Define $\chi_{G'} \in \{0,1\}^{n+|E|}$ as

$$(\chi_{G'})_i = \begin{cases} 1 & \text{if } i \in V(G') \\ 0 & \text{if } i \notin V(G') \end{cases} \qquad (\chi_{G'})_{ij} = \begin{cases} 1 & \text{if } ij \in E(G') \\ 0 & \text{if } ij \notin E(G') \end{cases}$$

The graphical model and its polytope

[Candogan-Ozdaglar-Parillo '18]

$n = \#$ types of goods, $a_i^* = \#$ items of type i, $a^* \in \mathbb{Z}_{\geq 0}^n$

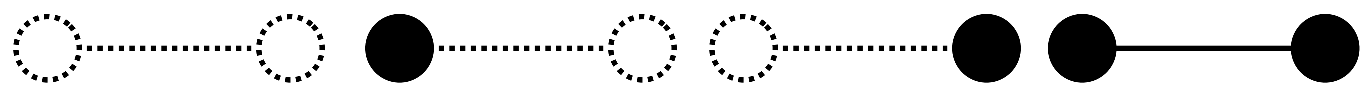
General assumptions:

1. Each bidder wants to buy ≤ 1 item per type.
2. Auctioneer wants to sell everything.

$G = ([n], E)$ graph, $G' \subseteq G$ induced subgraph. Define $\chi_{G'} \in \{0,1\}^{n+|E|}$ as

$$(\chi_{G'})_i = \begin{cases} 1 & \text{if } i \in V(G') \\ 0 & \text{if } i \notin V(G') \end{cases} \qquad (\chi_{G'})_{ij} = \begin{cases} 1 & \text{if } ij \in E(G') \\ 0 & \text{if } ij \notin E(G') \end{cases}$$

$$\begin{matrix} 1 \\ 2 \\ 12 \end{matrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$



The graphical model and its polytope

[Candogan-Ozdaglar-Parillo '18]

$n = \#$ types of goods, $a_i^* = \#$ items of type i, $a^* \in \mathbb{Z}^n_{\geq 0}$

General assumptions:

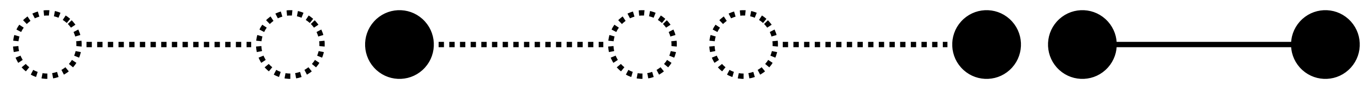
1. Each bidder wants to buy ≤ 1 item per type.
2. Auctioneer wants to sell everything.

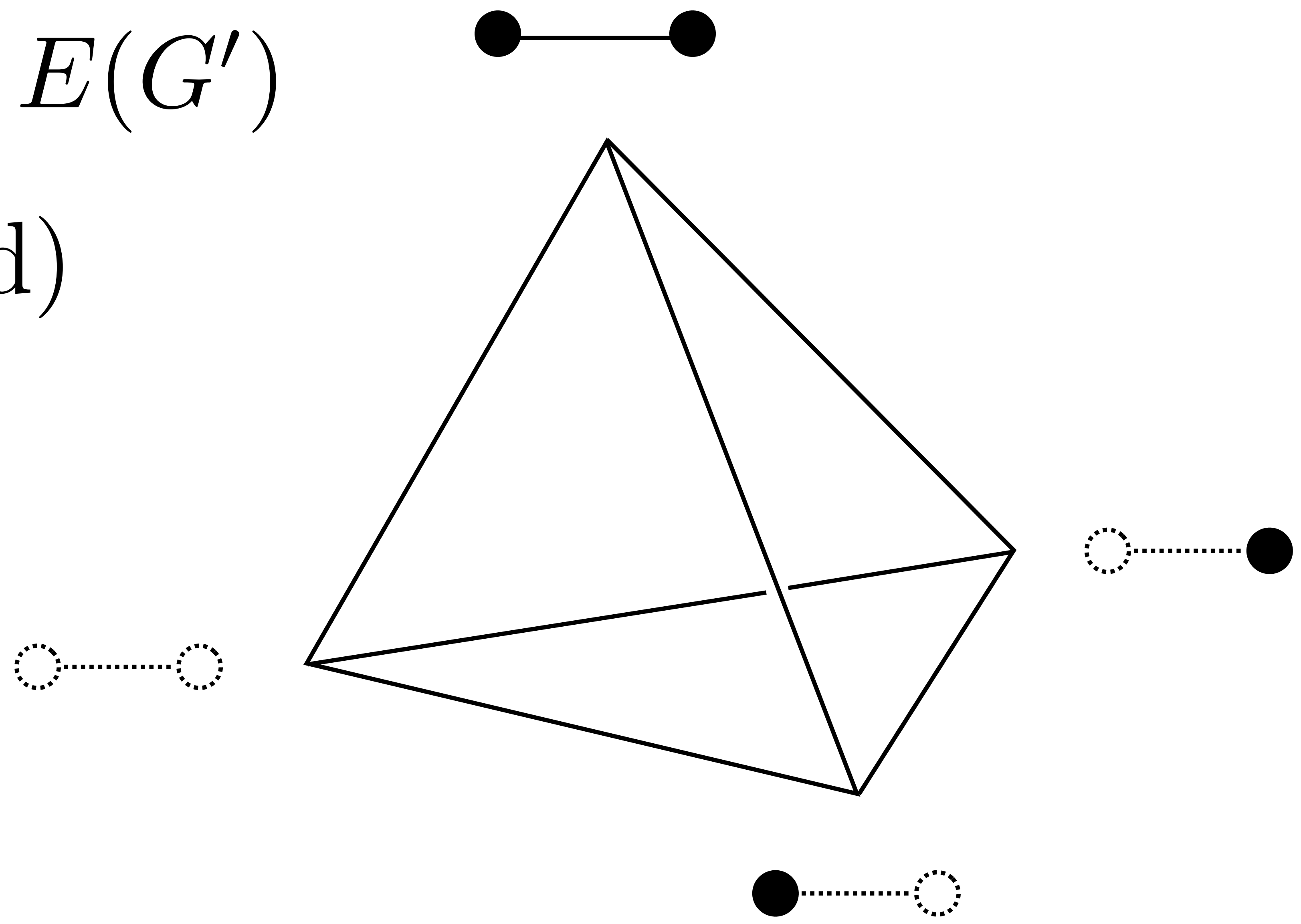
$G = ([n], E)$ graph, $G' \subseteq G$ induced subgraph. Define $\chi_{G'} \in \{0,1\}^{n+|E|}$ as

$$(\chi_{G'})_i = \begin{cases} 1 & \text{if } i \in V(G') \\ 0 & \text{if } i \notin V(G') \end{cases} \qquad (\chi_{G'})_{ij} = \begin{cases} 1 & \text{if } ij \in E(G') \\ 0 & \text{if } ij \notin E(G') \end{cases}$$

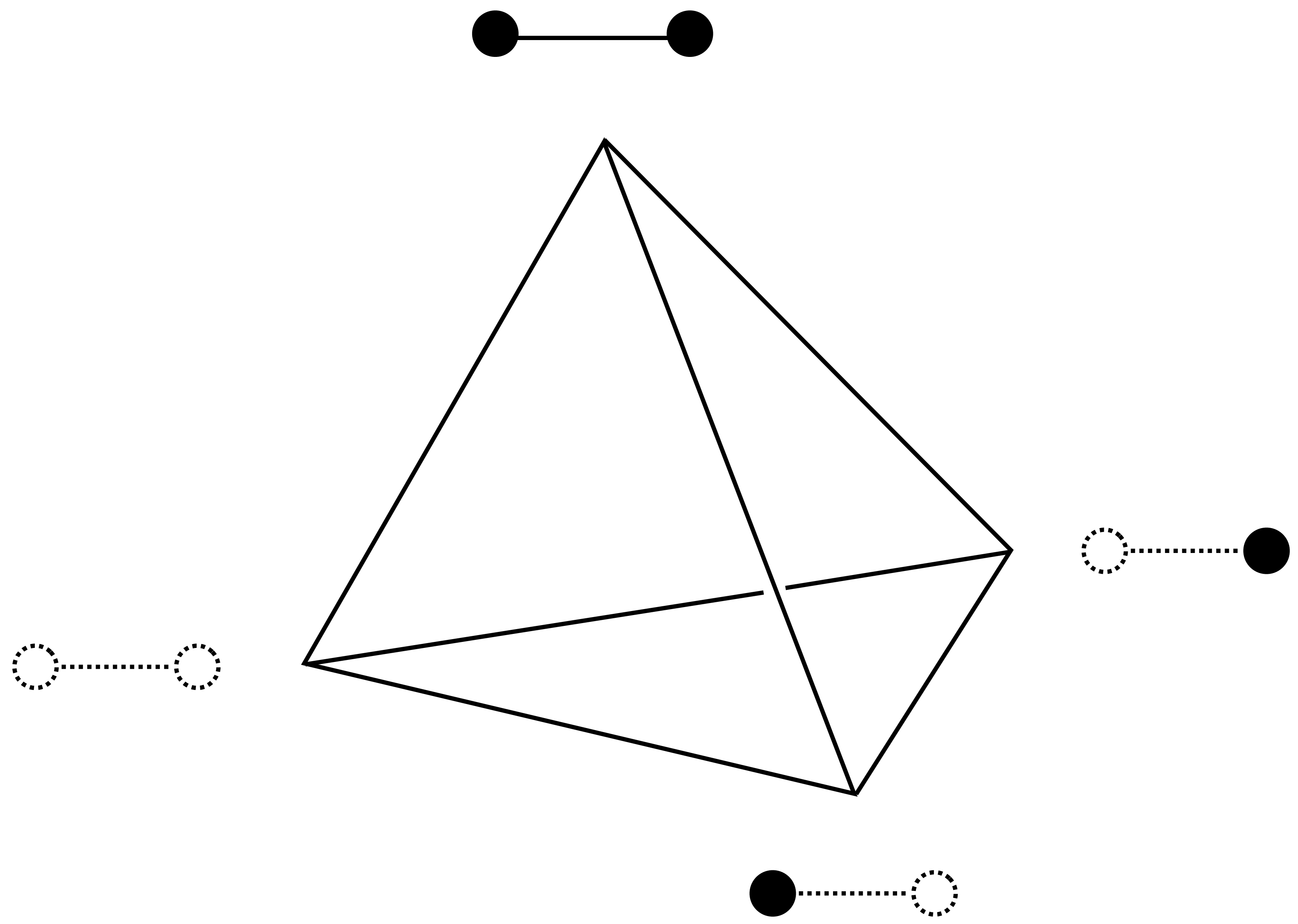
$$P(G) = \mathrm{conv}(\chi_{G'} \mid G' \subseteq G \text{ induced})$$

$$\begin{matrix} 1 \\ 2 \\ 12 \end{matrix} \quad \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$



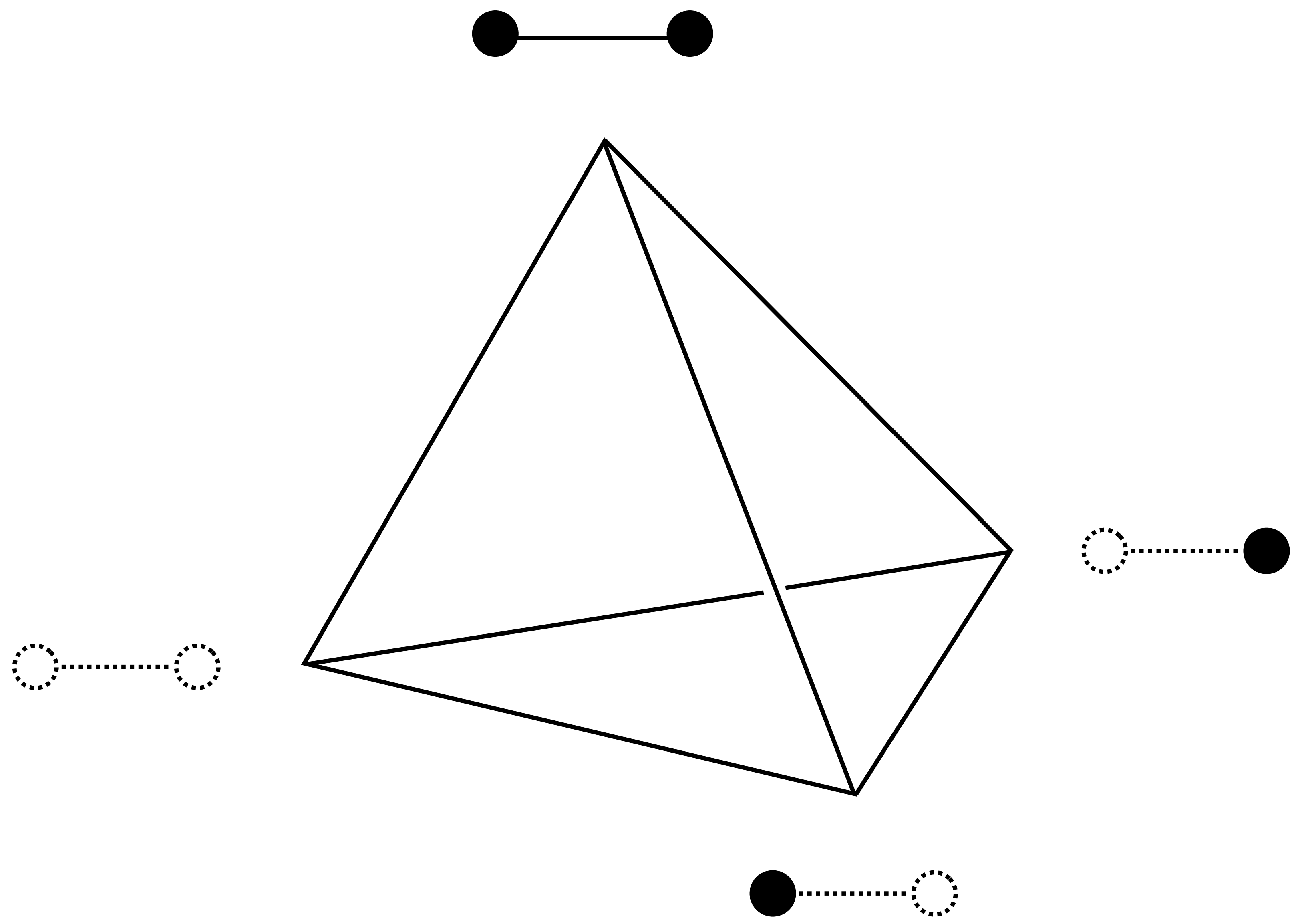


1. Bidding round



1. Bidding round

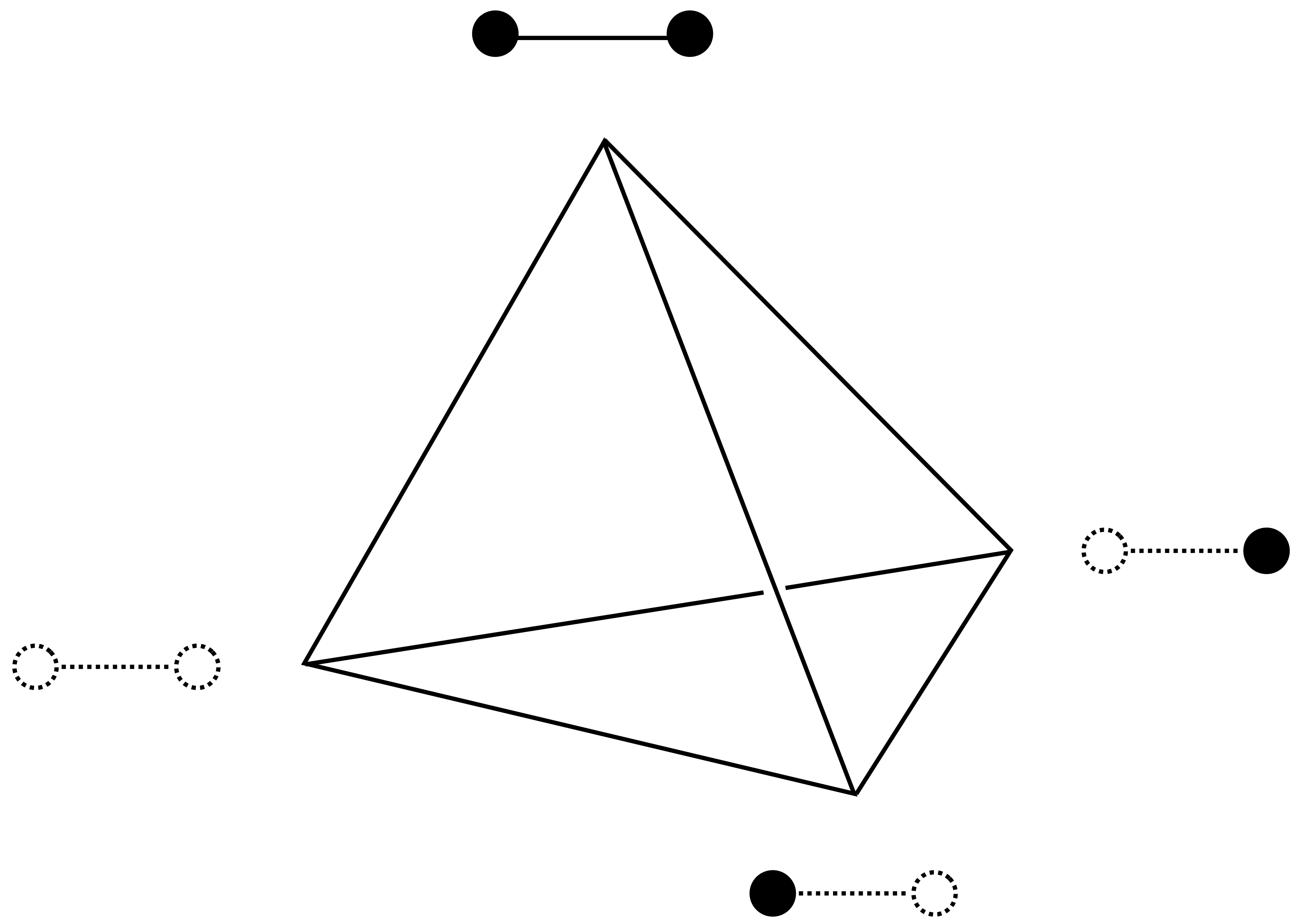
Bidder $b \in [m]$ communicates preferences to auctioneer



1. Bidding round

Bidder $b \in [m]$ communicates preferences to auctioneer

Valuation function $v^b : P \cap \mathbb{Z}^{n+|E|} \to \mathbb{R}$, $v^b(a) = \langle w^b, a \rangle$ for some $w^b \in \mathbb{R}^{n+|E|}$

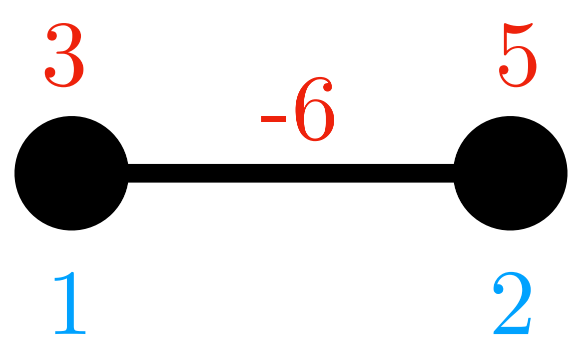


1. Bidding round

Bidder $b \in [m]$ communicates preferences to auctioneer

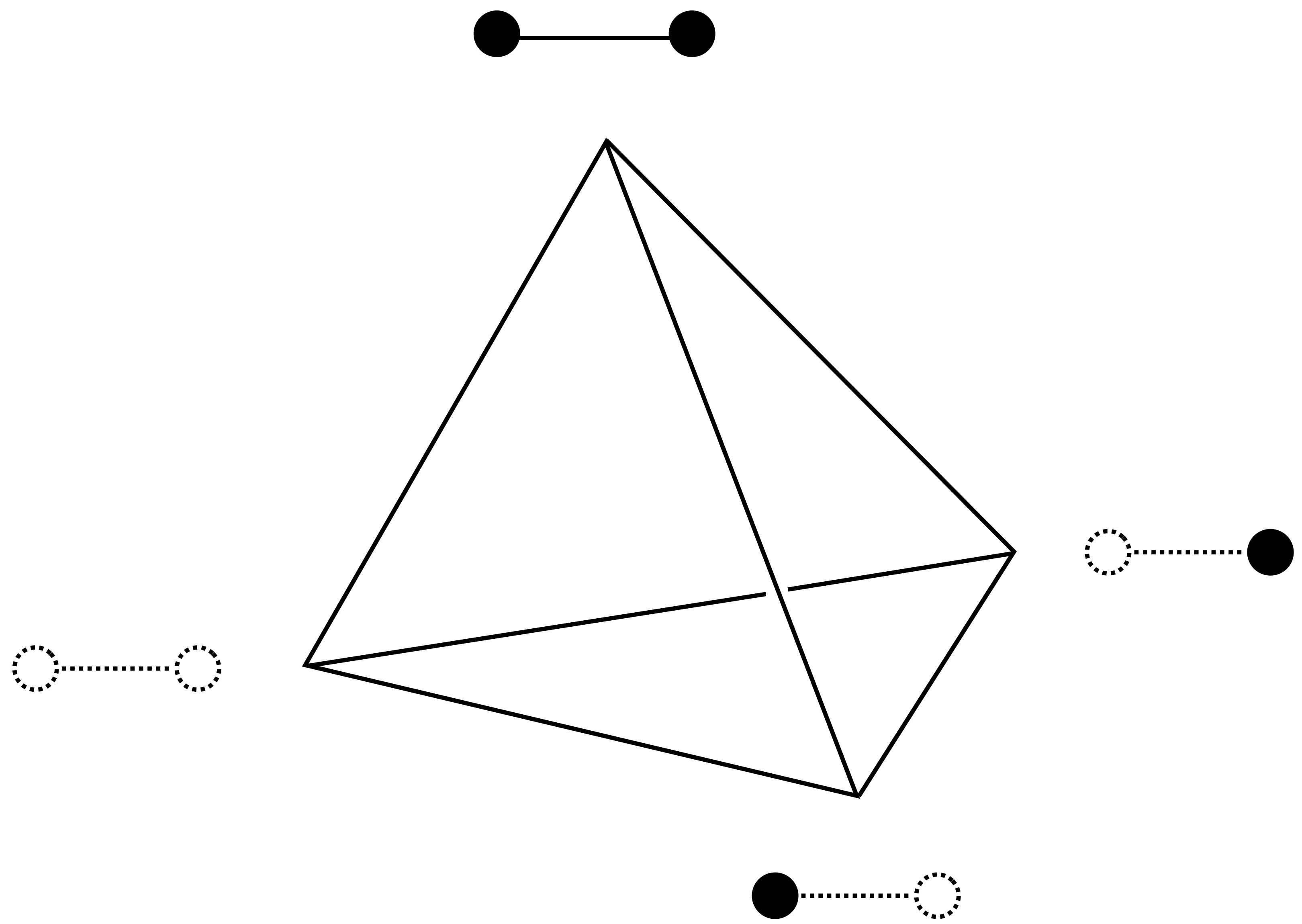
Valuation function $v^b : P \cap \mathbb{Z}^{n+|E|} \to \mathbb{R}$, $v^b(a) = \langle w^b, a \rangle$ for some $w^b \in \mathbb{R}^{n+|E|}$

$$w^b = \begin{pmatrix} 3 \\ 5 \\ -6 \end{pmatrix}$$



$$v^b\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right) = 0, \quad v^b\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) = 3,$$

$$v^b\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) = 5, \quad v^b\left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right) = 2$$

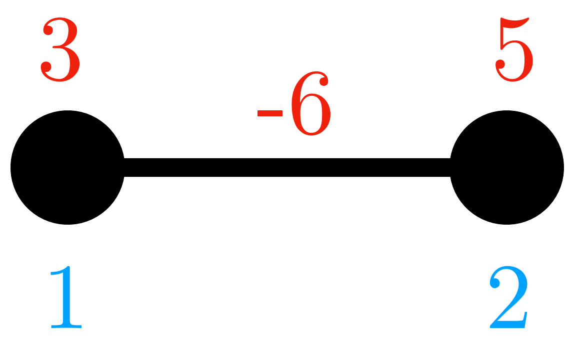


1. Bidding round

Bidder $b \in [m]$ communicates preferences to auctioneer

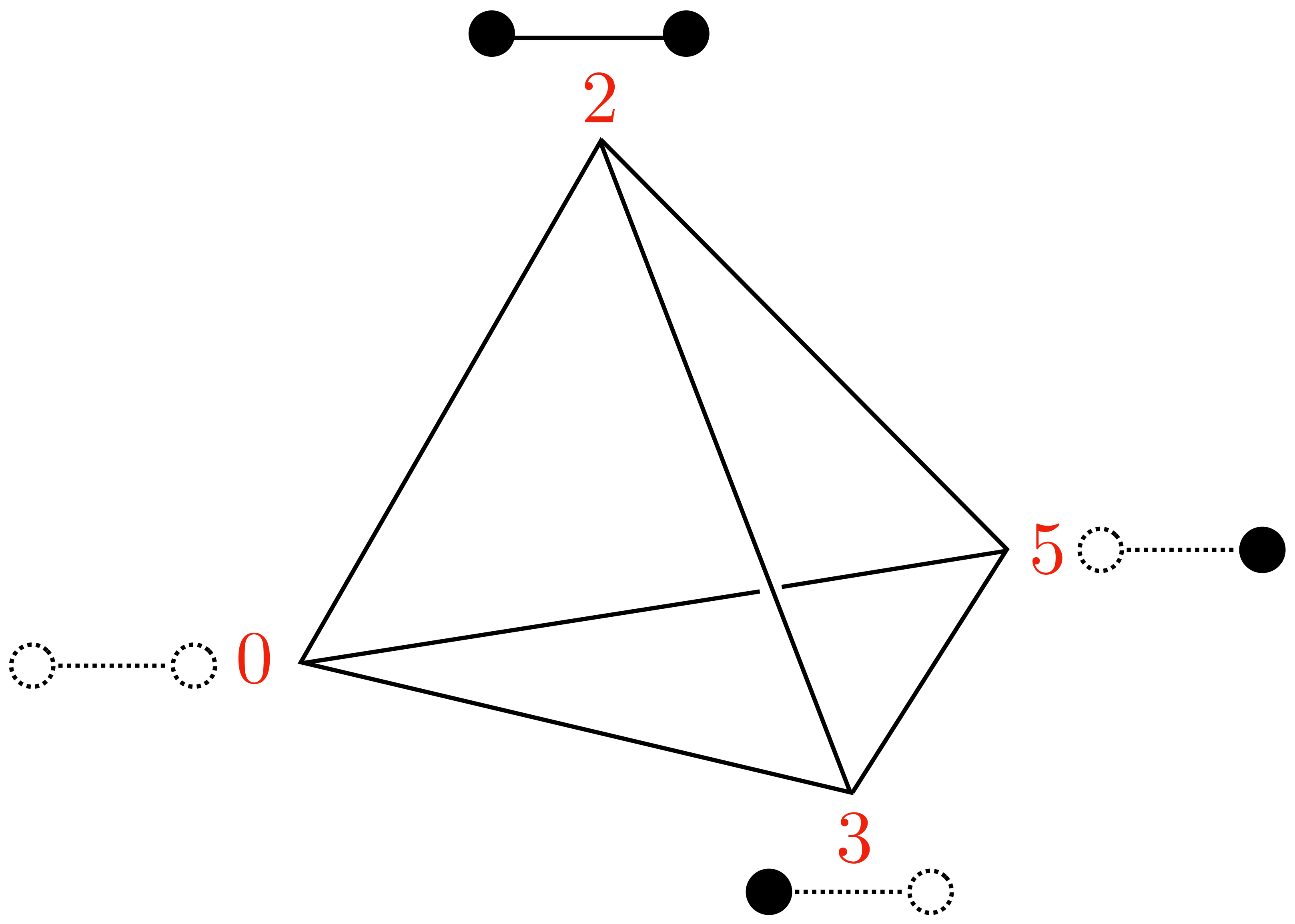
Valuation function $v^b : P \cap \mathbb{Z}^{n+|E|} \to \mathbb{R}$, $v^b(a) = \langle w^b, a \rangle$ for some $w^b \in \mathbb{R}^{n+|E|}$

$$w^b = \begin{pmatrix} 3 \\ 5 \\ -6 \end{pmatrix}$$



$$v^b\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right) = 0, \quad v^b\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) = 3,$$

$$v^b\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) = 5, \quad v^b\left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right) = 2$$

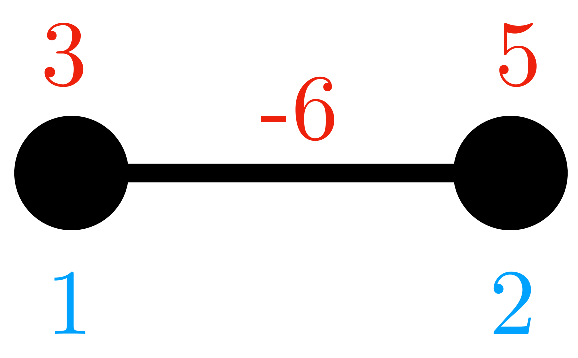


1. Bidding round

Bidder $b \in [m]$ communicates preferences to auctioneer

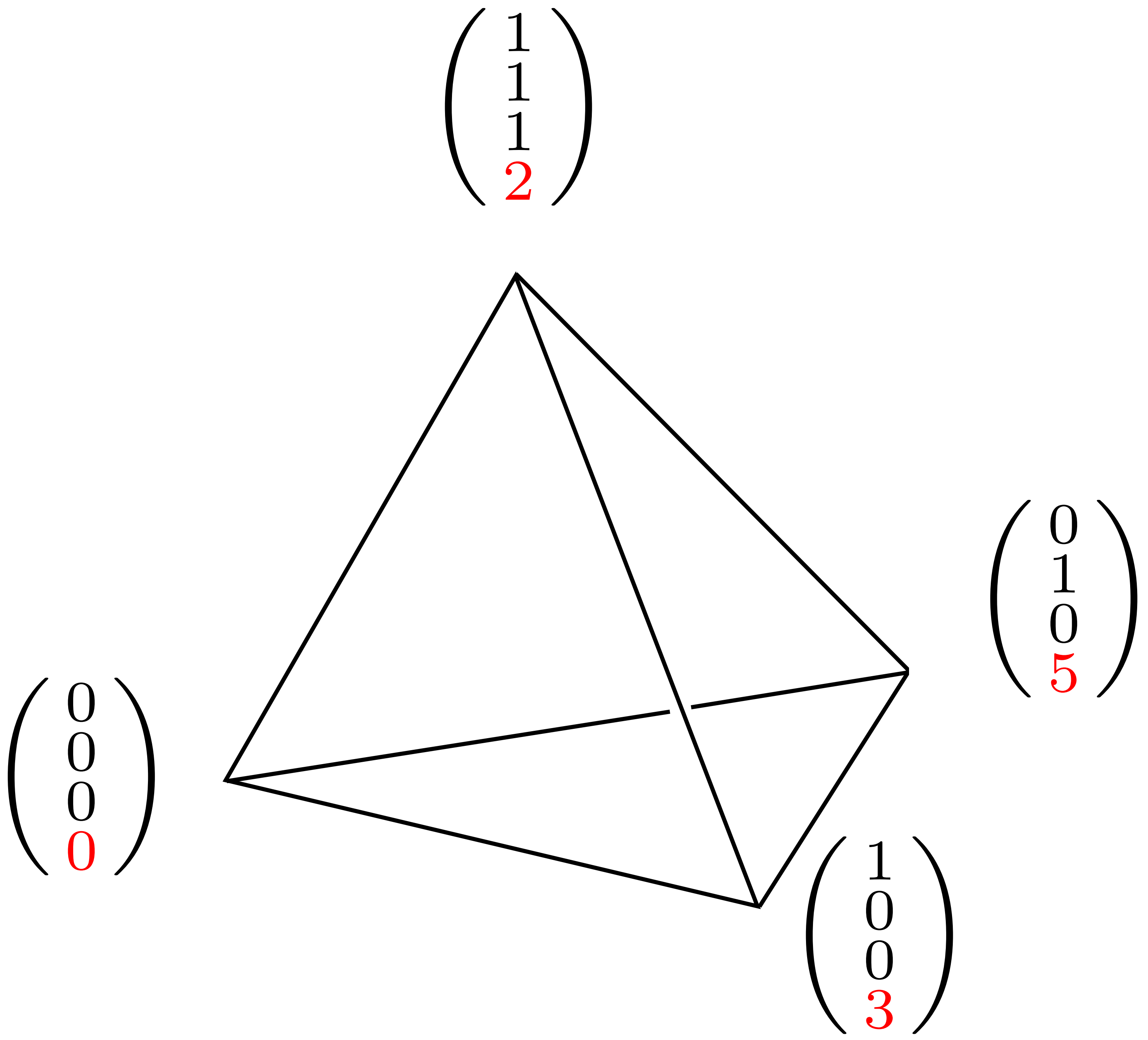
Valuation function $v^b : P \cap \mathbb{Z}^{n+|E|} \to \mathbb{R}$, $v^b(a) = \langle w^b, a \rangle$ for some $w^b \in \mathbb{R}^{n+|E|}$

$$w^b = \begin{pmatrix} 3 \\ 5 \\ -6 \end{pmatrix}$$



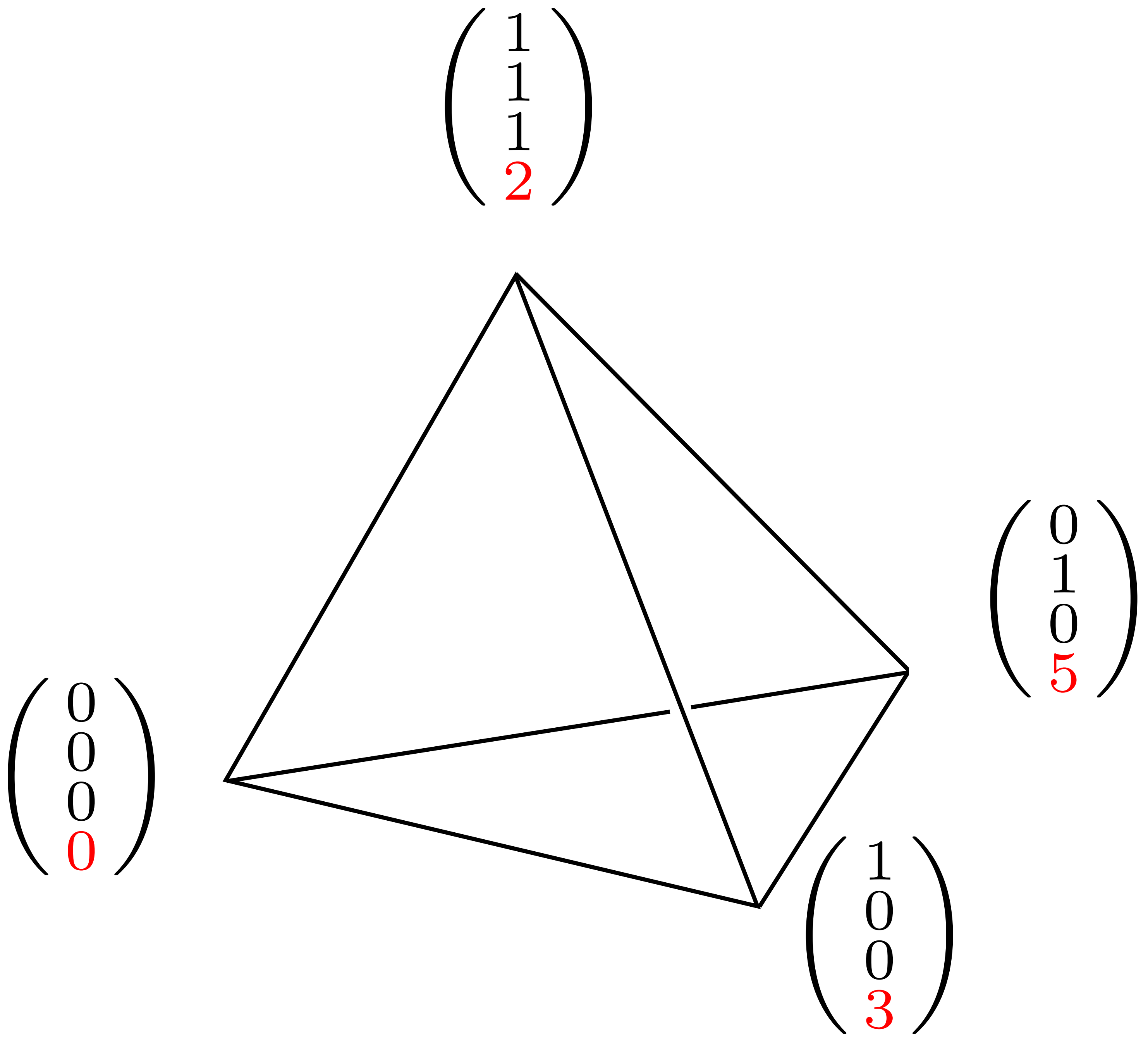
$$v^b\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right) = 0, \quad v^b\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) = 3,$$

$$v^b\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) = 5, \quad v^b\left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right) = 2$$



2. Auctioneer's decision

Auctioneer sets a price

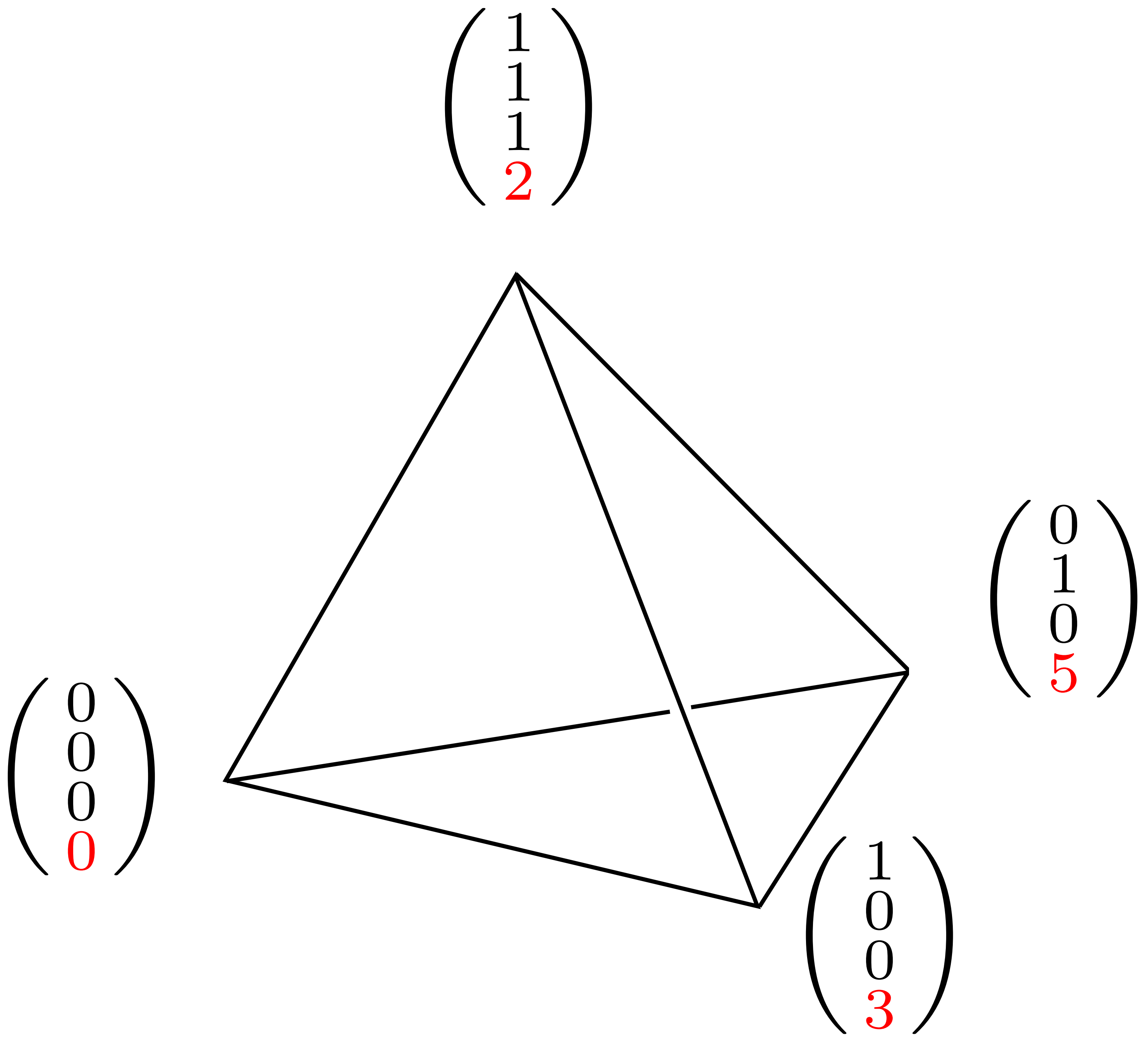


2. Auctioneer's decision

Auctioneer sets a price

Auctioneer computes the *demand set* of bidder *b* at price $p \in \mathbb{R}^{n+|E|}$:

$$D(v^b, p) = \underset{a \in \mathrm{vert}(P(G))}{\mathrm{argmax}} \quad \{v^b(a) - \langle p, a \rangle\}$$



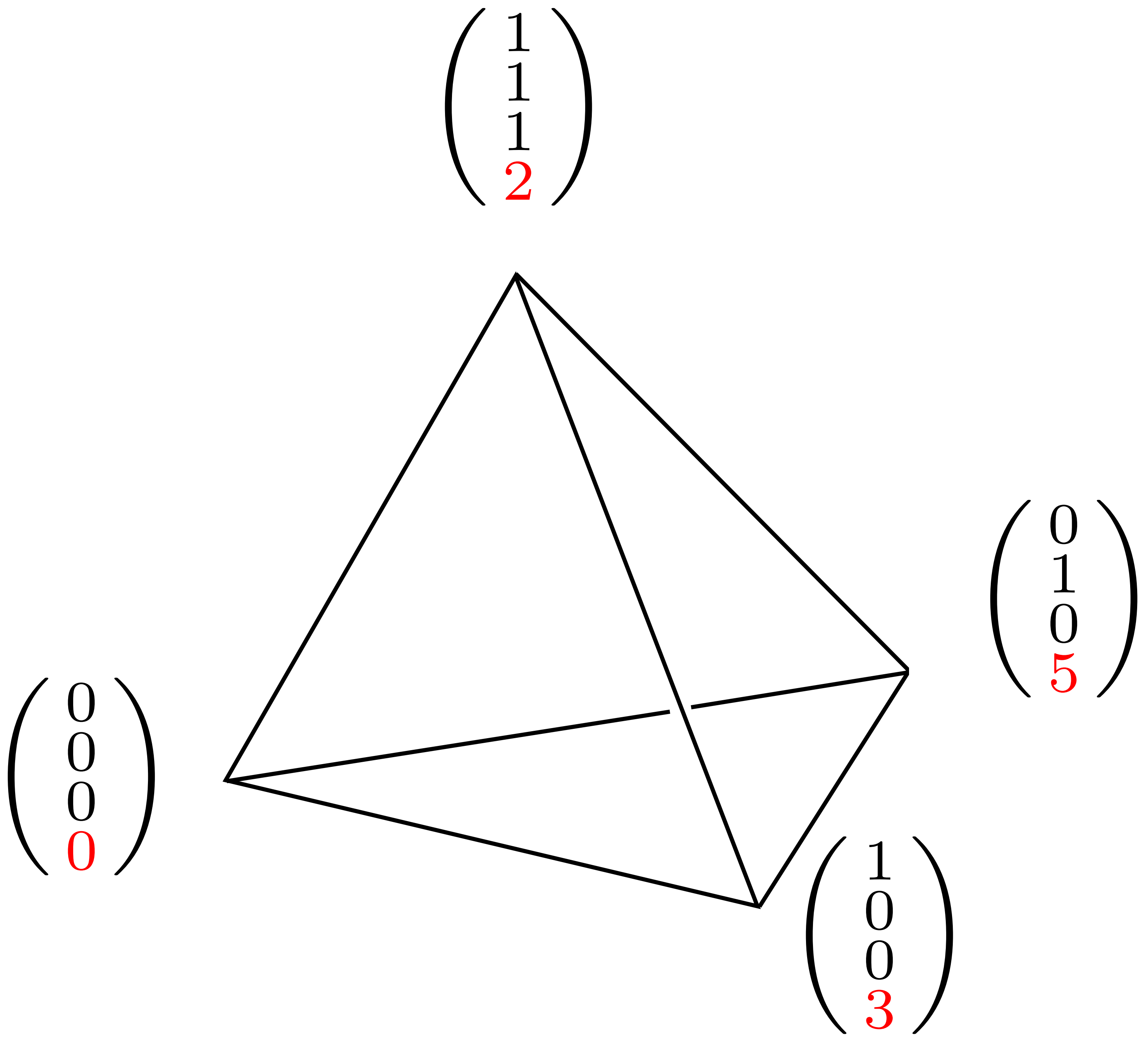
2. Auctioneer's decision

Auctioneer sets a price

Auctioneer computes the *demand set* of bidder *b* at price $p \in \mathbb{R}^{n+|E|}$:

$$D(v^b, p) = \underset{a \in \mathrm{vert}(P(G))}{\mathrm{argmax}} \quad \{v^b(a) - \langle p, a \rangle\}$$

Tropical aspects: coming soon!



2. Auctioneer's decision

Auctioneer sets a price

Auctioneer computes the *demand set* of bidder *b* at price $p \in \mathbb{R}^{n+|E|}$:

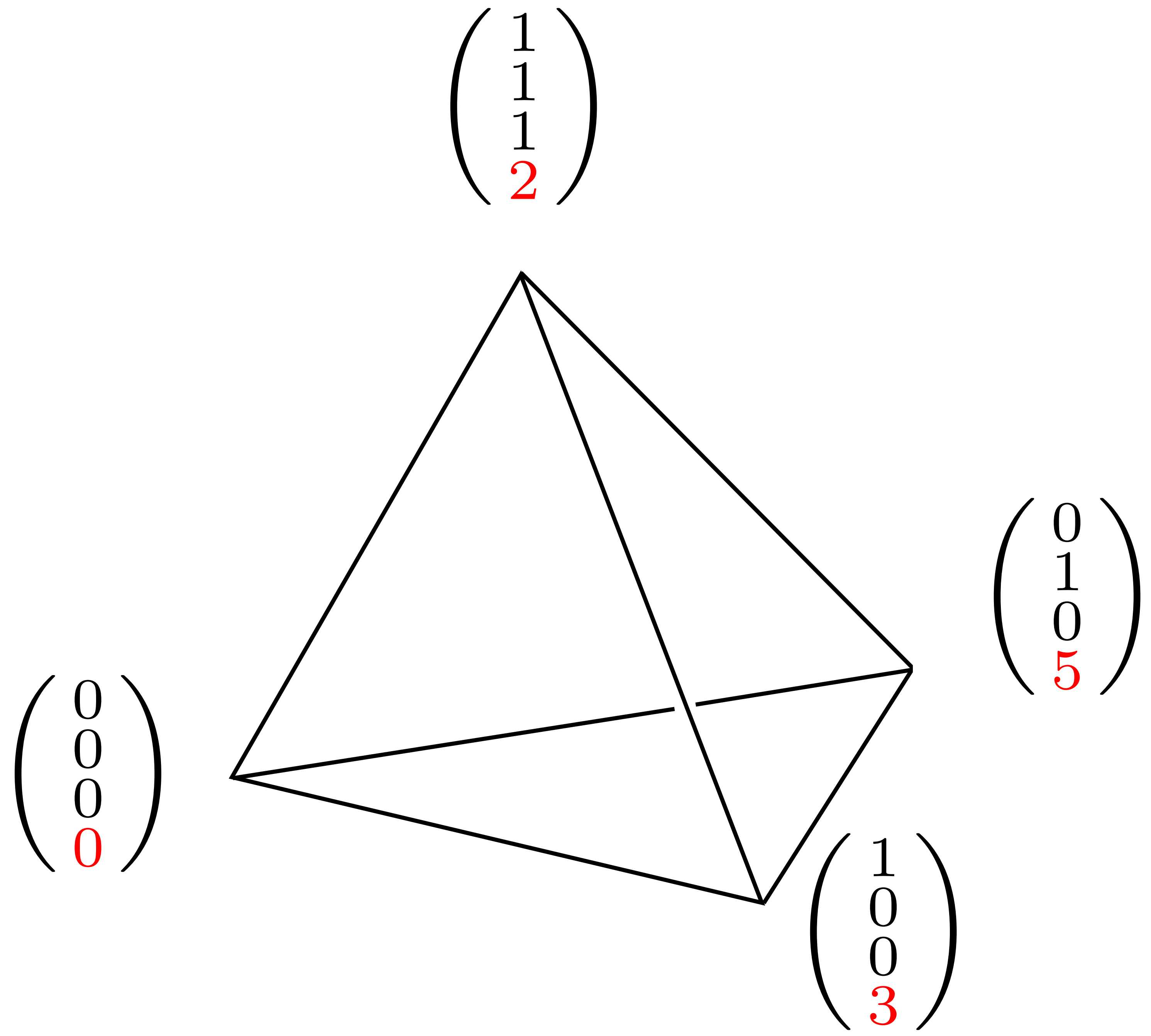
$$D(v^b, p) = \underset{a \in \mathrm{vert}(P(G))}{\operatorname{argmax}} \quad \{v^b(a) - \langle p, a \rangle\}$$

Tropical aspects: coming soon!

$$w^b = \begin{pmatrix} 3 \\ 5 \\ -6 \end{pmatrix}$$

$$p = \begin{pmatrix} 4 \\ 4 \\ -2 \end{pmatrix}$$

a	$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$



2. Auctioneer's decision

Auctioneer sets a price

Auctioneer computes the *demand set* of bidder *b* at price $p \in \mathbb{R}^{n+|E|}$:

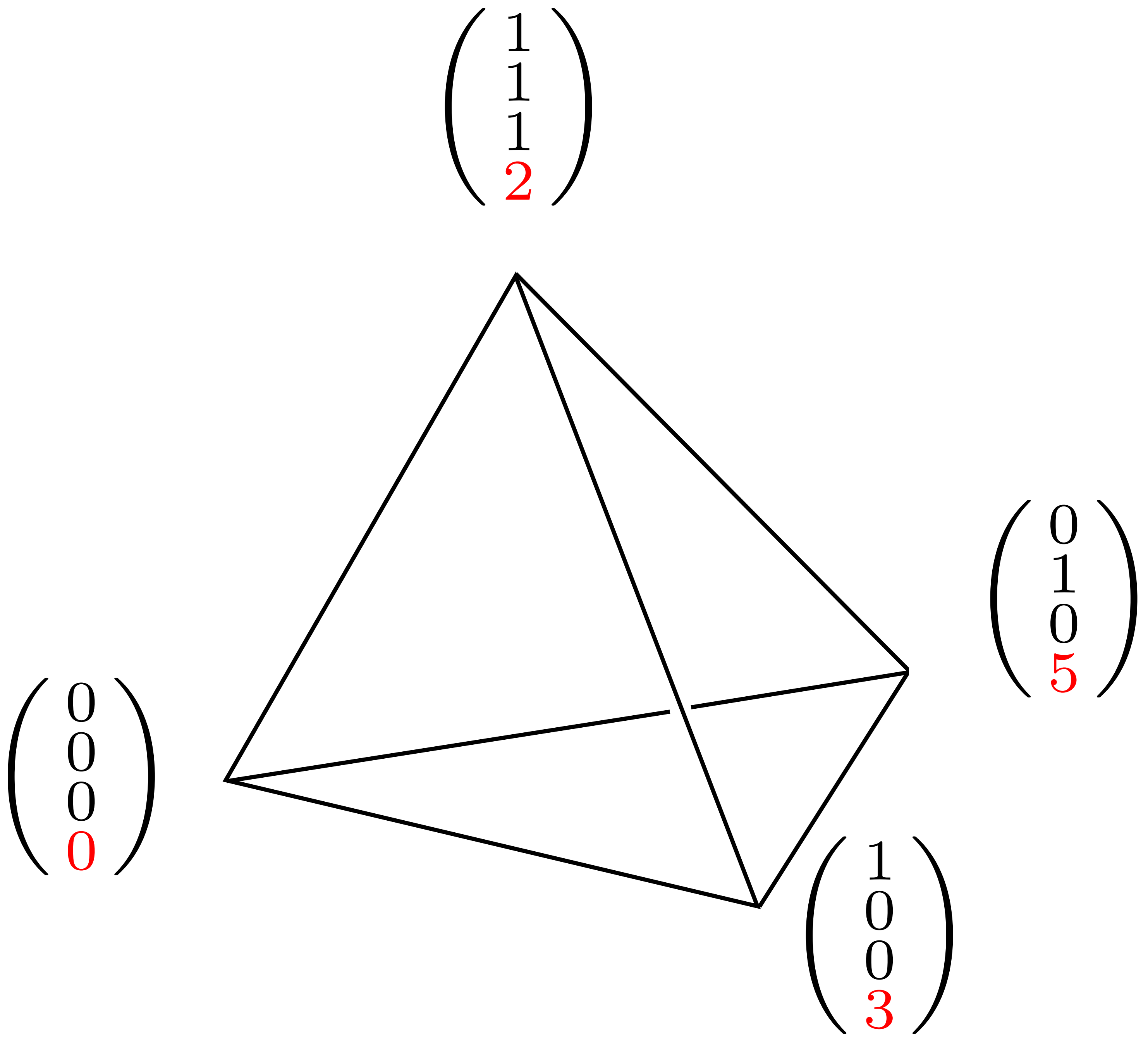
$$D(v^b, p) = \underset{a \in \mathrm{vert}(P(G))}{\mathrm{argmax}} \quad \{v^b(a) - \langle p, a \rangle\}$$

Tropical aspects: coming soon!

$$w^b = \begin{pmatrix} 3 \\ 5 \\ -6 \end{pmatrix}$$

$$p = \begin{pmatrix} 4 \\ 4 \\ -2 \end{pmatrix}$$

a	$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$
$v^b(a) = \langle w^b, a \rangle$	0	3	5	2



2. Auctioneer's decision

Auctioneer sets a price

Auctioneer computes the *demand set* of bidder *b* at price $p \in \mathbb{R}^{n+|E|}$:

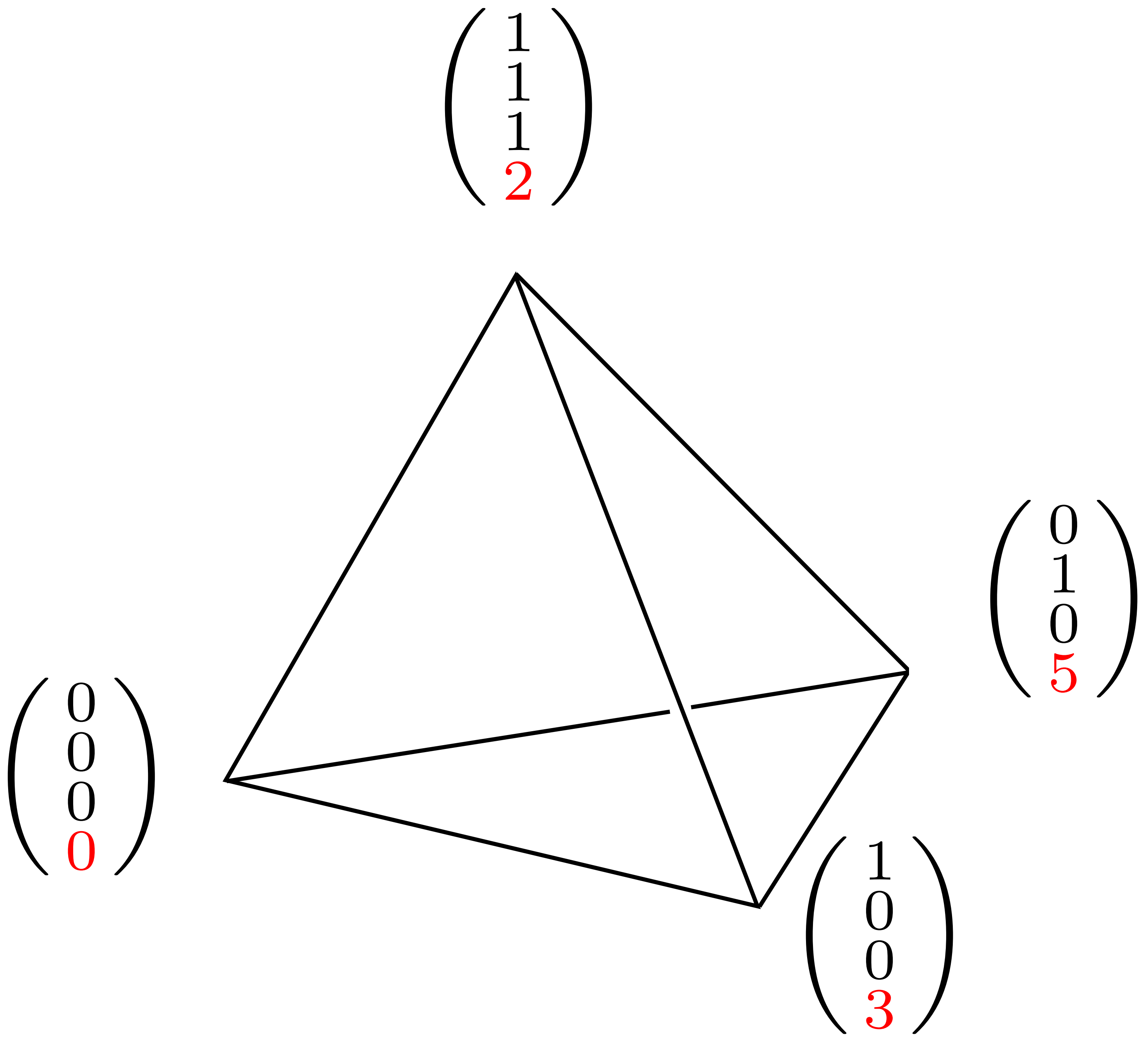
$$D(v^b, p) = \operatorname*{argmax}_{a \in \mathrm{vert}(P(G))} \{v^b(a) - \langle p, a \rangle\}$$

Tropical aspects: coming soon!

$$w^b = \begin{pmatrix} 3 \\ 5 \\ -6 \end{pmatrix}$$

$$p = \begin{pmatrix} 4 \\ 4 \\ -2 \end{pmatrix}$$

a	$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$
$v^b(a) = \langle w^b, a \rangle$	0	3	5	2
$\langle p, a \rangle$	0	4	4	6



2. Auctioneer's decision

Auctioneer sets a price

Auctioneer computes the *demand set* of bidder *b* at price $p \in \mathbb{R}^{n+|E|}$:

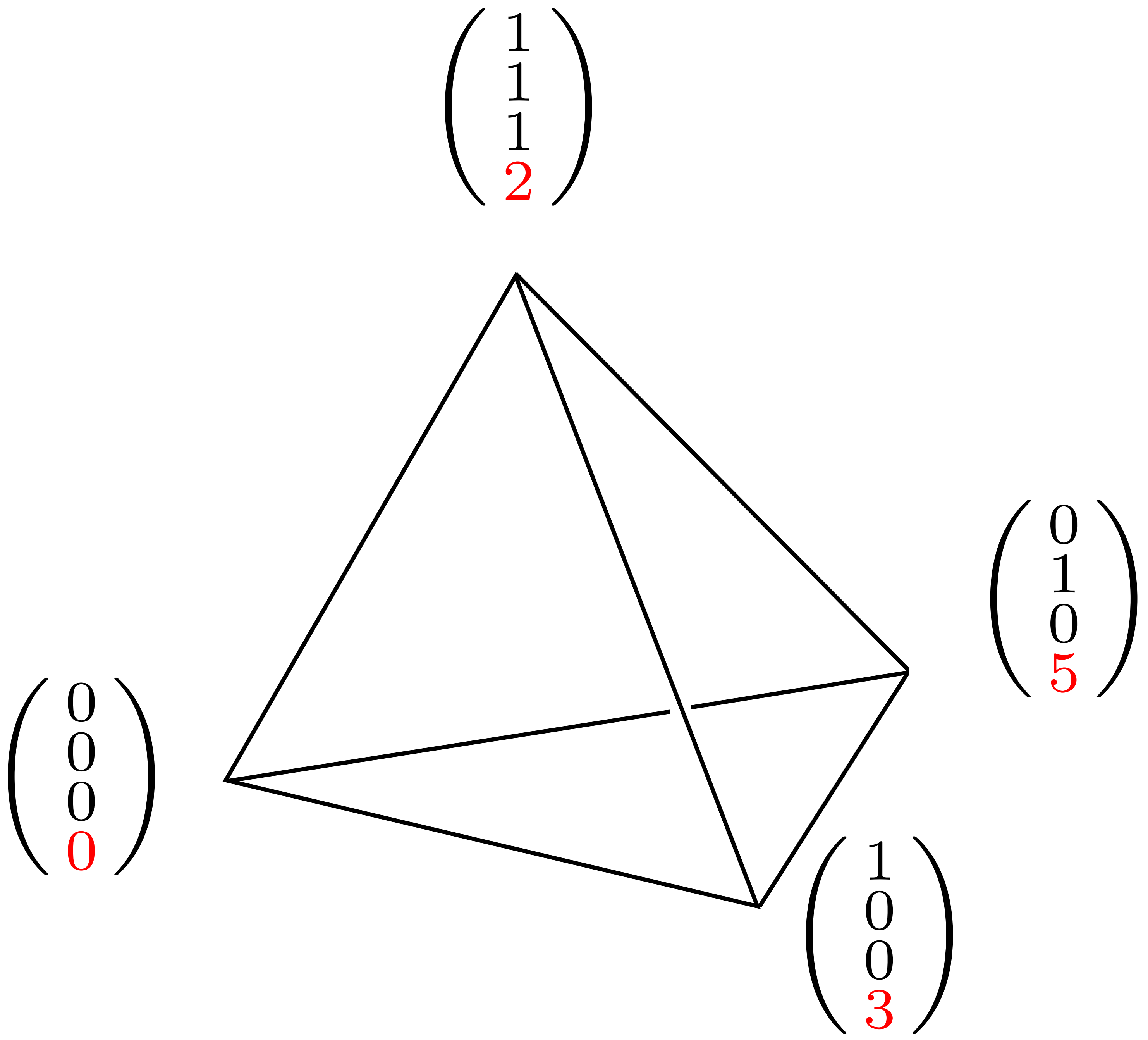
$$D(v^b, p) = \operatorname*{argmax}_{a \in \mathrm{vert}(P(G))} \{v^b(a) - \langle p, a \rangle\}$$

Tropical aspects: coming soon!

$$w^b = \begin{pmatrix} 3 \\ 5 \\ -6 \end{pmatrix}$$

$$p = \begin{pmatrix} 4 \\ 4 \\ -2 \end{pmatrix}$$

a	$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$
$v^b(a) = \langle w^b, a \rangle$	0	3	5	2
$\langle p, a \rangle$	0	4	4	6
$v^b(a) - \langle p, a \rangle$	0	-1	1	-4



2. Auctioneer's decision

Auctioneer sets a price

Auctioneer computes the *demand set* of bidder *b* at price $p \in \mathbb{R}^{n+|E|}$:

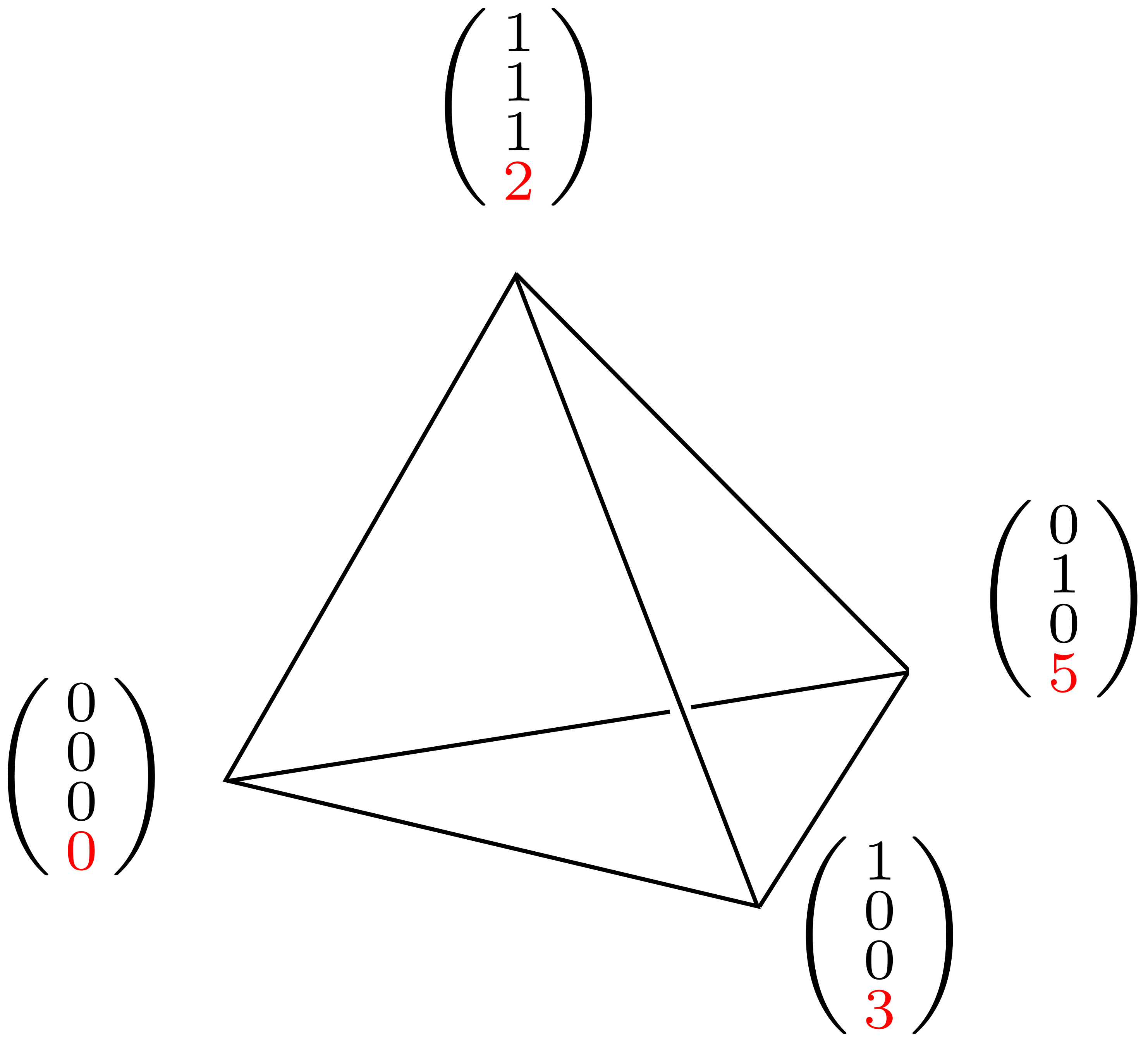
$$D(v^b, p) = \underset{a \in \text{vert}(P(G))}{\text{argmax}} \quad \{v^b(a) - \langle p, a \rangle\}$$

$$D(v^b, p) = \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$$w^b = \begin{pmatrix} 3 \\ 5 \\ -6 \end{pmatrix}$$

$$p = \begin{pmatrix} 4 \\ 4 \\ -2 \end{pmatrix}$$

a	$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$
$v^b(a) = \langle w^b, a \rangle$	0	3	5	2
$\langle p, a \rangle$	0	4	4	6
$v^b(a) - \langle p, a \rangle$	0	-1	1	-4



2. Auctioneer's decision

Auctioneer sets a price

Auctioneer computes the *demand set* of bidder *b* at price $p \in \mathbb{R}^{n+|E|}$:

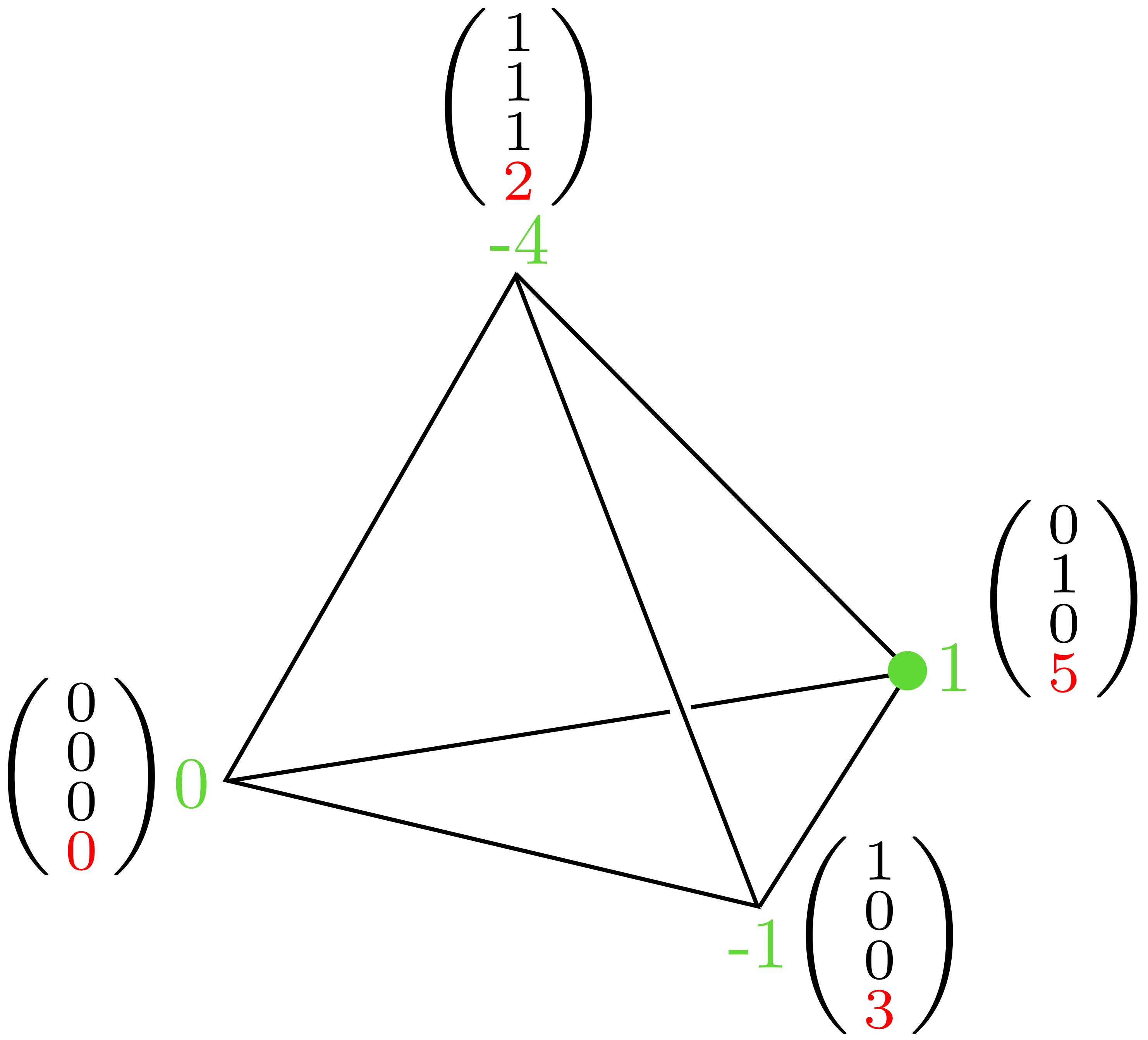
$$D(v^b, p) = \underset{a \in \mathrm{vert}(P(G))}{\mathrm{argmax}} \quad \{v^b(a) - \langle p, a \rangle\}$$

$$D(v^b, p) = \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$$w^b = \begin{pmatrix} 3 \\ 5 \\ -6 \end{pmatrix}$$

$$p = \begin{pmatrix} 4 \\ 4 \\ -2 \end{pmatrix}$$

a	$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$
$v^b(a) = \langle w^b, a \rangle$	0	3	5	2
$\langle p, a \rangle$	0	4	4	6
$v^b(a) - \langle p, a \rangle$	0	-1	1	-4



2. Auctioneer's decision

Auctioneer sets a price

Auctioneer computes the *demand set* of bidder *b* at price $p \in \mathbb{R}^{n+|E|}$:

$$D(v^b, p) = \underset{a \in \text{vert}(P(G))}{\text{argmax}} \{v^b(a) - \langle p, a\rangle\}$$

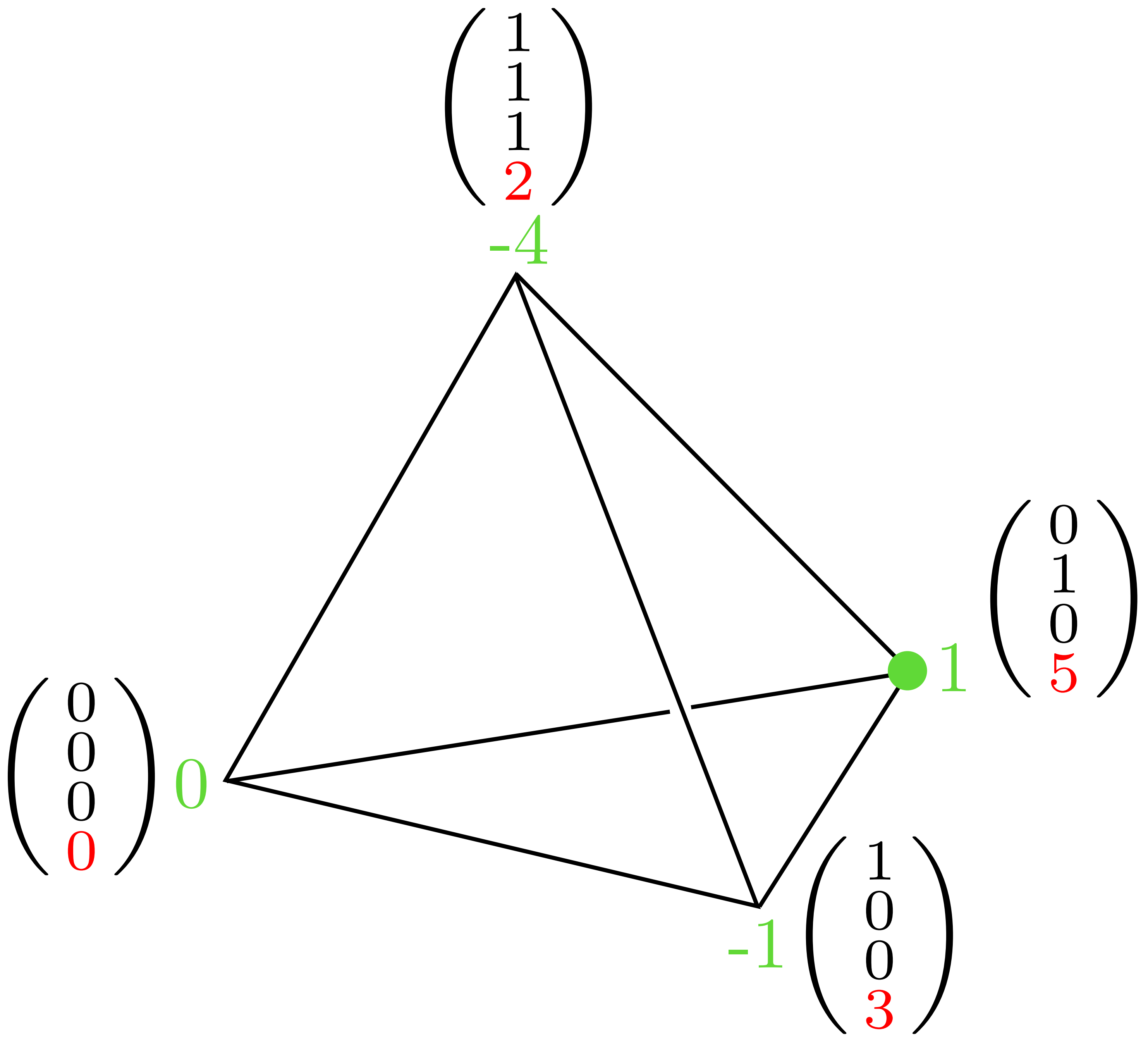
$$a \in D(v^b, p) \iff \langle \begin{pmatrix} a \\ v^b(a) \end{pmatrix}, \begin{pmatrix} -p \\ 1 \end{pmatrix} \rangle \text{ maximal}$$

$$D(v^b, p) = \{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \}$$

$$w^b = \begin{pmatrix} 3 \\ 5 \\ -6 \end{pmatrix}$$

$$p = \begin{pmatrix} 4 \\ 4 \\ -2 \end{pmatrix}$$

a	$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$
$v^b(a) = \langle w^b, a\rangle$	0	3	5	2
$\langle p, a\rangle$	0	4	4	6
$v^b(a) - \langle p, a\rangle$	0	-1	1	-4



2. Auctioneer's decision

Auctioneer sets a price

Auctioneer computes the *demand set* of bidder b at price $p \in \mathbb{R}^{n+|E|}$:

$$D(v^b, p) = \underset{a \in \mathrm{vert}(P(G))}{\mathrm{argmax}} \quad \{v^b(a) - \langle p, a \rangle\} = \mathrm{vert}(F^b) \text{ for some } F^b \preceq P(G)$$

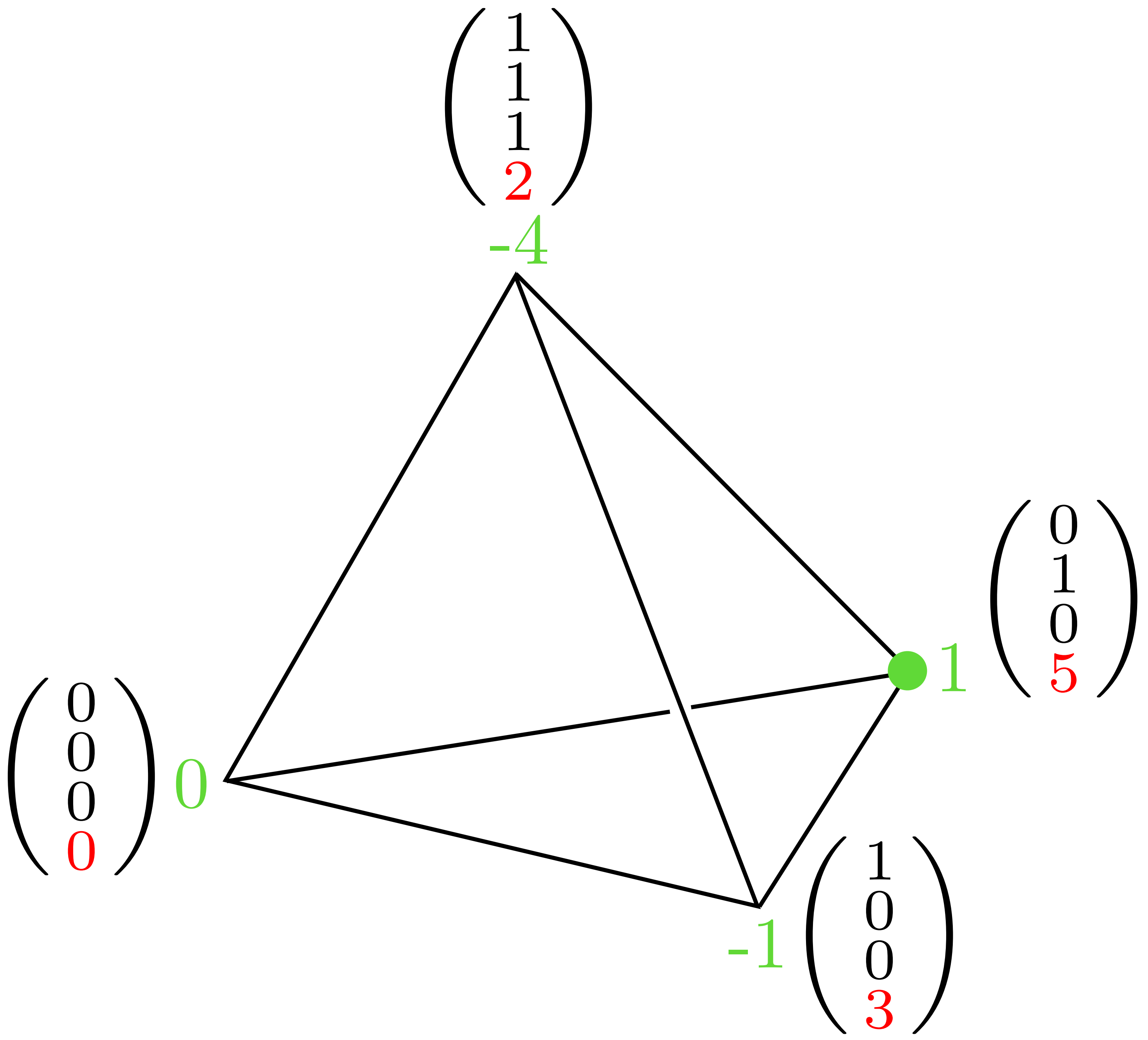
$$a \in D(v^b, p) \iff \langle \begin{pmatrix} a \\ v^b(a) \end{pmatrix}, \begin{pmatrix} -p \\ 1 \end{pmatrix} \rangle \text{ maximal}$$

$$D(v^b, p) = \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$$w^b = \begin{pmatrix} 3 \\ 5 \\ -6 \end{pmatrix}$$

$$p = \begin{pmatrix} 4 \\ 4 \\ -2 \end{pmatrix}$$

a	$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$
$v^b(a) = \langle w^b, a \rangle$	0	3	5	2
$\langle p, a \rangle$	0	4	4	6
$v^b(a) - \langle p, a \rangle$	0	-1	1	-4



2. Auctioneer's decision

Auctioneer sets a price

Auctioneer computes the *demand set* of bidder b at price $p \in \mathbb{R}^{n+|E|}$:

$$D(v^b, p) = \operatorname*{argmax}_{a \in \mathrm{vert}(P(G))} \{v^b(a) - \langle p, a \rangle\} = \mathrm{vert}(F^b) \text{ for some } F^b \preceq P(G)$$

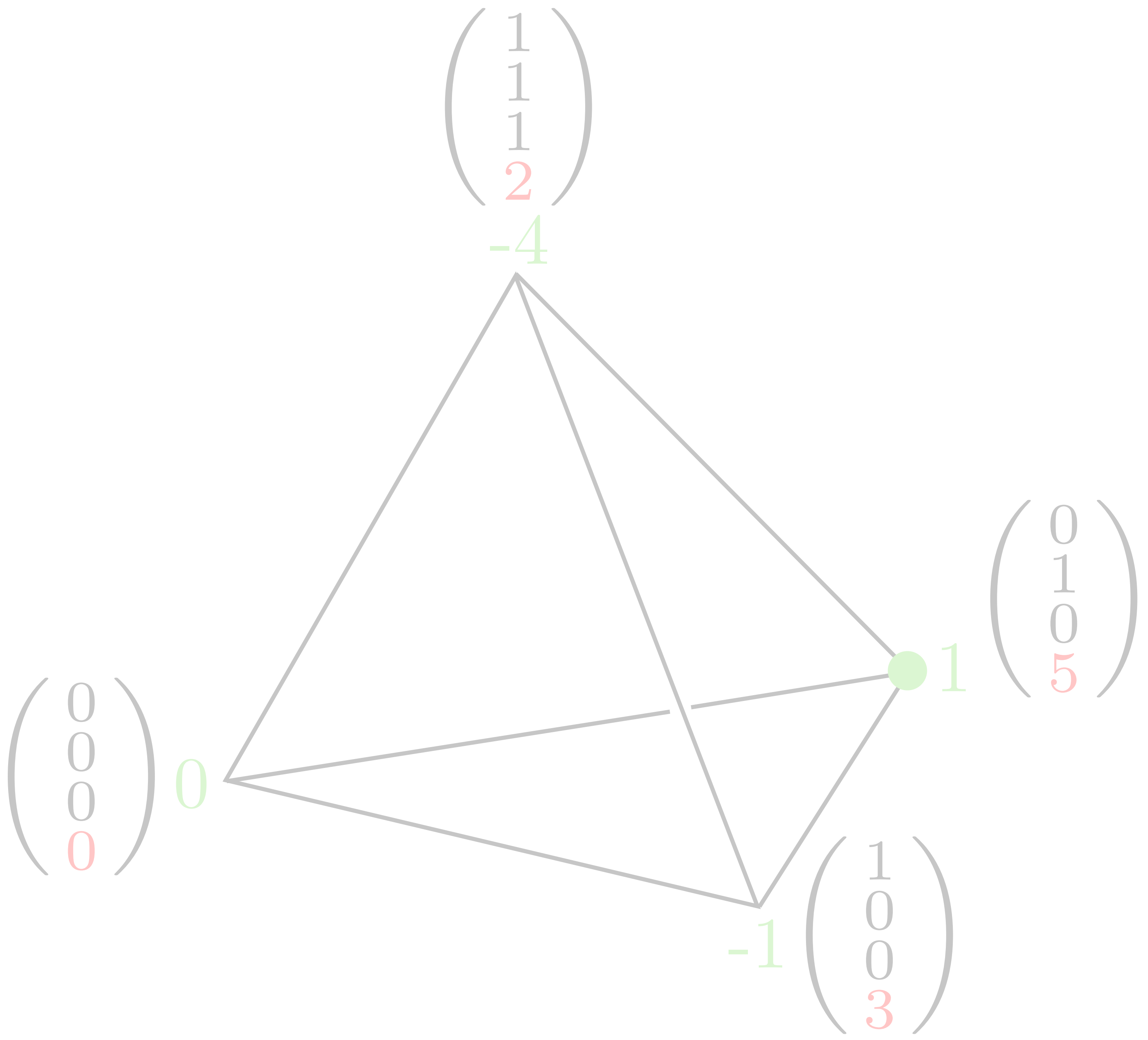
$$a \in D(v^b, p) \iff \langle \begin{pmatrix} a \\ v^b(a) \end{pmatrix}, \begin{pmatrix} -p \\ 1 \end{pmatrix} \rangle \text{ maximal}$$

$$D(v^b, p) = \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$$w^b = \begin{pmatrix} 3 \\ 5 \\ -6 \end{pmatrix}$$

$$p = \begin{pmatrix} 4 \\ 4 \\ -2 \end{pmatrix}$$

a	$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$
$v^b(a) = \langle w^b, a \rangle$	0	3	5	2
$\langle p, a \rangle$	0	4	4	6
$v^b(a) - \langle p, a \rangle$	0	-1	1	-4



2. Auctioneer's decision

Auctioneer sets a price

Auctioneer computes the *demand set* of bidder *b* at price $p \in \mathbb{R}^{n+|E|}$:

$$D(v^b, p) = \underset{a \in \mathrm{vert}(P(G))}{\operatorname{argmax}} \{v^b(a) - \langle p, a \rangle\} = \mathrm{vert}(F^b) \text{ for some } F^b \preceq P(G)$$

$$a \in D(v^b, p) \iff \langle \begin{pmatrix} a \\ v^b(a) \end{pmatrix}, \begin{pmatrix} -p \\ 1 \end{pmatrix} \rangle \text{ maximal}$$

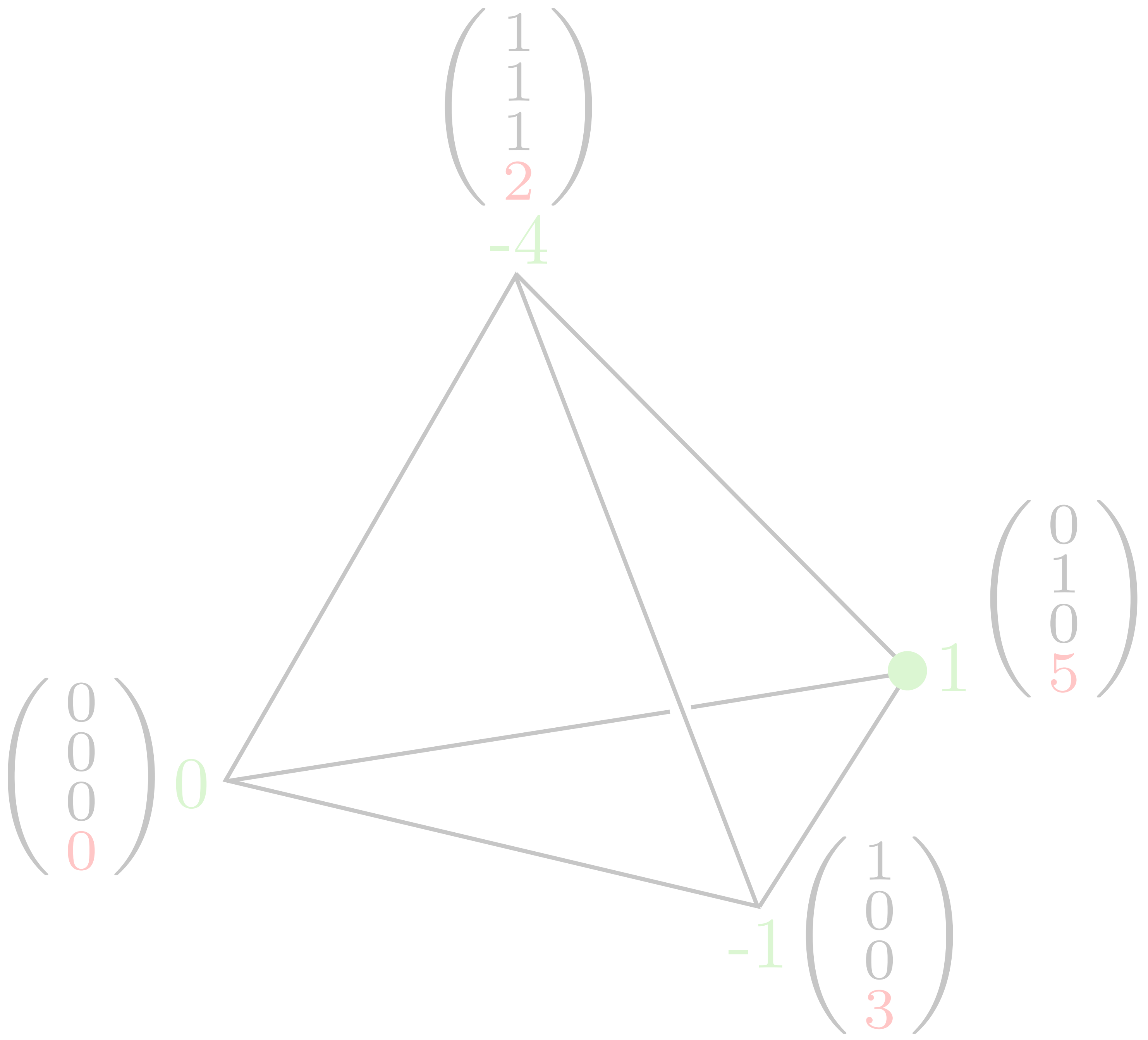
Auctioneer wants to find price $p \in \mathbb{R}^{n+|E|}$ and a distribution $a^b \in \mathrm{vert}(P(G)), b \in [m]$ s.t.

$$D(v^b, p) = \{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\}$$

$$w^b = \begin{pmatrix} 3 \\ 5 \\ -6 \end{pmatrix}$$

$$p = \begin{pmatrix} 4 \\ 4 \\ -2 \end{pmatrix}$$

a	$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$
$v^b(a) = \langle w^b, a \rangle$	0	3	5	2
$\langle p, a \rangle$	0	4	4	6
$v^b(a) - \langle p, a \rangle$	0	-1	1	-4



2. Auctioneer's decision

Auctioneer sets a price

Auctioneer computes the *demand set* of bidder b at price $p \in \mathbb{R}^{n+|E|}$:

$$D(v^b, p) = \underset{a \in \mathrm{vert}(P(G))}{\mathrm{argmax}} \{v^b(a) - \langle p, a \rangle\} = \mathrm{vert}(F^b) \text{ for some } F^b \preceq P(G)$$

$$a \in D(v^b, p) \iff \langle \begin{pmatrix} a \\ v^b(a) \end{pmatrix}, \begin{pmatrix} -p \\ 1 \end{pmatrix} \rangle \text{ maximal}$$

Auctioneer wants to find price $p \in \mathbb{R}^{n+|E|}$ and a distribution $a^b \in \mathrm{vert}(P(G)), b \in [m]$ s.t.

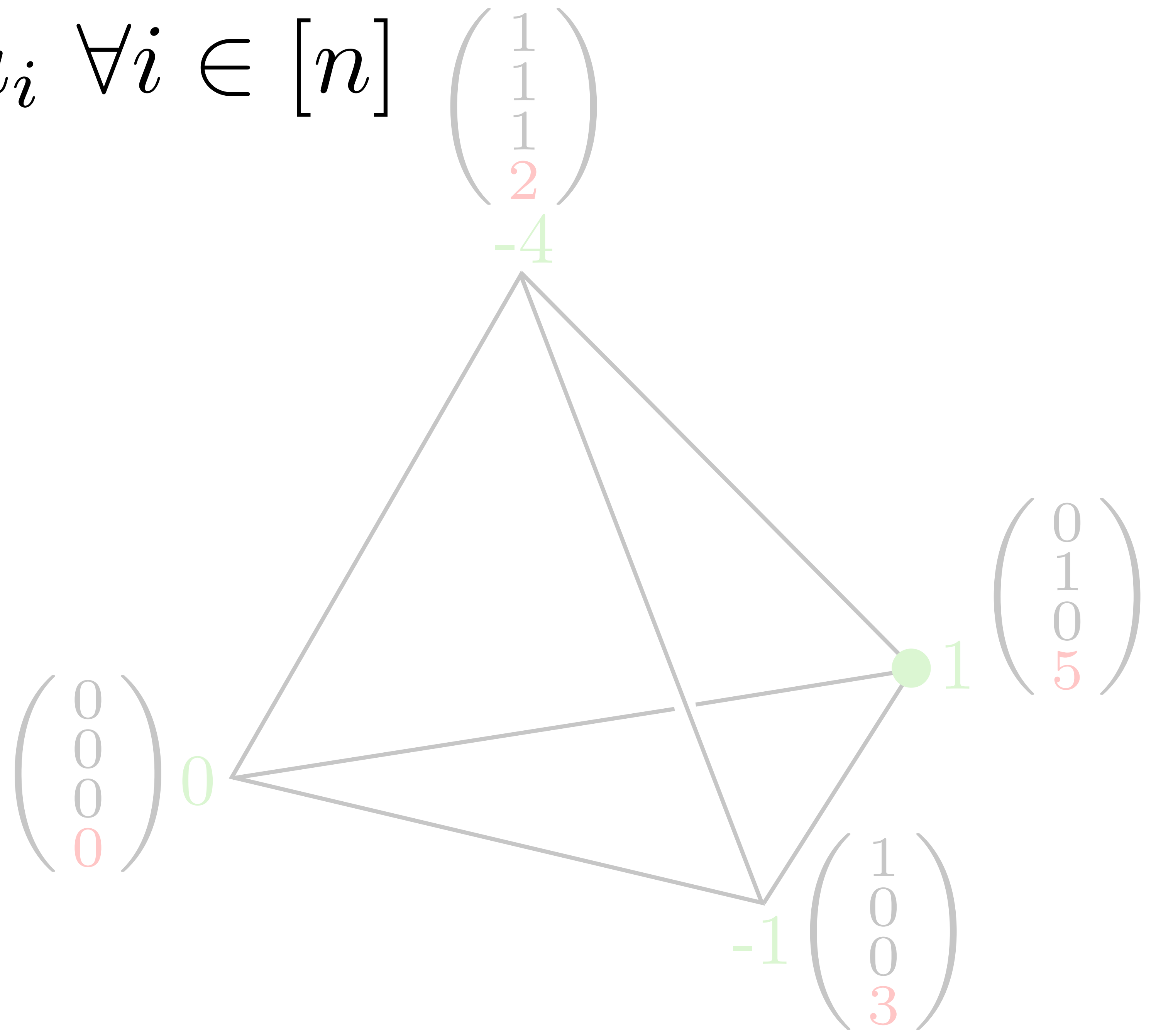
$$\forall\, b \in [m]\ \exists\, a^b \in D(v^b, p) : a = \sum_{b \in [m]} a^b \text{ and } a_i^* = a_i\ \forall i \in [n]$$

$$D(v^b, p) = \{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\}$$

$$w^b = \begin{pmatrix} 3 \\ 5 \\ -6 \end{pmatrix}$$

$$p = \begin{pmatrix} 4 \\ 4 \\ -2 \end{pmatrix}$$

a	$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$
$v^b(a) = \langle w^b, a \rangle$	0	3	5	2
$\langle p, a \rangle$	0	4	4	6
$v^b(a) - \langle p, a \rangle$	0	-1	1	-4



2. Auctioneer's decision

Auctioneer sets a price

Auctioneer computes the *demand set* of bidder *b* at price $p \in \mathbb{R}^{n+|E|}$:

$$D(v^b, p) = \operatorname*{argmax}_{a \in \text{vert}(P(G))} \{v^b(a) - \langle p, a \rangle\} = \text{vert}(F^b) \text{ for some } F^b \preceq P(G)$$

$$a \in D(v^b, p) \iff \langle \begin{pmatrix} a \\ v^b(a) \end{pmatrix}, \begin{pmatrix} -p \\ 1 \end{pmatrix} \rangle \text{ maximal}$$

Auctioneer wants to find price $p \in \mathbb{R}^{n+|E|}$ and a distribution $a^b \in \text{vert}(P(G)), b \in [m]$ s.t.

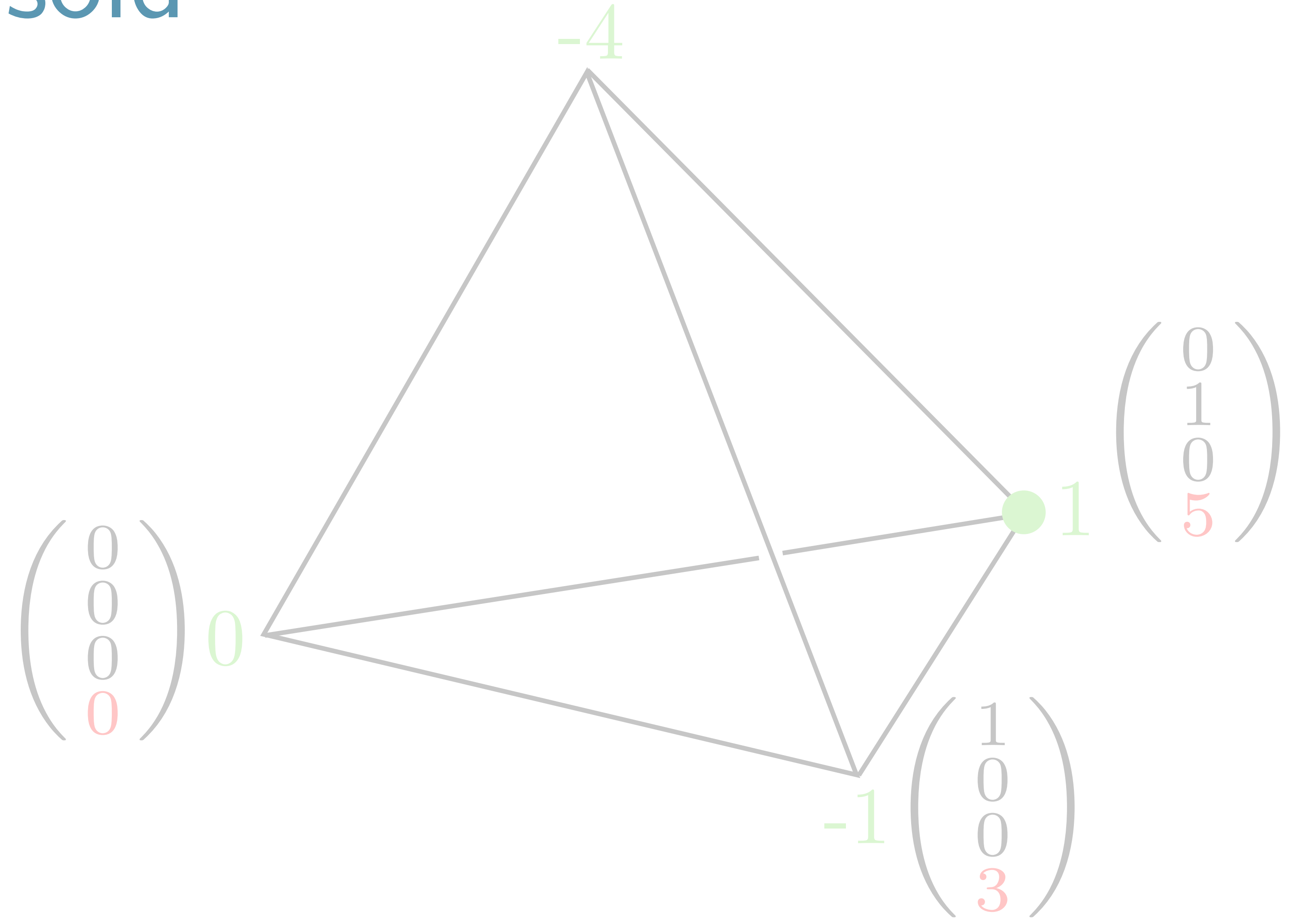
$$\underbrace{\forall\, b \in [m]\ \exists\, a^b \in D(v^b, p)}_{\text{all bidders are happy}} : \underbrace{a = \sum_{b \in [m]} a^b \text{ and } a_i^* = a_i\ \forall i \in [n]}_{\text{all items are sold}}$$

$$D(v^b, p) = \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$$w^b = \begin{pmatrix} 3 \\ 5 \\ -6 \end{pmatrix}$$

$$p = \begin{pmatrix} 4 \\ 4 \\ -2 \end{pmatrix}$$

a	$(0,0,0)^T$	$(1,0,0)^T$	$(0,1,0)^T$	$(1,1,1)^T$
$v^b(a) = \langle w^b, a \rangle$	0	3	5	2
$\langle p, a \rangle$	0	4	4	6
$v^b(a) - \langle p, a \rangle$	0	-1	1	-4



Competitive equilibrium

Definitions

Let $\pi : mP \cap \mathbb{Z}^{n+|E|} \to \mathbb{Z}^n$ be the coordinate projection. $(m = \#\ \text{bidders})$

Competitive equilibrium

Definitions

Let $\pi : mP \cap \mathbb{Z}^{n+|E|} \to \mathbb{Z}^n$ be the coordinate projection. $(m = \#\ \text{bidders})$

Definition.

Given valuations $\{v^b \mid b \in [m]\}$, a *competitive equilibrium exists* if there exist

$$p \in \mathbb{R}^{n+|E|}, a \in \sum_{b \in [m]} D(v^b, p) \text{ such that } a \in \pi^{-1}(a^*) \ (\text{i.e. } a_i^* = a_i \ \forall i \in [n]).$$

Competitive equilibrium

Definitions

Let $\pi : mP \cap \mathbb{Z}^{n+|E|} \to \mathbb{Z}^n$ be the coordinate projection. ($m = \#$ bidders)

Definition.
Given valuations $\{v^b \mid b \in [m]\}$, a *competitive equilibrium exists* if there exist $p \in \mathbb{R}^{n+|E|}, a \in \sum_{b\in[m]} D(v^b, p)$ such that $a \in \pi^{-1}(a^*)$ (i.e. $a_i^* = a_i \ \forall i \in [n]$).

Definition.
A competitive equilibrium is *guaranteed to exist* if for any set of valuations $\{v^b \mid b \in [m]\}$ there exists $p \in \mathbb{R}^{n+|E|}, a \in \sum_{b\in[m]} D(v^b, p)$ such that $a \in \pi^{-1}(a^*)$.

Competitive equilibrium

and lattice polytopes

Definition.

A competitive equilibrium is *guaranteed to exist* if for any set of valuations $\{v^b \mid b \in [m]\}$ there exists $p \in \mathbb{R}^{n+|E|}, a \in \sum_{b\in[m]} D(v^b, p)$ such that $a \in \pi^{-1}(a^*)$.

Competitive equilibrium

and lattice polytopes

Definition.
A competitive equilibrium is *guaranteed to exist* if for any set of valuations $\{v^b \mid b \in [m]\}$ there exists $p \in \mathbb{R}^{n+|E|}, a \in \sum_{b\in[m]} D(v^b, p)$ such that $a \in \pi^{-1}(a^*)$.

Lemma (B.-Haase-Tran, '21+).
Let $a^* \in \mathbb{Z}^n_{\geq 0}$ and $a \in \pi^{-1}(a^*)$. Then TFAE:

Competitive equilibrium

and lattice polytopes

Definition.
A competitive equilibrium is *guaranteed to exist* if for any set of valuations $\{v^b \mid b \in [m]\}$ there exists $p \in \mathbb{R}^{n+|E|}, a \in \sum_{b\in[m]} D(v^b, p)$ such that $a \in \pi^{-1}(a^*)$.

Lemma (B.-Haase-Tran, '21+).
Let $a^* \in \mathbb{Z}^n_{\geq 0}$ and $a \in \pi^{-1}(a^*)$. Then TFAE:

a) $\forall \{v^b \mid b \in [m]\} \; \exists \, p \in \mathbb{R}^{n+|E|} : \; a \in \sum_{b\in[m]} D(v^b, p)$

Competitive equilibrium
and lattice polytopes

Definition.
A competitive equilibrium is *guaranteed to exist* if for any set of valuations $\{v^b \mid b \in [m]\}$ there exists $p \in \mathbb{R}^{n+|E|}, a \in \sum_{b\in[m]} D(v^b, p)$ such that $a \in \pi^{-1}(a^*)$.

Lemma (B.-Haase-Tran, '21+).
Let $a^* \in \mathbb{Z}^n_{\geq 0}$ and $a \in \pi^{-1}(a^*)$. Then TFAE:

a) $\forall \ \{v^b \mid b \in [m]\} \ \exists \ p \in \mathbb{R}^{n+|E|} : \ a \in \sum_{b\in[m]} D(v^b, p)$

b) $\forall \ F^1, \ldots, F^m \preceq P(G) : \ \text{if } a \in \sum_{b\in[m]} F^b \text{ then } a \in \sum_{b\in[m]} \text{vert}(F^b)$

Competitive equilibrium

and lattice polytopes

Definition.
A competitive equilibrium is *guaranteed to exist* if for any set of valuations $\{v^b \mid b \in [m]\}$ there exists $p \in \mathbb{R}^{n+|E|}, a \in \sum_{b\in[m]} D(v^b, p)$ such that $a \in \pi^{-1}(a^*)$.

Lemma (B.-Haase-Tran, '21+).
Let $a^* \in \mathbb{Z}^n_{\geq 0}$ and $a \in \pi^{-1}(a^*)$. Then TFAE:

a) $\forall \{v^b \mid b \in [m]\} \ \exists \ p \in \mathbb{R}^{n+|E|} : \ a \in \sum_{b\in[m]} D(v^b, p)$

b) $\forall \ F^1, \ldots, F^m \preceq P(G) : \ \text{if } a \in \sum_{b\in[m]} F^b \text{ then } a \in \sum_{b\in[m]} \text{vert}(F^b)$

In particular, then a CE is guaranteed to exist.

Tropical Intermezzo

Tropical Intermezzo

Notation: $a \oplus b = \max\{a, b\}, a \odot b = a + b$

Tropical Intermezzo

Notation: $a \oplus b = \max\{a, b\}, a \odot b = a + b$

$$f_v(x) = \bigoplus_{a \in P(G) \cap \mathbb{Z}^d} v(a) \odot x^{\odot a}$$ tropical Laurent polynomial,

Tropical Intermezzo

Notation: $a \oplus b = \max\{a, b\}, a \odot b = a + b$

$f_v(x) = \bigoplus_{a \in P(G) \cap \mathbb{Z}^d} v(a) \odot x^{\odot a}$ tropical Laurent polynomial, $f_v(-p) = \max_{a \in P(G) \cap \mathbb{Z}^d} \{v(a) - \langle p, a \rangle\}$

Tropical Intermezzo

Notation: $a \oplus b = \max\{a, b\}, a \odot b = a + b$

$f_v(x) = \bigoplus_{a \in P(G) \cap \mathbb{Z}^d} v(a) \odot x^{\odot a}$ tropical Laurent polynomial, $f_v(-p) = \max_{a \in P(G) \cap \mathbb{Z}^d} \{v(a) - \langle p, a \rangle\}$

$\mathcal{T}(f_v)$ tropical hypersurface, i.e. $p \in -\mathcal{T}(f_v) \iff |D_v(p)| \geq 2.$

Tropical Intermezzo

Notation: $a \oplus b = \max\{a, b\}, a \odot b = a + b$

$f_v(x) = \bigoplus_{a \in P(G) \cap \mathbb{Z}^d} v(a) \odot x^{\odot a}$ tropical Laurent polynomial, $f_v(-p) = \max_{a \in P(G) \cap \mathbb{Z}^d} \{v(a) - \langle p, a \rangle\}$

$\mathcal{T}(f_v)$ tropical hypersurface, i.e. $p \in -\mathcal{T}(f_v) \iff |D_v(p)| \geq 2.$

$\mathcal{T}(f_v)$ induces the *price regions* $\{p \mid D_v(p) = D_v(p')\}$ on $\mathbb{R}^d$.

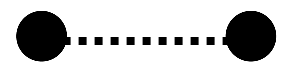
Tropical Intermezzo

Notation: $a \oplus b = \max\{a, b\}, a \odot b = a + b$

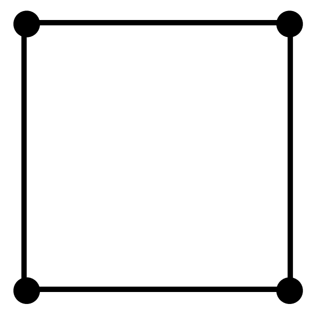
$f_v(x) = \bigoplus_{a \in P(G) \cap \mathbb{Z}^d} v(a) \odot x^{\odot a}$ tropical Laurent polynomial, $f_v(-p) = \max_{a \in P(G) \cap \mathbb{Z}^d} \{v(a) - \langle p, a \rangle\}$

$\mathcal{T}(f_v)$ tropical hypersurface, i.e. $p \in -\mathcal{T}(f_v) \iff |D_v(p)| \geq 2$.

$\mathcal{T}(f_v)$ induces the *price regions* $\{p \mid D_v(p) = D_v(p')\}$ on $\mathbb{R}^d$.



G



$P(G)$

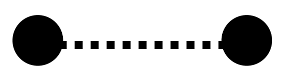
Tropical Intermezzo

Notation: $a \oplus b = \max\{a, b\}, a \odot b = a + b$

$f_v(x) = \bigoplus_{a \in P(G) \cap \mathbb{Z}^d} v(a) \odot x^{\odot a}$ tropical Laurent polynomial, $f_v(-p) = \max_{a \in P(G) \cap \mathbb{Z}^d} \{v(a) - \langle p, a \rangle\}$

$\mathcal{T}(f_v)$ tropical hypersurface, i.e. $p \in -\mathcal{T}(f_v) \iff |D_v(p)| \geq 2$.

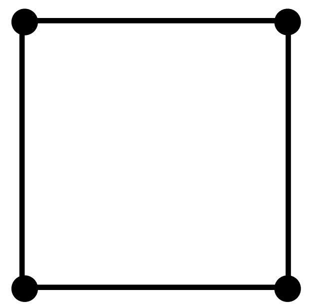
$\mathcal{T}(f_v)$ induces the *price regions* $\{p \mid D_v(p) = D_v(p')\}$ on $\mathbb{R}^d$.



G

$w^1 = \binom{3}{5}$

$$f_{v^1}(x) = 0 \odot x^{\odot \binom{0}{0}} \oplus 3 \odot x^{\odot \binom{1}{0}} \oplus 5 \odot x^{\odot \binom{0}{1}} \oplus 8 \odot x^{\odot \binom{1}{1}}$$



$P(G)$

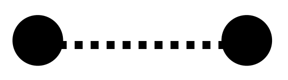
Tropical Intermezzo

Notation: $a \oplus b = \max\{a, b\}, a \odot b = a + b$

$f_v(x) = \bigoplus_{a \in P(G) \cap \mathbb{Z}^d} v(a) \odot x^{\odot a}$ tropical Laurent polynomial, $f_v(-p) = \max_{a \in P(G) \cap \mathbb{Z}^d} \{v(a) - \langle p, a \rangle\}$

$\mathcal{T}(f_v)$ tropical hypersurface, i.e. $p \in -\mathcal{T}(f_v) \iff |D_v(p)| \geq 2$.

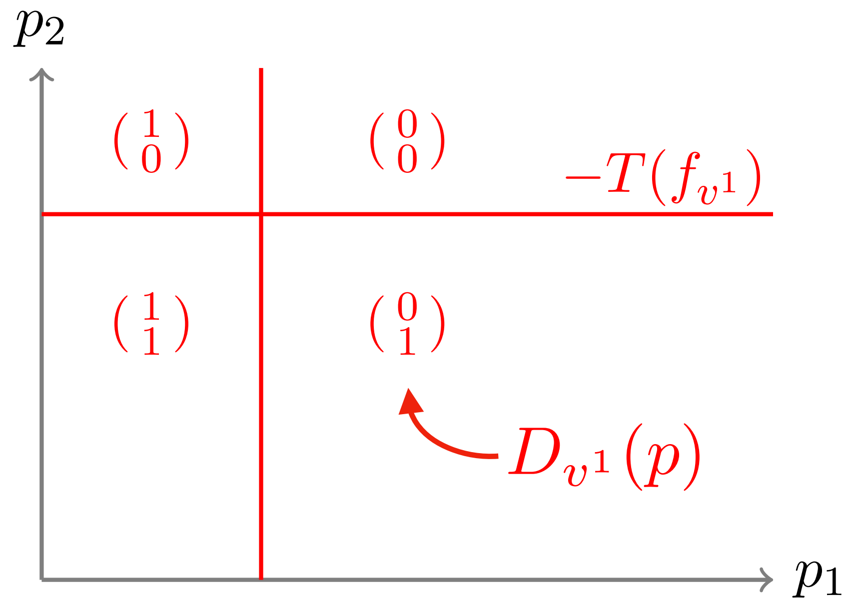
$\mathcal{T}(f_v)$ induces the *price regions* $\{p \mid D_v(p) = D_v(p')\}$ on $\mathbb{R}^d$.

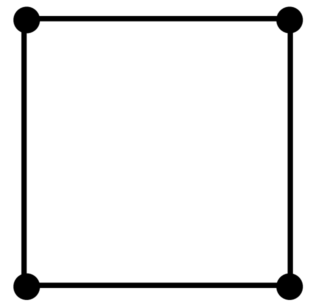


G

$w^1 = \binom{3}{5}$

$$f_{v^1}(x) = 0 \odot x^{\odot \binom{0}{0}} \oplus 3 \odot x^{\odot \binom{1}{0}} \oplus 5 \odot x^{\odot \binom{0}{1}} \oplus 8 \odot x^{\odot \binom{1}{1}}$$





$P(G)$

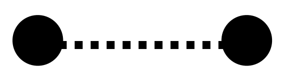
Tropical Intermezzo

Notation: $a \oplus b = \max\{a, b\}, a \odot b = a + b$

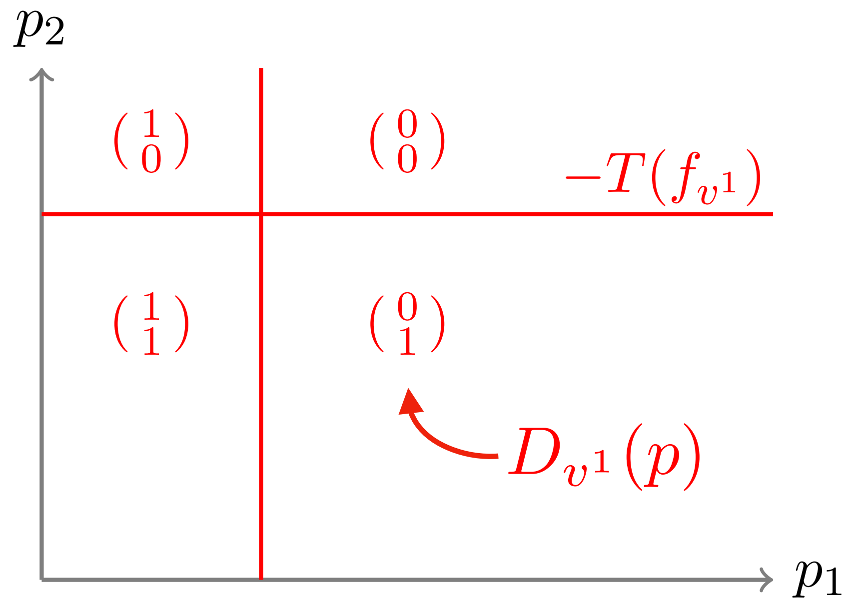
$f_v(x) = \bigoplus_{a \in P(G) \cap \mathbb{Z}^d} v(a) \odot x^{\odot a}$ tropical Laurent polynomial, $f_v(-p) = \max_{a \in P(G) \cap \mathbb{Z}^d} \{v(a) - \langle p, a \rangle\}$

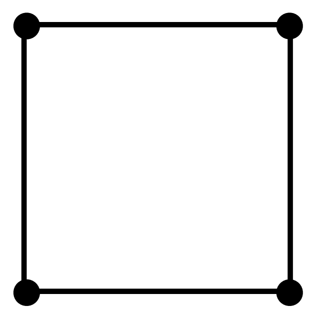
$\mathcal{T}(f_v)$ tropical hypersurface, i.e. $p \in -\mathcal{T}(f_v) \iff |D_v(p)| \geq 2$.

$\mathcal{T}(f_v)$ induces the *price regions* $\{p \mid D_v(p) = D_v(p')\}$ on $\mathbb{R}^d$.

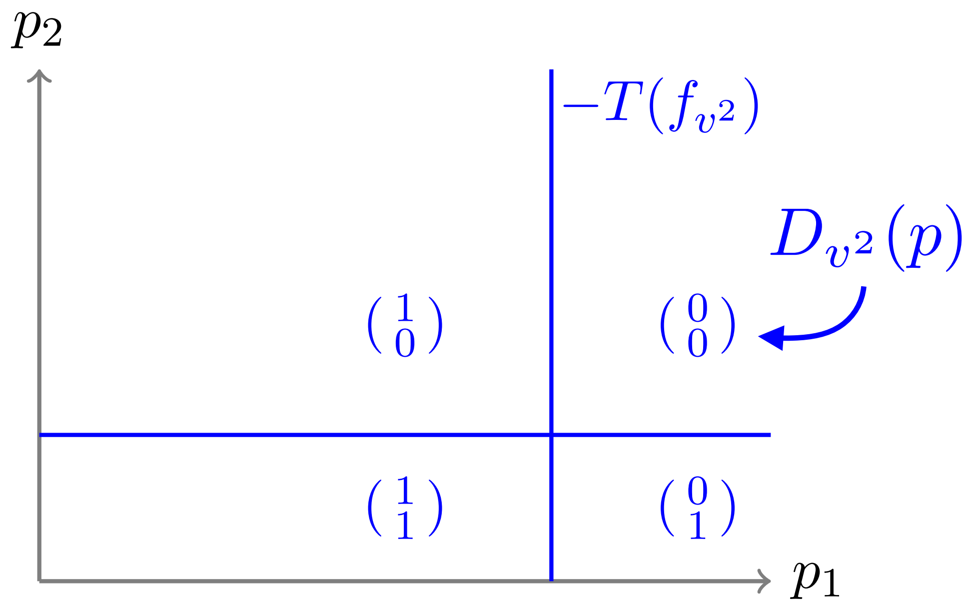


$w^1 = \binom{3}{5}$ $\quad f_{v^1}(x) = 0 \odot x^{\odot \binom{0}{0}} \oplus 3 \odot x^{\odot \binom{1}{0}} \oplus 5 \odot x^{\odot \binom{0}{1}} \oplus 8 \odot x^{\odot \binom{1}{1}}$





$w^2 = \binom{7}{2}$ $\quad f_{v^2}(x) = 0 \odot x^{\odot \binom{0}{0}} \oplus 7 \odot x^{\odot \binom{1}{0}} \oplus 2 \odot x^{\odot \binom{0}{1}} \oplus 9 \odot x^{\odot \binom{1}{1}}$



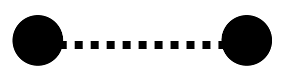
Tropical Intermezzo

Notation: $a \oplus b = \max\{a, b\}, a \odot b = a + b$

$f_v(x) = \bigoplus_{a \in P(G) \cap \mathbb{Z}^d} v(a) \odot x^{\odot a}$ tropical Laurent polynomial, $f_v(-p) = \max_{a \in P(G) \cap \mathbb{Z}^d} \{v(a) - \langle p, a \rangle\}$

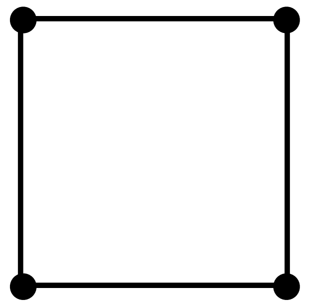
$\mathcal{T}(f_v)$ tropical hypersurface, i.e. $p \in -\mathcal{T}(f_v) \iff |D_v(p)| \geq 2$.

$\mathcal{T}(f_v)$ induces the *price regions* $\{p \mid D_v(p) = D_v(p')\}$ on $\mathbb{R}^d$.



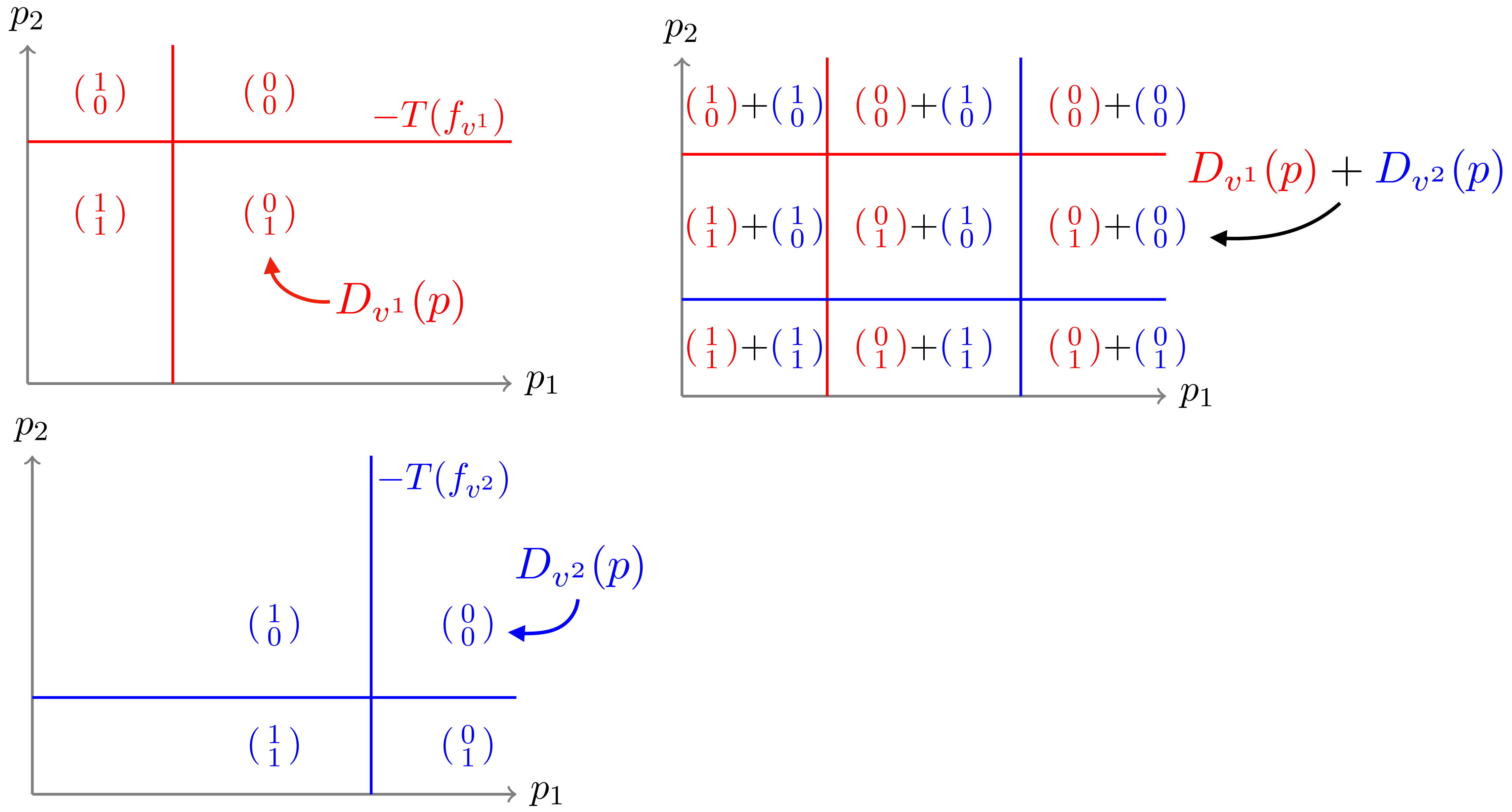
G

$w^1 = \binom{3}{5}$ $\quad f_{v^1}(x) = 0 \odot x^{\odot \binom{0}{0}} \oplus 3 \odot x^{\odot \binom{1}{0}} \oplus 5 \odot x^{\odot \binom{0}{1}} \oplus 8 \odot x^{\odot \binom{1}{1}}$



$P(G)$

$w^2 = \binom{7}{2}$ $\quad f_{v^2}(x) = 0 \odot x^{\odot \binom{0}{0}} \oplus 7 \odot x^{\odot \binom{1}{0}} \oplus 2 \odot x^{\odot \binom{0}{1}} \oplus 9 \odot x^{\odot \binom{1}{1}}$



Tropical Intermezzo

Notation: $a \oplus b = \max\{a, b\}, a \odot b = a + b$

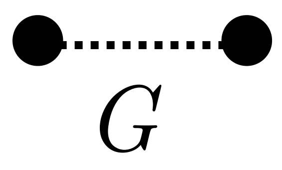
$f_v(x) = \bigoplus_{a \in P(G) \cap \mathbb{Z}^d} v(a) \odot x^{\odot a}$ tropical Laurent polynomial, $f_v(-p) = \max_{a \in P(G) \cap \mathbb{Z}^d} \{v(a) - \langle p, a \rangle\}$

$\mathcal{T}(f_v)$ tropical hypersurface, i.e. $p \in -\mathcal{T}(f_v) \iff |D_v(p)| \geq 2.$

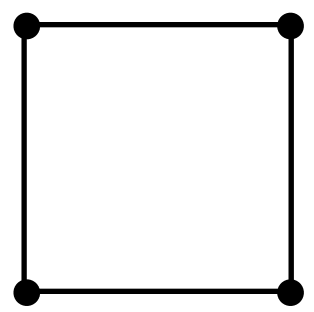
$\mathcal{T}(f_v)$ induces the *price regions* $\{p \mid D_v(p) = D_v(p')\}$ on $\mathbb{R}^d$.

$f_{v^1}(x) \odot f_{v^2}(x) = f_V(x)$

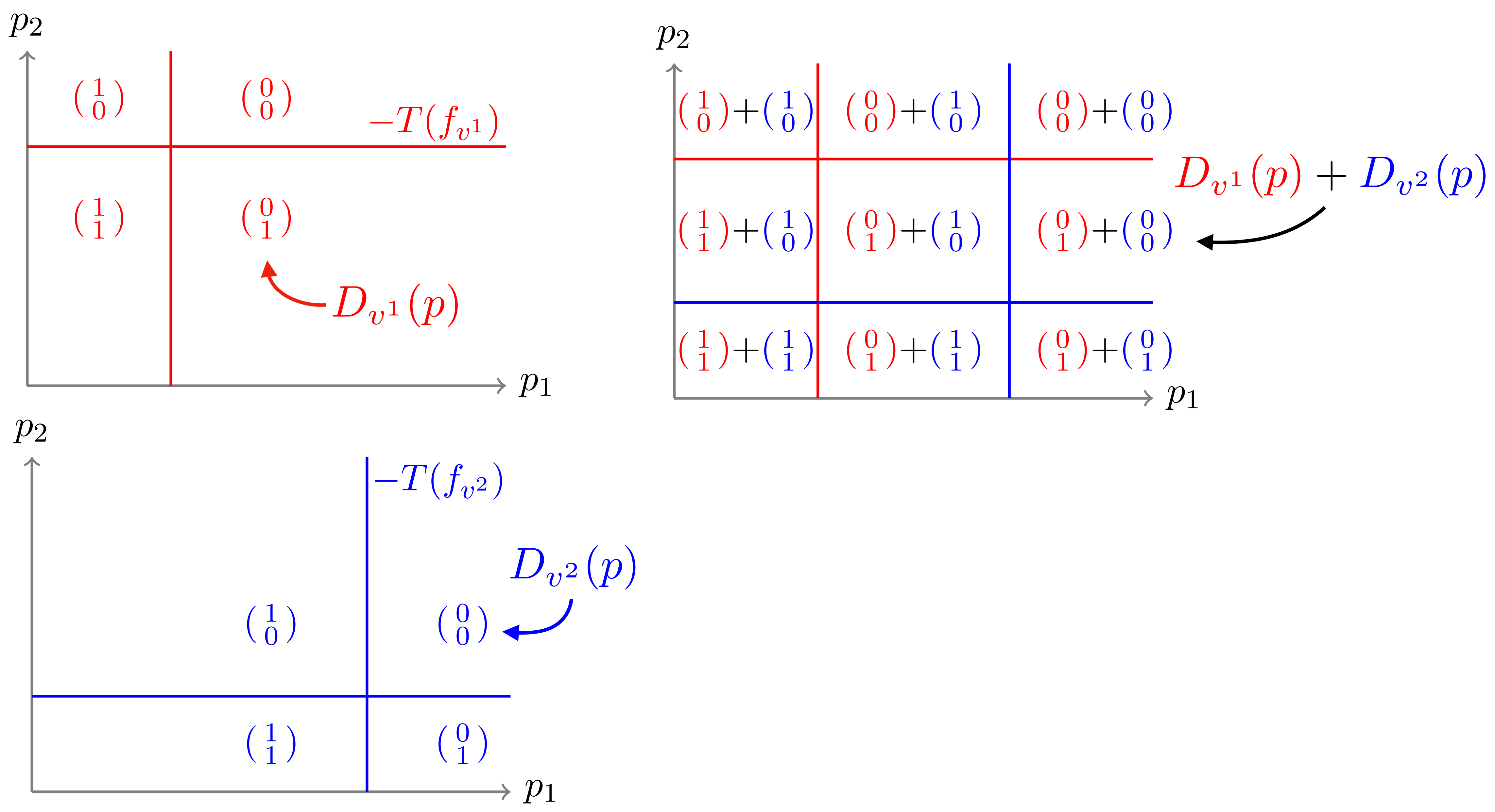
$\implies D_{v^1}(p) + D_{v^2}(p) = D_V(p)$



$w^1 = \binom{3}{5}$ $\quad f_{v^1}(x) = 0 \odot x^{\odot\binom{0}{0}} \oplus 3 \odot x^{\odot\binom{1}{0}} \oplus 5 \odot x^{\odot\binom{0}{1}} \oplus 8 \odot x^{\odot\binom{1}{1}}$



$w^2 = \binom{7}{2}$ $\quad f_{v^2}(x) = 0 \odot x^{\odot\binom{0}{0}} \oplus 7 \odot x^{\odot\binom{1}{0}} \oplus 2 \odot x^{\odot\binom{0}{1}} \oplus 9 \odot x^{\odot\binom{1}{1}}$



Tropical Intermezzo

Notation: $a \oplus b = \max\{a, b\}, a \odot b = a + b$

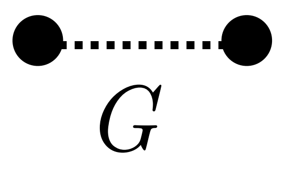
$f_v(x) = \bigoplus_{a \in P(G)\cap\mathbb{Z}^d} v(a) \odot x^{\odot a}$ tropical Laurent polynomial, $f_v(-p) = \max_{a \in P(G)\cap\mathbb{Z}^d} \{v(a) - \langle p, a\rangle\}$

$\mathcal{T}(f_v)$ tropical hypersurface, i.e. $p \in -\mathcal{T}(f_v) \iff |D_v(p)| \geq 2$.

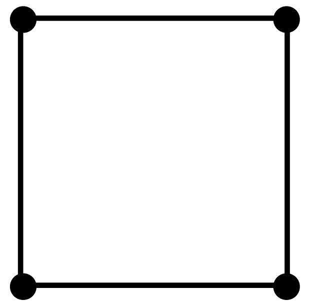
$\mathcal{T}(f_v)$ induces the *price regions* $\{p \mid D_v(p) = D_v(p')\}$ on $\mathbb{R}^d$.

$f_{v^1}(x) \odot f_{v^2}(x) = f_V(x)$

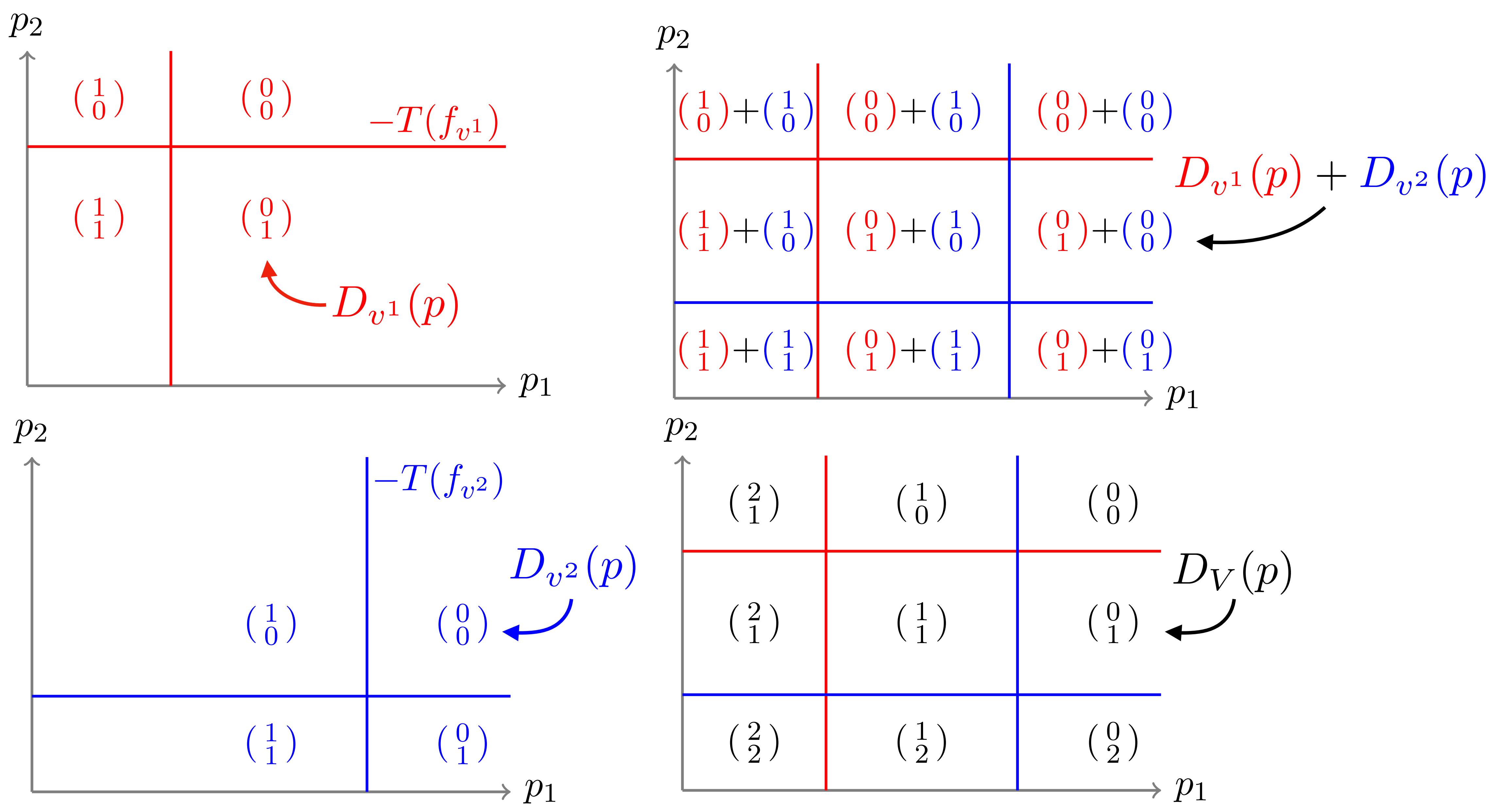
$\implies D_{v^1}(p) + D_{v^2}(p) = D_V(p)$



$w^1 = \binom{3}{5}$ $\quad f_{v^1}(x) = 0 \odot x^{\odot\binom{0}{0}} \oplus 3 \odot x^{\odot\binom{1}{0}} \oplus 5 \odot x^{\odot\binom{0}{1}} \oplus 8 \odot x^{\odot\binom{1}{1}}$



$w^2 = \binom{7}{2}$ $\quad f_{v^2}(x) = 0 \odot x^{\odot\binom{0}{0}} \oplus 7 \odot x^{\odot\binom{1}{0}} \oplus 2 \odot x^{\odot\binom{0}{1}} \oplus 9 \odot x^{\odot\binom{1}{1}}$



Mixed regular subdivisions

Aggregate valuation function:

$$V(a) = \max\{ \sum_{b\in[m]} v^b(a^b) \mid a^b \in P(G) \cap \mathbb{Z}^d, \sum_{b\in[m]} a^b = a\}$$

Mixed regular subdivisions

Aggregate valuation function:

$$V(a) = \max\{ \sum_{b \in [m]} v^b(a^b) \mid a^b \in P(G) \cap \mathbb{Z}^d, \sum_{b \in [m]} a^b = a\}$$

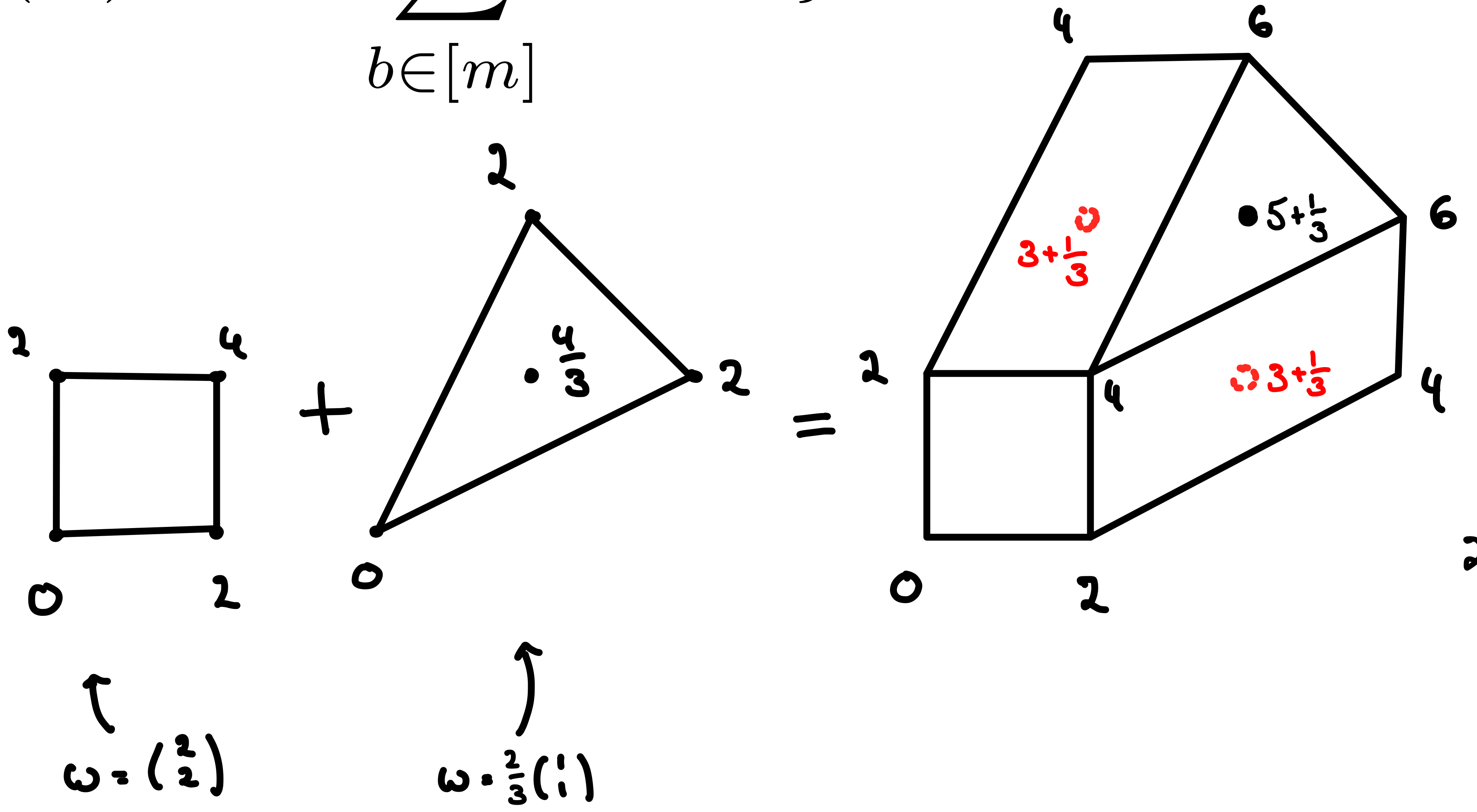
$\implies$ mixed regular subdivision on
$P(G) + \ldots + P(G) = mP(G)$
with faces $\sum_{b \in [m]} F^b$.

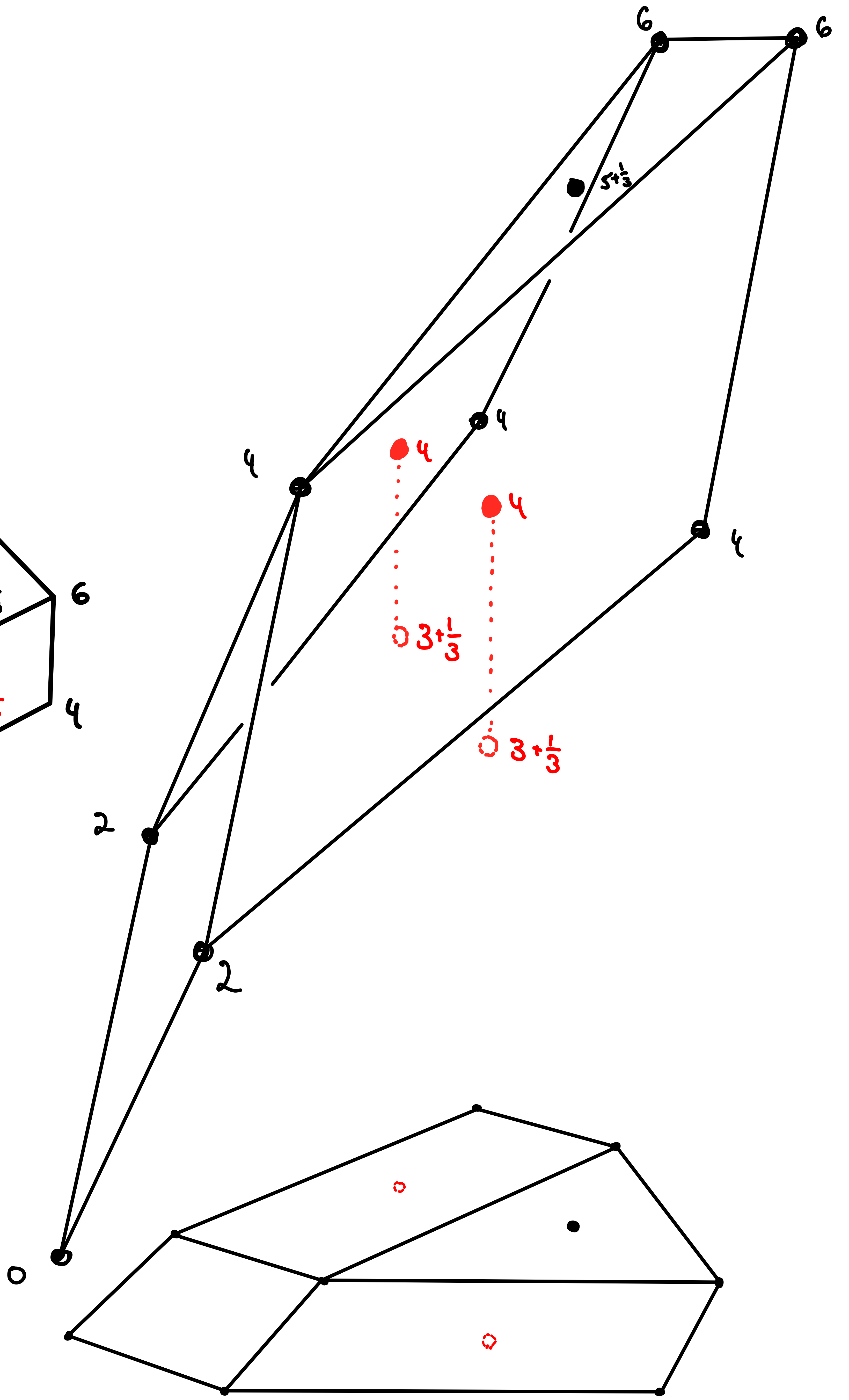
Mixed regular subdivisions

Aggregate valuation function:

$$V(a) = \max\{ \sum_{b\in[m]} v^b(a^b) \mid a^b \in P(G) \cap \mathbb{Z}^d, \sum_{b\in[m]} a^b = a \}$$

$\implies$ mixed regular subdivision on $P(G)+\ldots+P(G)=mP(G)$ with faces $\sum_{b\in[m]} F^b$.





Mixed regular subdivisions

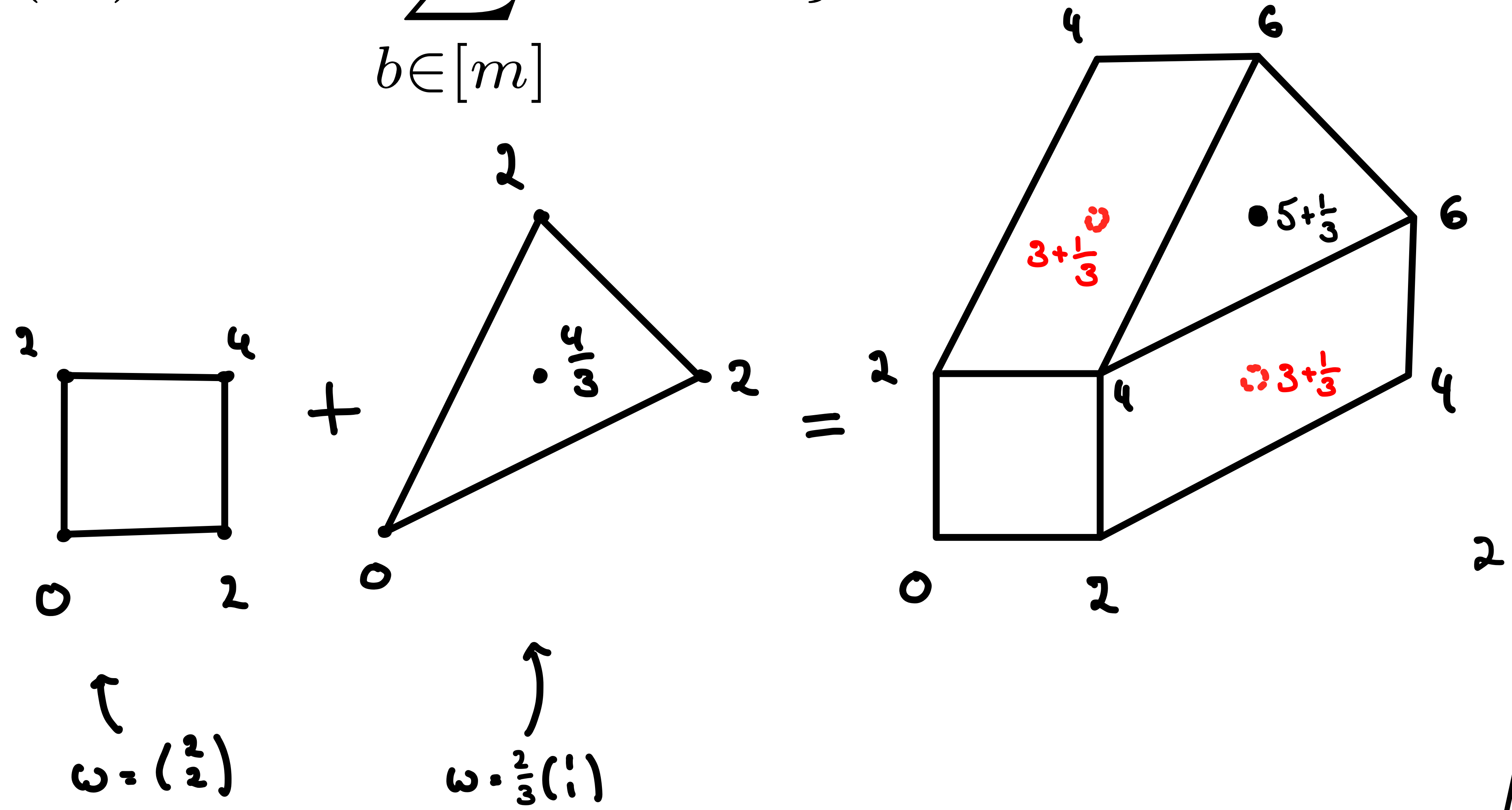
Aggregate valuation function:

$$V(a) = \max\{\sum_{b\in[m]} v^b(a^b) \mid a^b \in P(G)\cap\mathbb{Z}^d, \sum_{b\in[m]} a^b = a\}$$

$\Longrightarrow$ mixed regular subdivision on

$$P(G){+}\dots{+}P(G){=}mP(G)$$

with faces $\sum_{b\in[m]} F^b$.

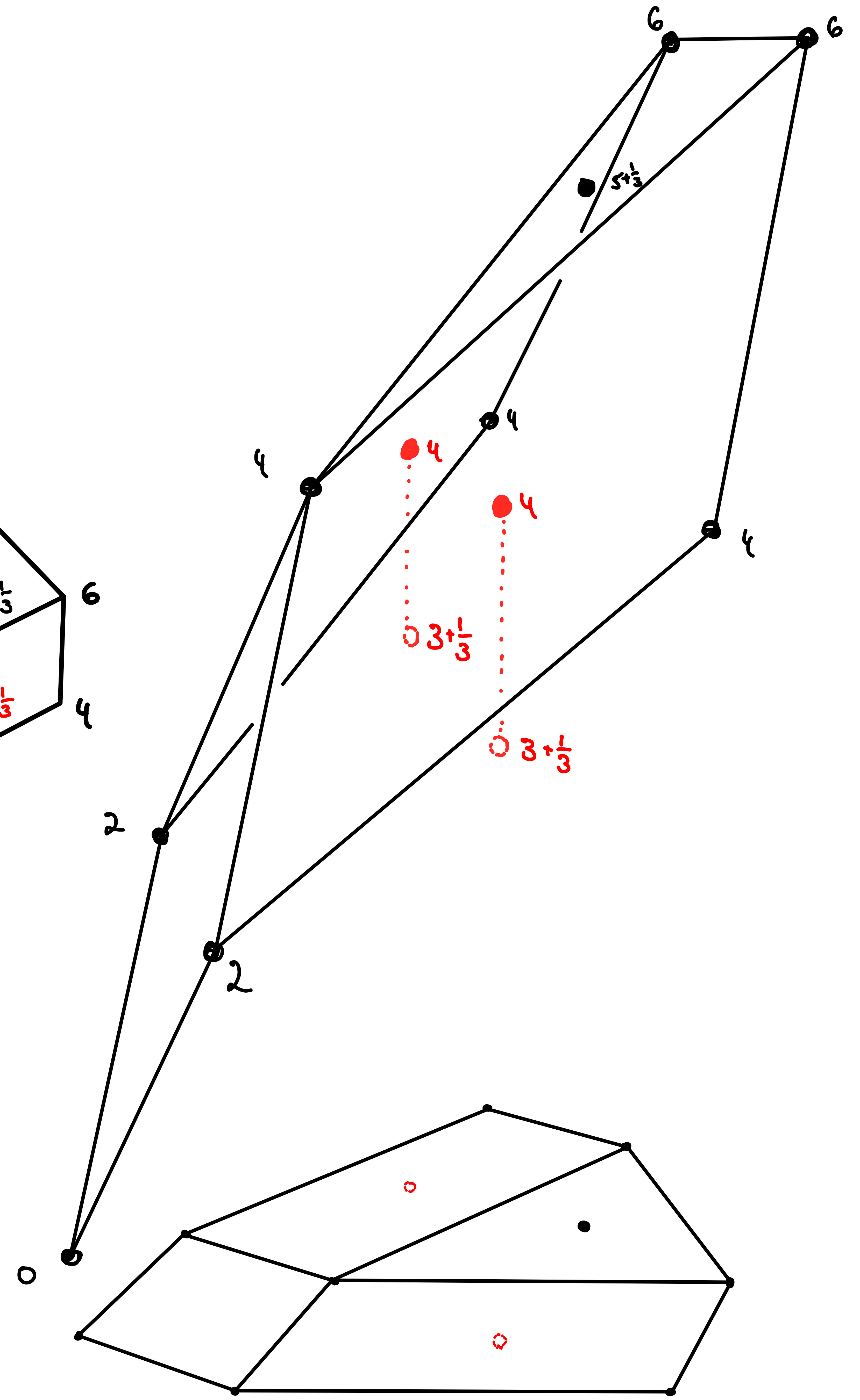


$\forall\ F^1,\dots,F^m \preceq P(G)$:

$$\text{if } a \in \sum_{b\in[m]} F^b \text{ then } a \in \sum_{b\in[m]} \text{vert}(F^b)$$

face in mixed regular subdivision

Points that are always in the upper convex hull of the lifted $mP(G)$



The complete graph

and 0/1-bundles

The complete graph
and 0/1-bundles

Definition / Proposition (de Simone, '90)

Let $G = K_n$. The polytope $P(K_n)$ is the *correlation polytope (boolean quadric polytope)*. $P(K_n) \cong$ cut polytope, but not lattice isomorphic!

The complete graph

and 0/1-bundles

Definition / Proposition (de Simone, '90)

Let $G = K_n$. The polytope $P(K_n)$ is the *correlation polytope (boolean quadric polytope)*. $P(K_n) \cong$cut polytope, but not lattice isomorphic!

Theorem (B.-Haase-Tran, '21$^+$)

Let $a^* \in \{0,1\}^n$. Then $\forall a \in \pi^{-1}(a^*)$ such that

$$\forall F^1, \ldots, F^m \preceq P(K_n) \text{ holds: if } a \in \sum_{b \in [m]} F^b \text{ then } a \in \sum_{b \in [m]} \text{vert}(F^b).$$

Reminder

Let $a^* \in \mathbb{Z}^n_{\geq 0}$. A CE is guaranteed to exist if $\exists\ a \in \pi^{-1}(a^*)$ such that

$$\forall\ F^1, \ldots, F^m \preceq P(G) : \text{ if } a \in \sum_{b \in [m]} F^b \text{ then } a \in \sum_{b \in [m]} \text{vert}(F^b).$$

The complete graph

and arbitrary bundles

The complete graph
and arbitrary bundles

Example.

$G = K_4, a^* = (2,2,2,2)$. There are edges e_1, e_2, e_3, e_4 of $P(K_4)$ s.t.
$a = (2,2,2,2,\ 1,1,1,1,1,1)$ is the sum of midpoints, but not sum of vertices.

The complete graph

and arbitrary bundles

Example.

$G = K_4, a^* = (2,2,2,2)$. There are edges e_1, e_2, e_3, e_4 of $P(K_4)$ s.t. $a = (2,2,2,2,\ 1,1,1,1,1,1)$ is the sum of midpoints, but not sum of vertices.

($P(K_n)$ is not IDP for $n \geq 4$)

The complete graph

and arbitrary bundles

Example.

$G = K_4, a^* = (2,2,2,2)$. There are edges e_1, e_2, e_3, e_4 of $P(K_4)$ s.t.
$a = (2,2,2,2,\ 1,1,1,1,1,1)$ is the sum of midpoints, but not sum of vertices.

($P(K_n)$ is not IDP for $n \geq 4$)

CE is still guaranteed to exist: $a = (2,2,2,2,\ 2,2,2,2,2,2)$

The complete graph

and arbitrary bundles

Example.

$G = K_4, a^* = (2,2,2,2)$. There are edges e_1, e_2, e_3, e_4 of $P(K_4)$ s.t.
$a = (2,2,2,2,\ 1,1,1,1,1,1)$ is the sum of midpoints, but not sum of vertices.

($P(K_n)$ is not IDP for $n \geq 4$)

CE is still guaranteed to exist: $a = (2,2,2,2,\ 2,2,2,2,2,2)$

Theorem (B.-Haase-Tran, '21$^+$)

Let $a^* \in \mathbb{Z}^n_{\geq 0}$. Then $\exists\ a \in \pi^{-1}(a^*)$ such that

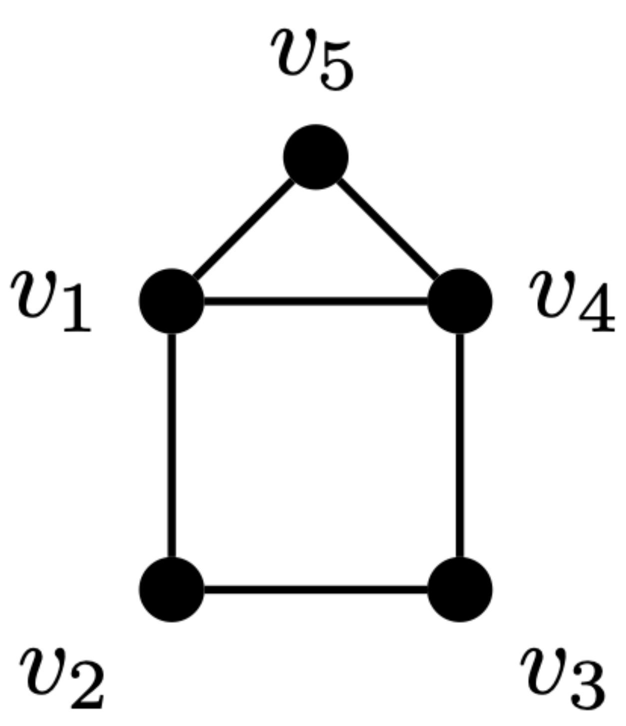
$$\forall\ F^1, \dots, F^m \preceq P(G) \text{ holds: if } a \in \sum_{b \in [m]} F^b \text{ then } a \in \sum_{b \in [m]} \operatorname{vert}(F^b).$$

Other graphs

where CE might not exist

Other graphs
where CE might not exist

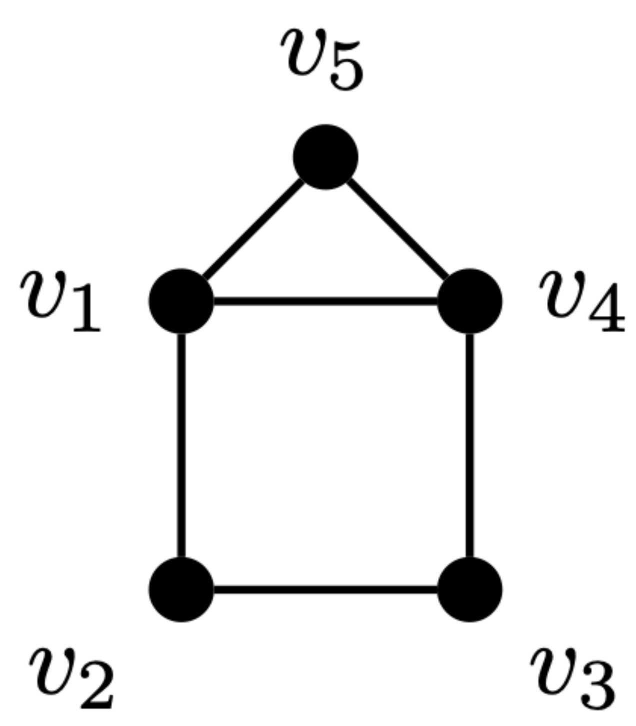
Example.



Other graphs

where CE might not exist

Example.



$a^* = (1,1,1,1,1)$. There are edges e_1, e_2, e_3, e_4 of $P(G)$ s.t.

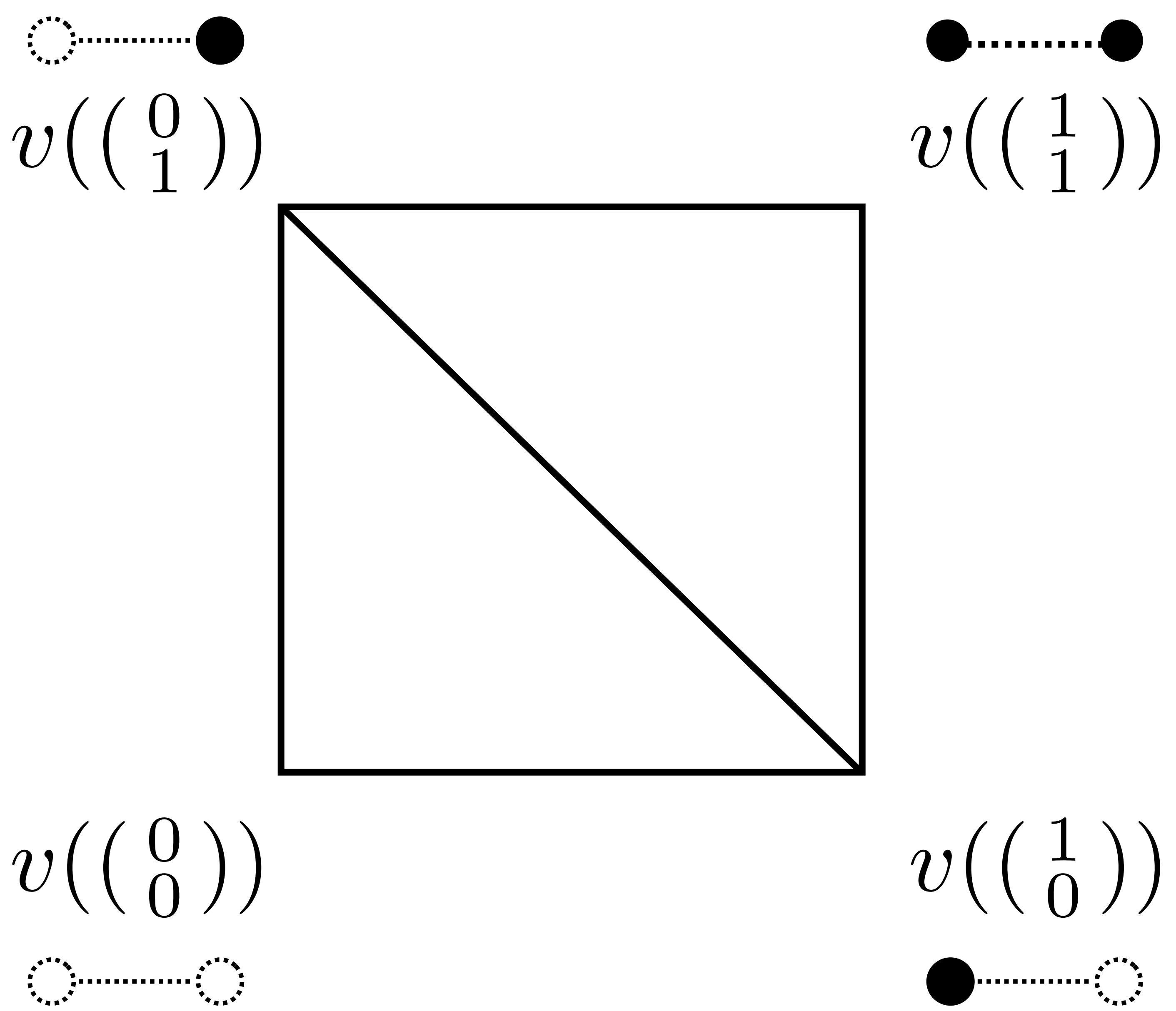
$$\pi^{-1}(a^*) \cap \sum_{i=1}^{4} e_i = \{(1,1,1,1,\ 0,0,0,0,0,0)\}$$

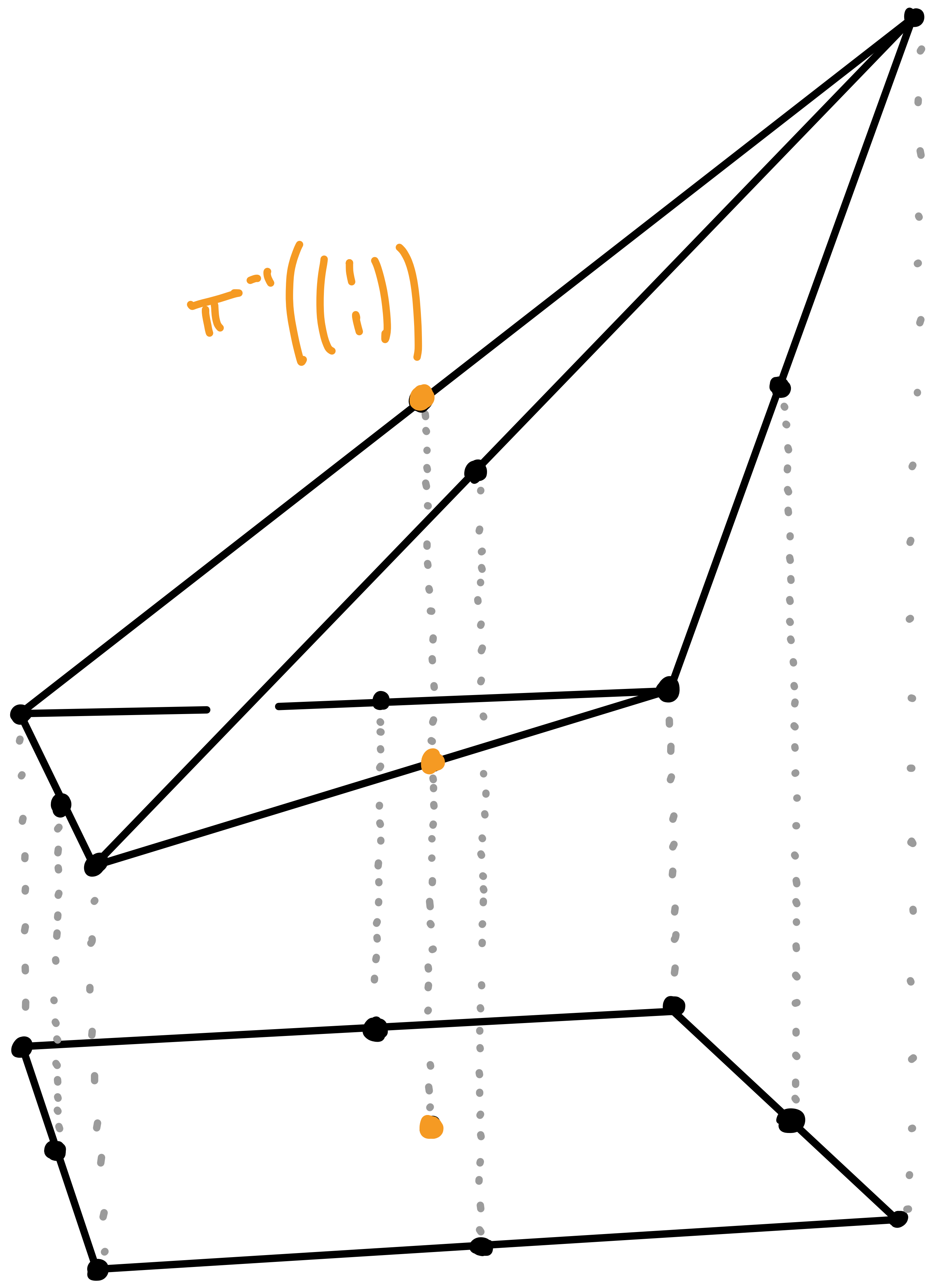
and

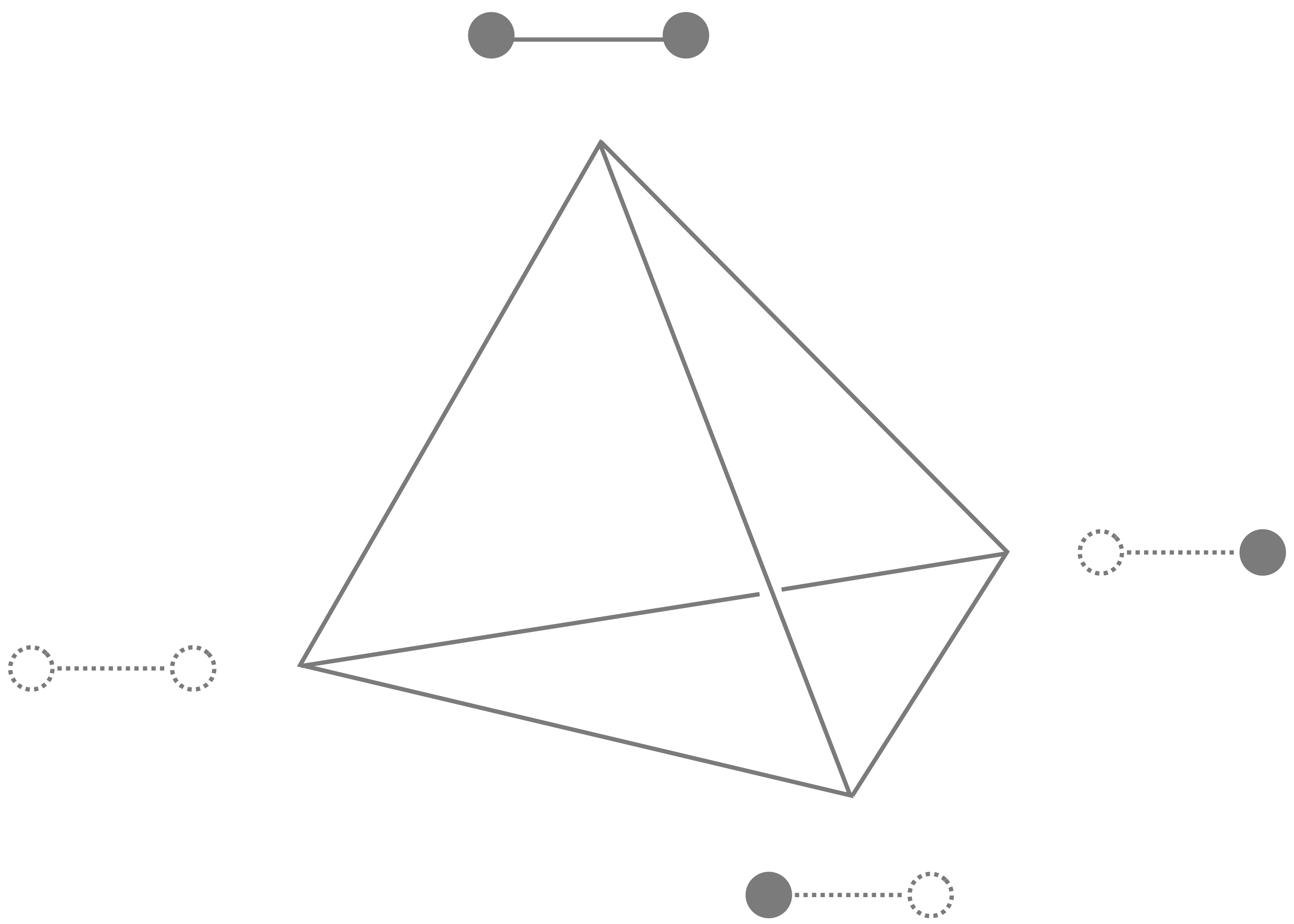
$$\pi^{-1}(a^*) \cap \sum_{i=1}^{4} \mathrm{vert}(e_i) = \emptyset.$$

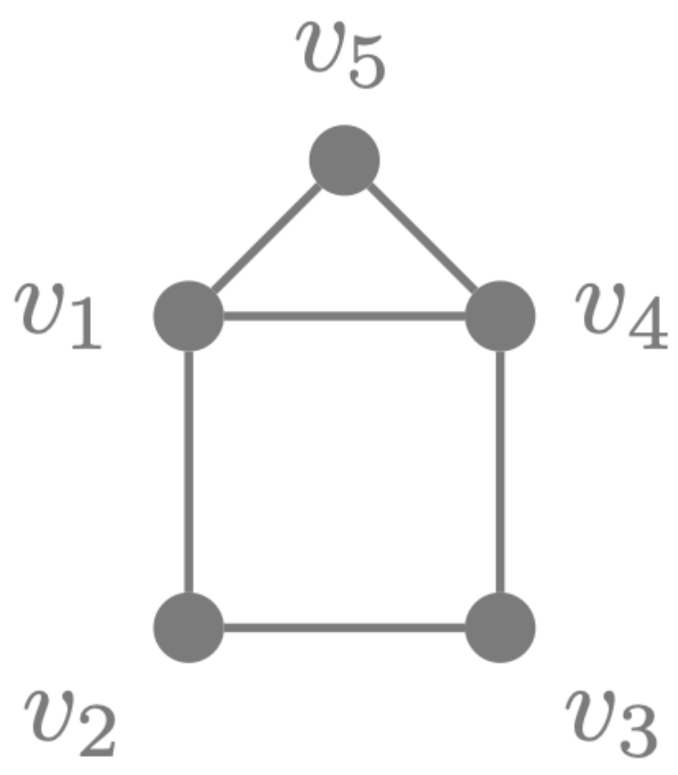
Comparison: classical approach

Non-linear valuations on the cube









Thank you!