MAX PLANCK INSTITUTE
FOR MATHEMATICS IN THE SCIENCES

Tropical Positivity and Semialgebraic Sets From Polytopes

PhD Defense

Marie-Charlotte Brandenburg

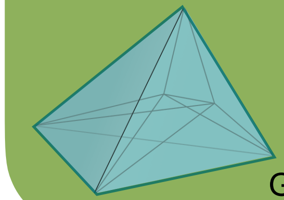
Universität Leipzig
May 10, 2023

Tropical Geometry

Semialgebraic Sets

Polyhedral Geometry

TROPICAL POSITIVITY AND DETERMINANTAL VARIETIES



joint work with
Georg Loho and Rainer Sinn

POLYHEDRAL GEOMETRY

TROPICAL GEOMETRY

SEMIALGEBRAIC SETS

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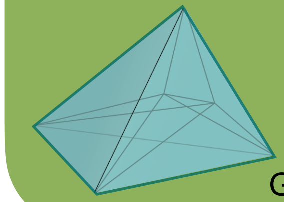

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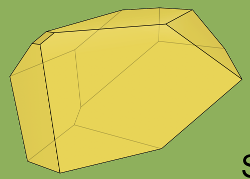
POLYHEDRAL GEOMETRY

VOLUME POLYNOMIALS OF TROPICAL POLYTOPES



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Semialgebraic Sets

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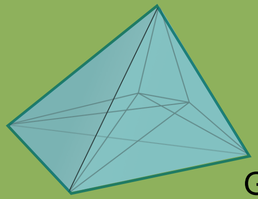

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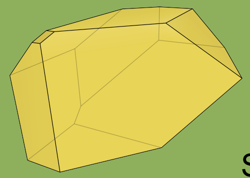
Volume Polynomials of Tropical Polytopes



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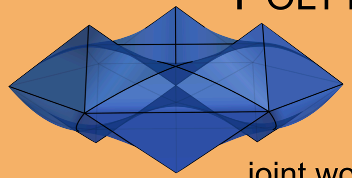
Volumes

Intersection Bodies of Polytopes



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Tropical Geometry

Semialgebraic Sets

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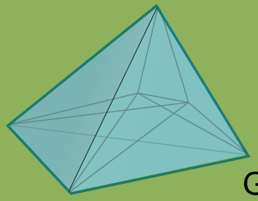

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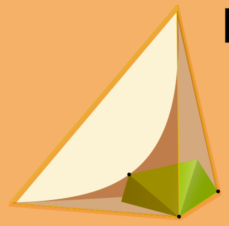
Correlated Equilibrium Polytopes



joint work with
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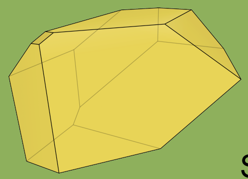
Volume Polynomials of Tropical Polytopes



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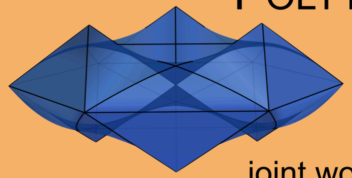

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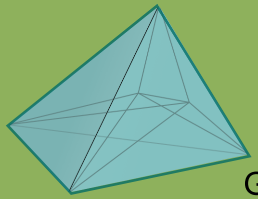

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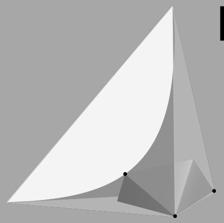

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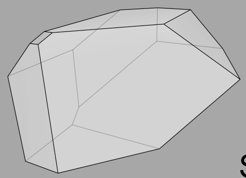

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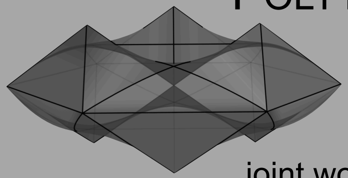

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[1] M. Brandenburg, G. Loho, and R. Sinn. "Tropical Positivity and Determinantal Varieties". To appear in *Algebraic Combinatorics* (2023).

Tropical Geometry

Tropical Semiring $(\mathbb{T}, \oplus, \odot) = (\mathbb{R} \cup \{\infty\}, \min, +)$

$$a \oplus b = \min(a, b)$$

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Geometry over the tropical semiring

- tropical polynomials
- tropical hypersurfaces
- tropical varieties
- tropical linear spaces
- tropical polytopes
- tropical rank of a matrix
- ….

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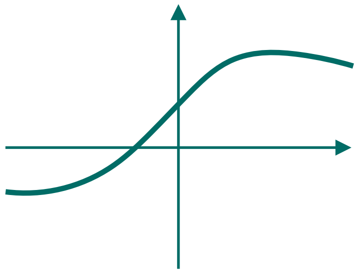
Connections e.g. to

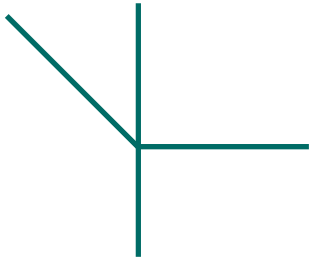
- Algebraic Geometry
- Optimization
- Economics
- Machine-Learning

Classical and tropical Geometry



algebraic variety
V

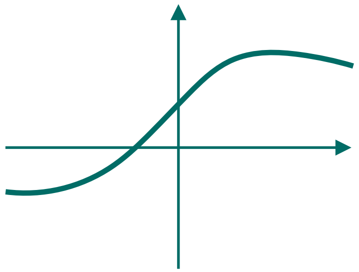
tropicalization



tropical variety
$\mathrm{trop}(V)$

„combinatorial shadow of V“

Classical and tropical Geometry



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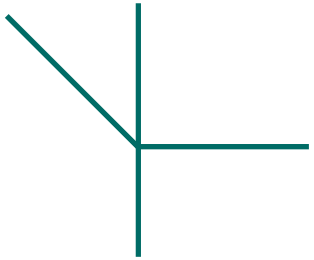

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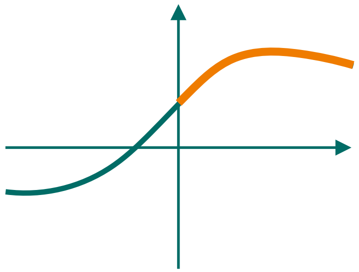

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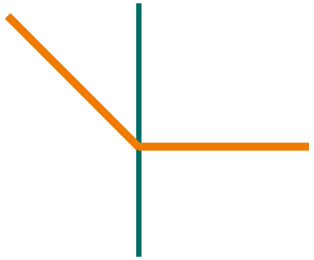

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If $V \subseteq K^d$ and K_+^d is the „positive orthant“, what is trop$^+(V) :=$ trop$(V \cap K_+^d)$?

Tropical Positivity: Challenges

classical algebraic variety: $V(\langle f_1, \ldots, f_n \rangle) = \bigcap_{i=1}^{n} V(f_i)$

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Definition (B.-Loho-Sinn): A finite set P is a set of positive generators if

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Determinantal Varieties and positive generators

Proposition:

positive generators $\not\Rightarrow$ tropical basis

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Identification of positive generators:

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Determinantal varieties:

$V^r_{d\times n} = \{A \in K^{d\times n} \mid \mathrm{rk}(A) \leq r\}$

$I = \langle (r+1)\times(r+1) - \text{minors}\rangle$

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The $(r+1)\times(r+1)$-minors form a tropical basis of trop$(V^r_{d\times n}) \Longleftrightarrow$

- $r \leq 2$, or
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$\rightarrow$ tropical point configurations

Tropical point configurations

classically: $A \in V^r_{d\times n} \Longrightarrow$ columns of $A \leftrightarrow n$ points on r-dim'l linear space in K^d

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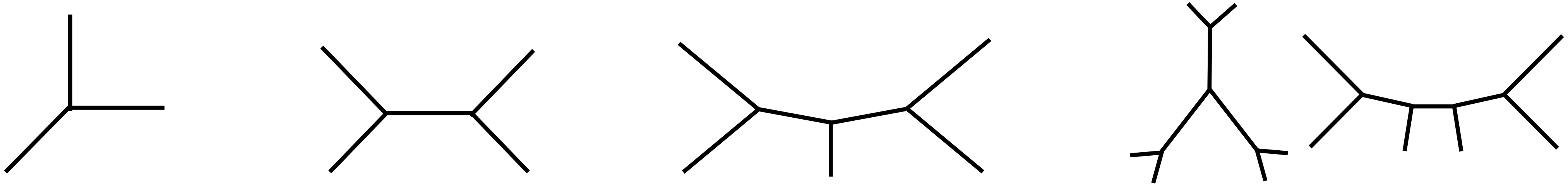
Rank 2: Points on tropical lines

2 3 4 5



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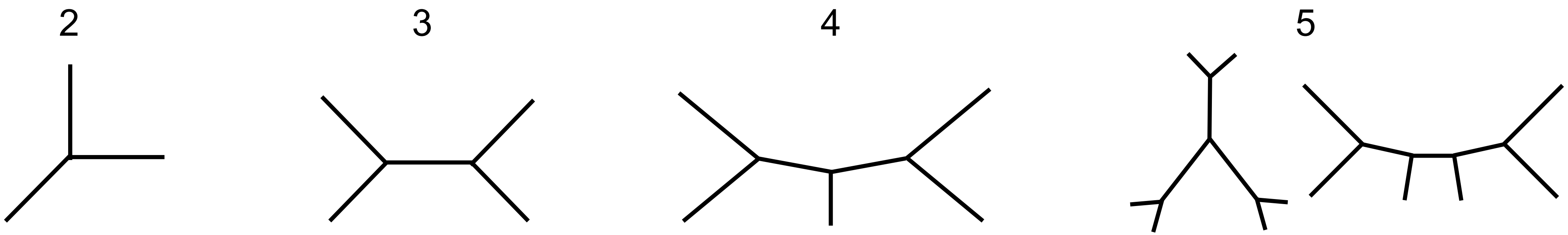

Theorem (B.-Loho-Sinn):

$A \in \text{trop}^+(V^2_{d\times n}) \Longleftrightarrow$ points form a „consecutive chain" on a tropical line

$\Longleftrightarrow$ the ass. bicolored phylogenetic tree [MY09] is a caterpillar tree

Rank 3: Necessary conditions for $A \in \text{trop}^+(V^3_{d\times n})$

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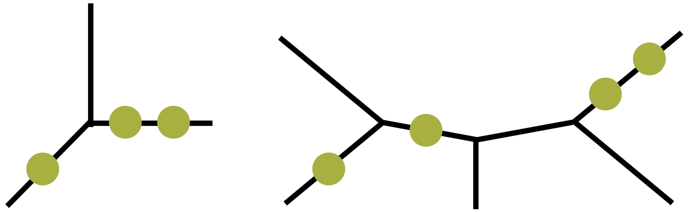
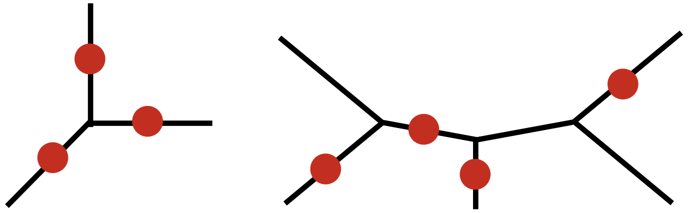


Not Positive



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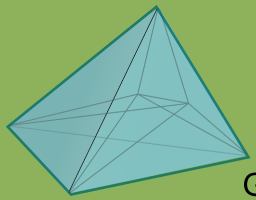

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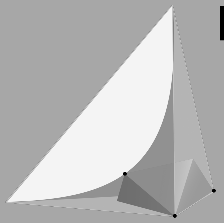

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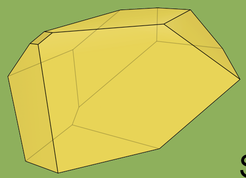

joint work with
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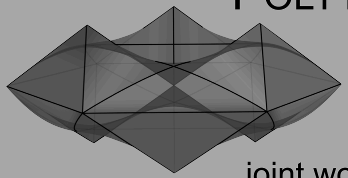
VOLUMES

INTERSECTION BODIES OF POLYTOPES



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[2]M. Brandenburg, S. Elia, and L. Zhang. "Multivariate volume, Ehrhart- and h^*-polynomials of polytropes". *Journal of Symbolic Computation* 144 (Jan. 2023) pp. 209-230.

Tropical Polytopes

convex hull: $\operatorname{conv}(v_1, \ldots, v_n) = \{ \sum_{i=1}^n \lambda_i v_i \mid 0 \leq \lambda_i \leq 1, \sum_{i=1}^n \lambda_i = 1\} \subseteq \mathbb{R}^d$

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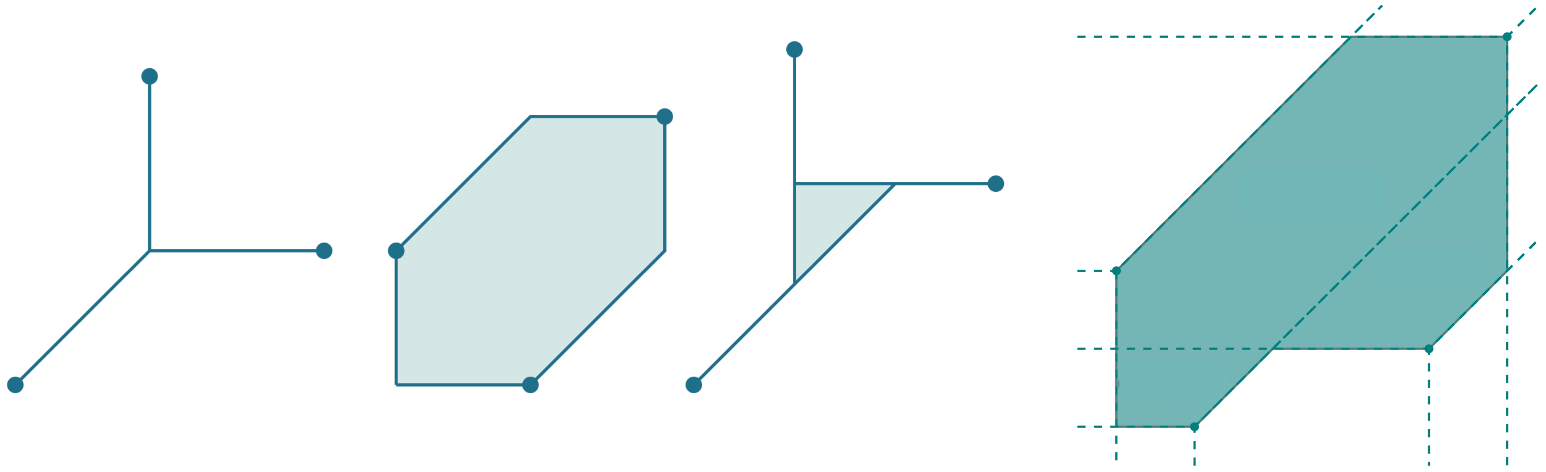

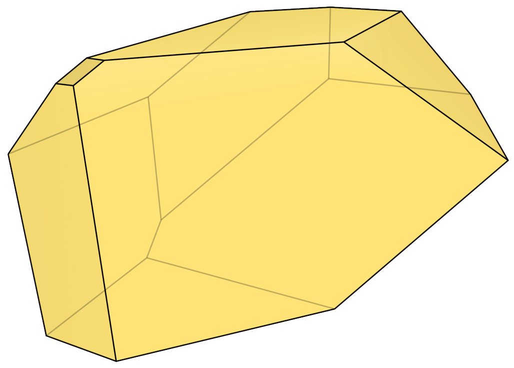

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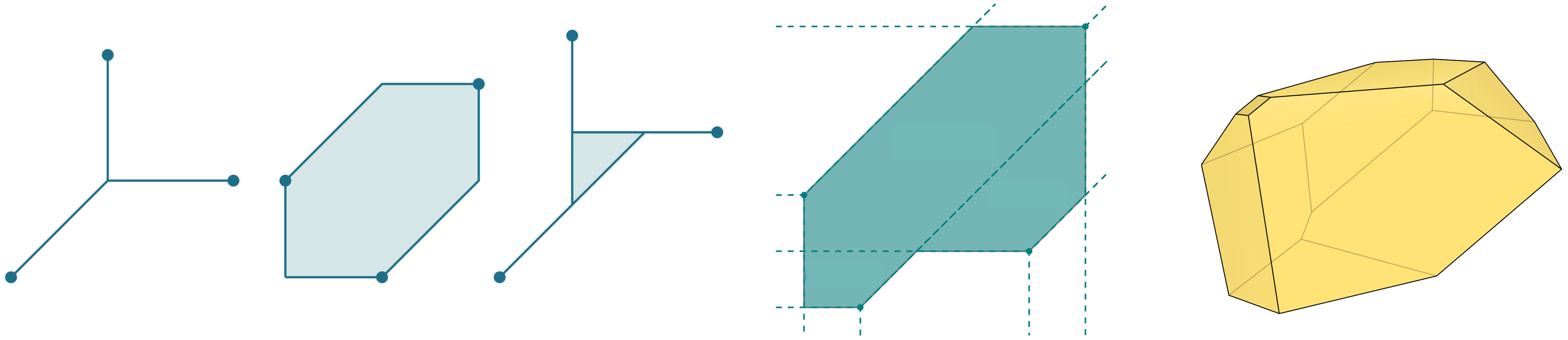
A polytrope is a tropical polytope which is also classically convex.

A polytrope is maximal if it has $\binom{2d}{d}$ vertices (as classical polytope).

$\longrightarrow$ "building blocks" of tropical polytopes



Volumes of Polytropes

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B.-Elia-Zhang: computation of all multivariate volume, Ehrhart and h^*-polynomials for polytropes of dimension ≤ 4

Volumes of Polytropes

[Tra17] Classification of combinatorial types of maximal polytropes of dim ≤ 4

dimension	2	3	4	≥ 5
# combinatorial types	1	6	27 248	?

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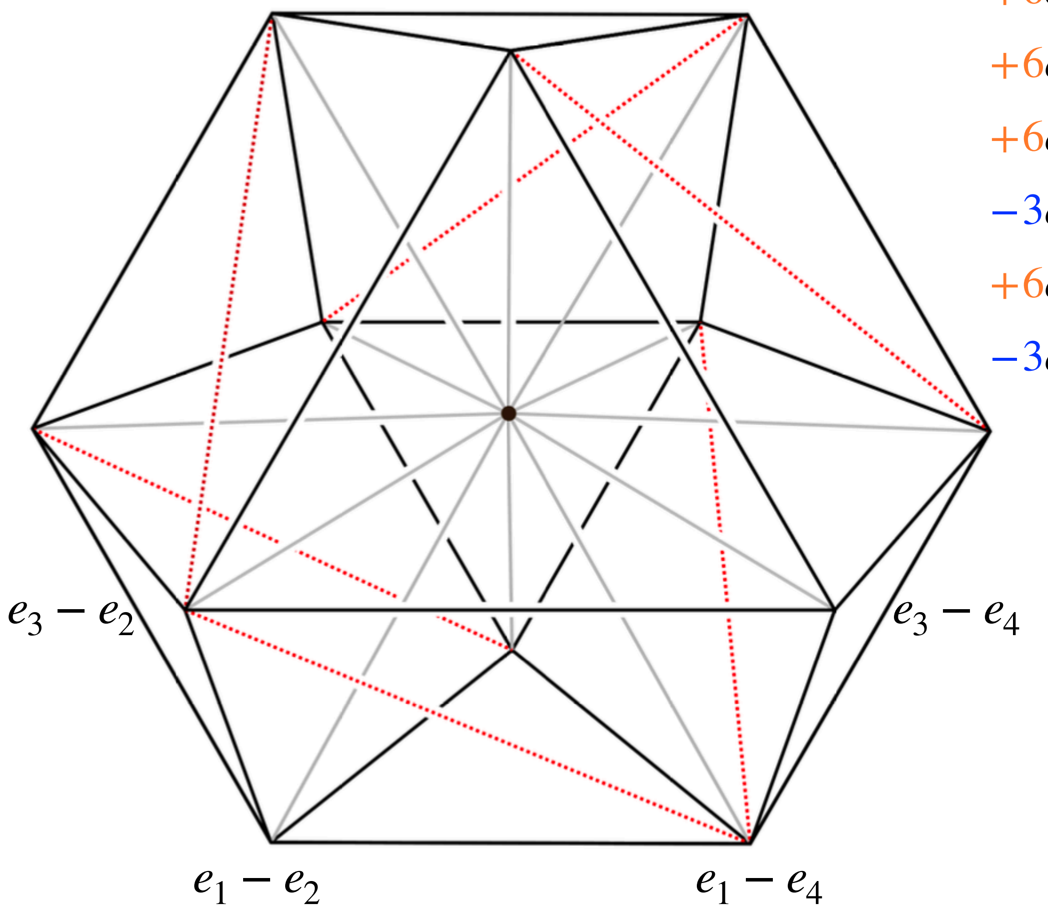
Volumes of Polytropes



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&2c_{12}^3 - 3c_{12}^2c_{13} + c_{13}^3 - 3c_{12}^2c_{14} + 6c_{12}c_{13}c_{14} - 3c_{13}^2c_{14} + c_{21}^3 - 3c_{13}^2c_{23} + 6c_{13}c_{14}c_{23} - 3c_{14}^2c_{23} \\
&- 3c_{14}c_{23}^2 - 3c_{21}c_{23}^2 + c_{23}^3 - 3c_{21}^2c_{24} + 6c_{14}c_{23}c_{24} + 6c_{21}c_{23}c_{24} - 3c_{14}c_{24}^2 - 3c_{23}c_{24}^2 + c_{24}^3 - 3c_{21}^2c_{31} \\
&+ 6c_{21}c_{24}c_{31} - 3c_{24}^2c_{31} - 3c_{24}c_{31}^2 + c_{31}^3 - 3c_{12}^2c_{32} + 6c_{12}c_{14}c_{32} - 3c_{14}^2c_{32} - 3c_{31}^2c_{32} - 3c_{14}c_{32}^2 \\
&+ 6c_{14}c_{24}c_{34} + 6c_{24}c_{31}c_{34} + 6c_{14}c_{32}c_{34} + 6c_{31}c_{32}c_{34} - 3c_{14}c_{34}^2 - 3c_{24}c_{34}^2 - 3c_{31}c_{34}^2 - 3c_{32}c_{34}^2 + 2c_{34}^3 \\
&+ 6c_{21}c_{31}c_{41} - 3c_{31}^2c_{41} + 6c_{31}c_{32}c_{41} - 3c_{32}^2c_{41} - 3c_{21}c_{41}^2 - 3c_{32}c_{41}^2 + c_{41}^3 - 3c_{12}^2c_{42} + 6c_{12}c_{13}c_{42} \\
&- 3c_{13}^2c_{42} + 6c_{12}c_{32}c_{42} + 6c_{32}c_{41}c_{42} - 3c_{13}c_{42}^2 - 3c_{32}c_{42}^2 - 3c_{41}c_{42}^2 + c_{42}^3 - 3c_{21}^2c_{43} + 6c_{13}c_{23}c_{43} \\
&+ 6c_{21}c_{23}c_{43} - 3c_{23}^2c_{43} + 6c_{21}c_{41}c_{43} - 3c_{41}^2c_{43} + 6c_{13}c_{42}c_{43} + 6c_{41}c_{42}c_{43} - 3c_{13}c_{43}^2 - 3c_{21}c_{43}^2 \\
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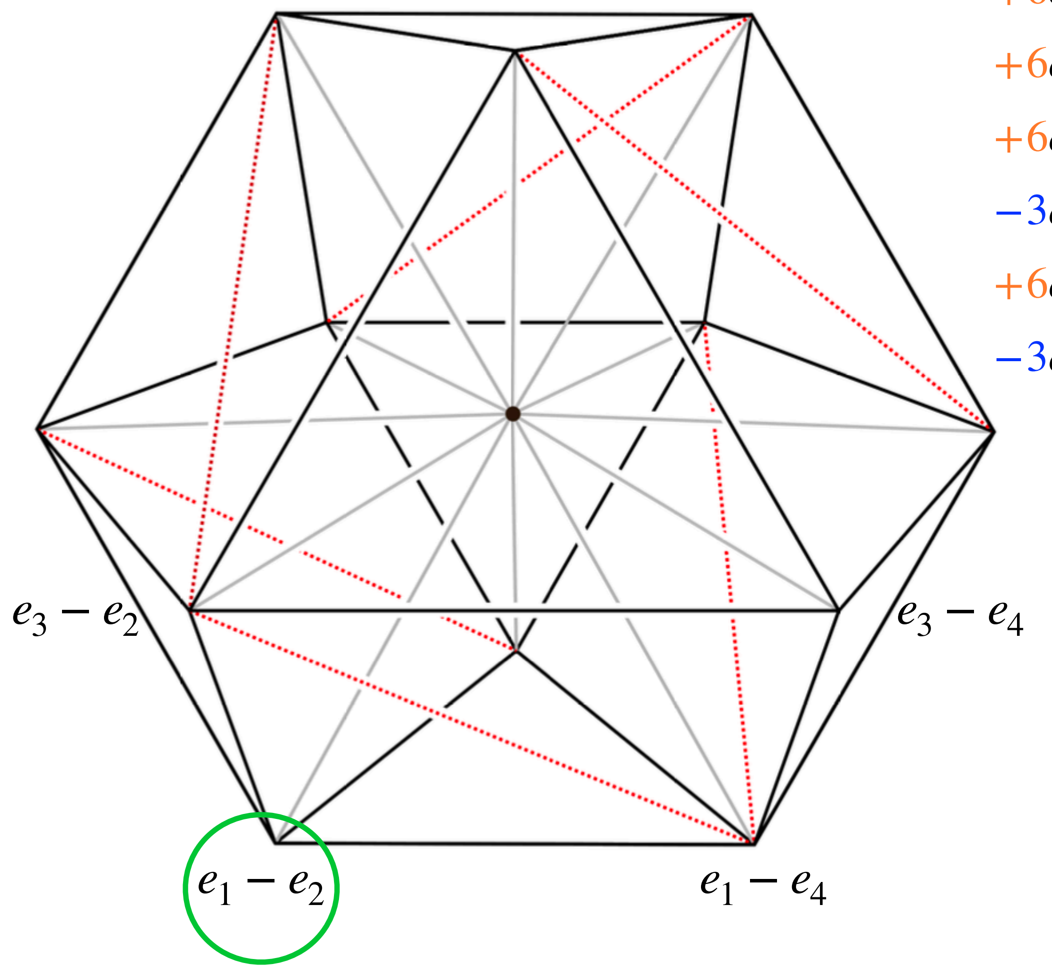
Volumes of Polytropes



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Volumes of Polytropes

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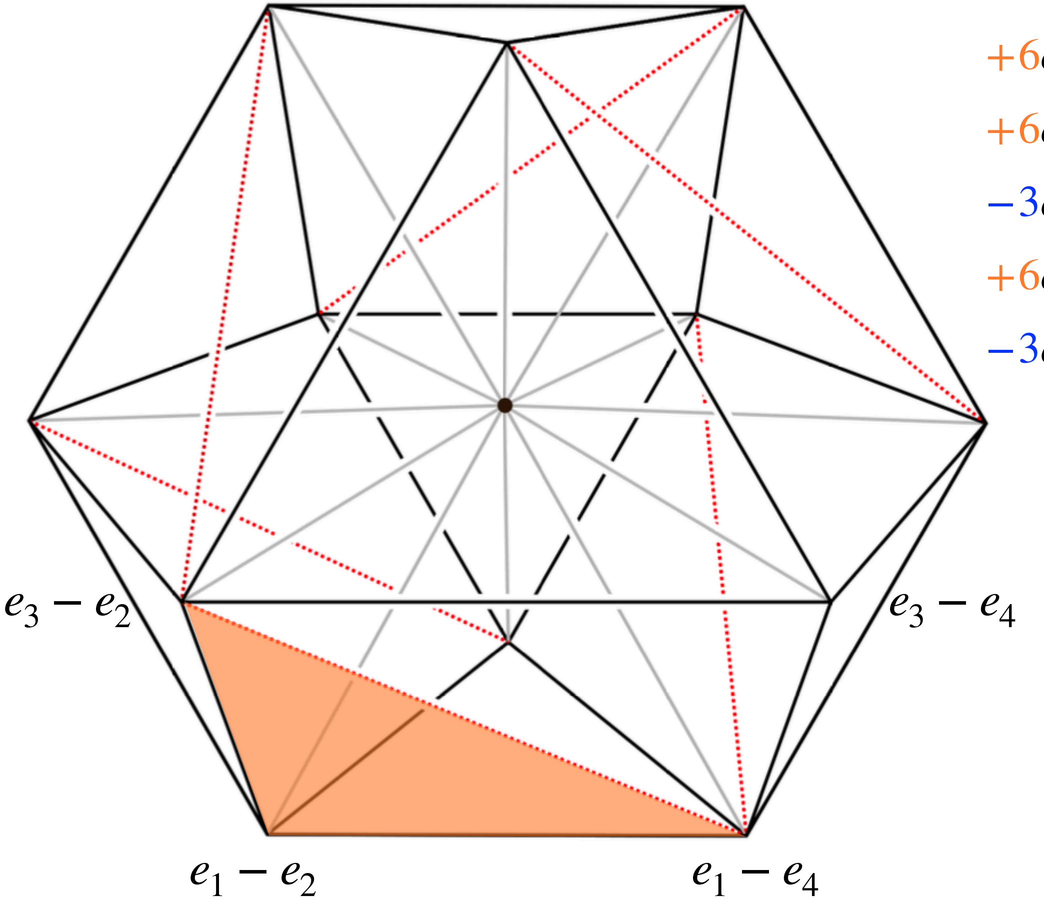

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Volumes of Polytropes

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&+6c_{21}c_{24}c_{31}-3c_{24}^2c_{31}-3c_{24}c_{31}^2+c_{31}^3-3c_{12}^2c_{32}+6c_{12}c_{14}c_{32}-3c_{14}^2c_{32}-3c_{31}^2c_{32}-3c_{14}c_{32}^2\\
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&-3c_{13}^2c_{42}+6c_{12}c_{32}c_{42}+6c_{32}c_{41}c_{42}-3c_{13}c_{42}^2-3c_{32}c_{42}^2-3c_{41}c_{42}^2+c_{42}^3-3c_{21}^2c_{43}+6c_{13}c_{23}c_{43}\\
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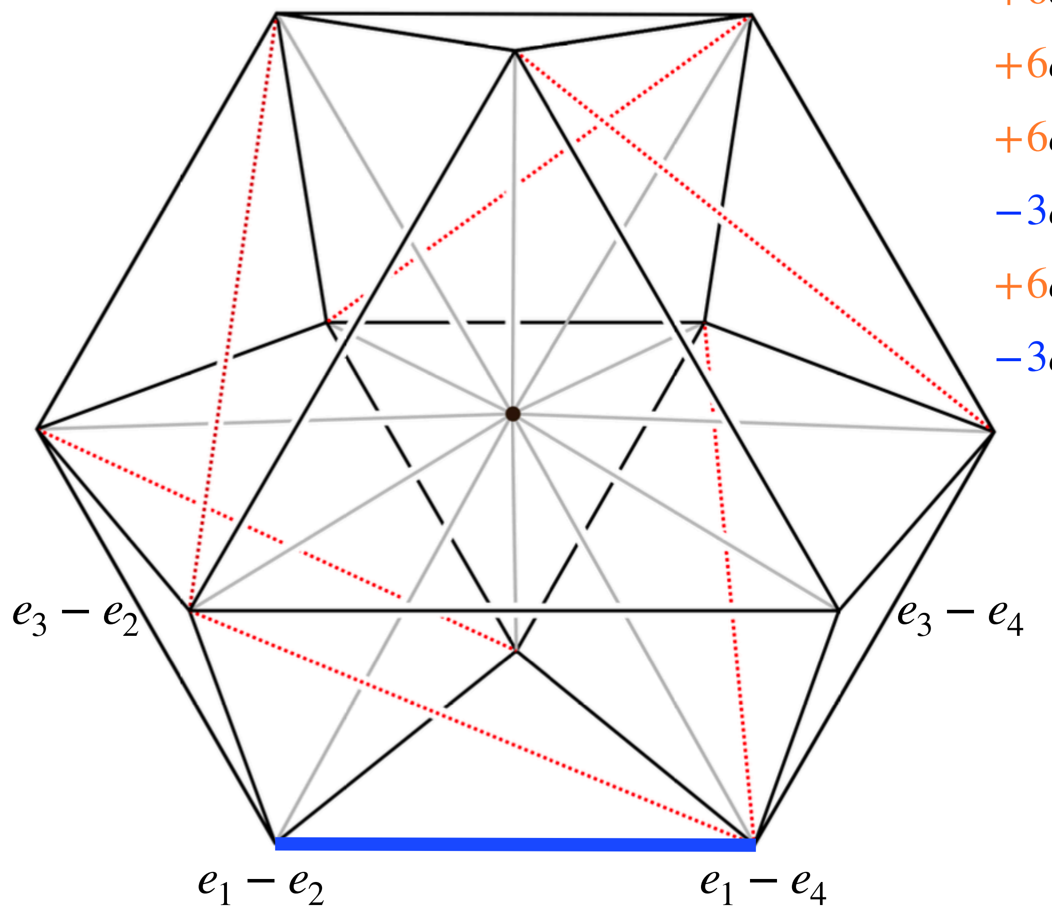

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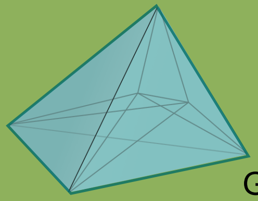
TROPICAL GEOMETRY

SEMIALGEBRAIC SETS

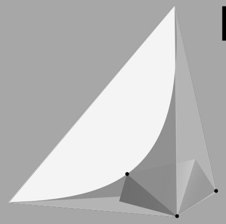
TROPICAL POSITIVITY AND DETERMINANTAL VARIETIES



joint work with
Georg Loho and Rainer Sinn

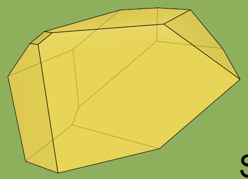
POLYHEDRAL GEOMETRY

CORRELATED EQUILIBRIUM POLYTOPES



joint work with
Benjamin Hollering and Irem Portakal

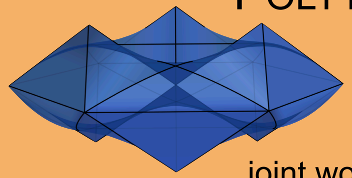
VOLUME POLYNOMIALS OF TROPICAL POLYTOPES



joint work with
Sophia Elia and Leon Zhang

VOLUMES

INTERSECTION BODIES OF POLYTOPES



joint work with Katalin Berlow,
Chiara Meroni, and Isabelle Shankar[3]

[3]K. Berlow, M. Brandenburg, C. Meroni, and I. Shankar. "Intersection Bodies of Polytopes". *Beiträge zur Algebra und Geometrie* 63.2 (June 2022) pp. 419-439.
M. Brandenburg and C. Meroni. *Intersection Bodies of Polytopes: Translations and Convexity*. 2023. arXiv: 2302.11764

Intersection Bodies of Polytopes

Definition:

The intersection body of $P \subseteq \mathbb{R}^d$ is $IP = \{x \in \mathbb{R}^d \mid \rho(x) \geq 1\}$, where $\rho(x) = \frac{1}{\|x\|}\mathrm{vol}(P \cap x^{\perp})$.

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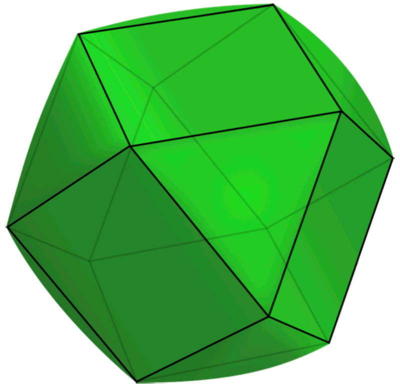
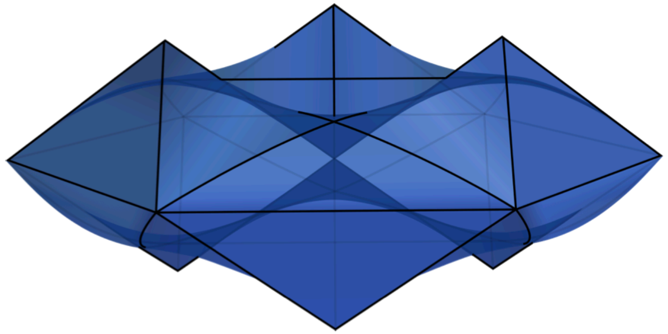


cube
$[-1,1]^3$



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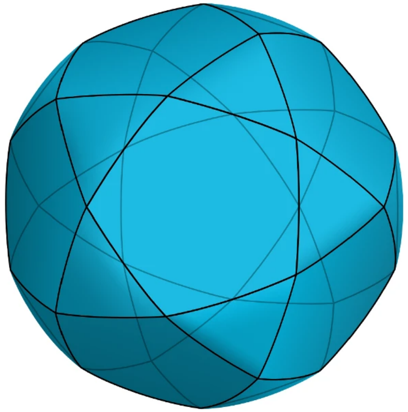

icosahedron

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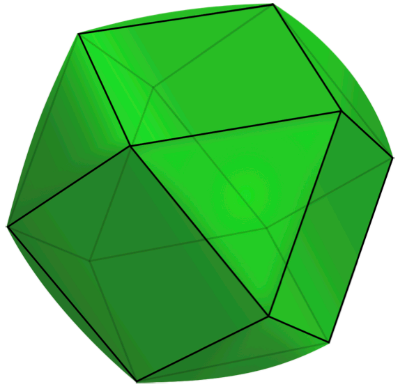

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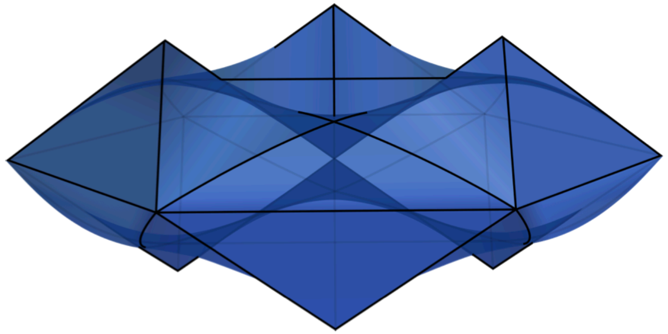

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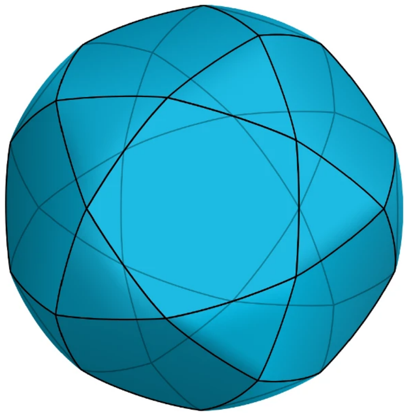


icosahedron

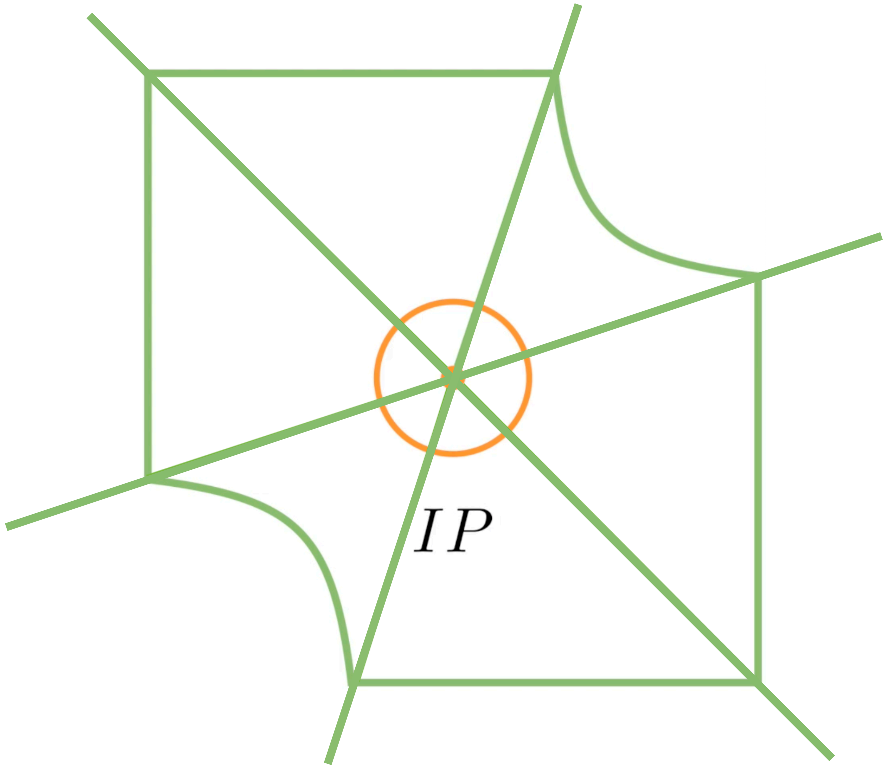
Can we describe the boundary structure of IP?

Semialgebraic Intersection Bodies

Theorem (Berlow-B.-Meroni-Shankar):

There exists a central hyperplane arrangement $\mathscr{H}(P)$ such that within each chamber C of $\mathscr{H}(P)$, $\rho(x)$ is a rational function in variables $x_1, \ldots, x_d$:

$$\rho(x) = \frac{1}{\|x\|}\mathrm{vol}(P \cap x^{\perp}) = \frac{p_C(x)}{q_C(x)} \quad \text{for } x \in C.$$



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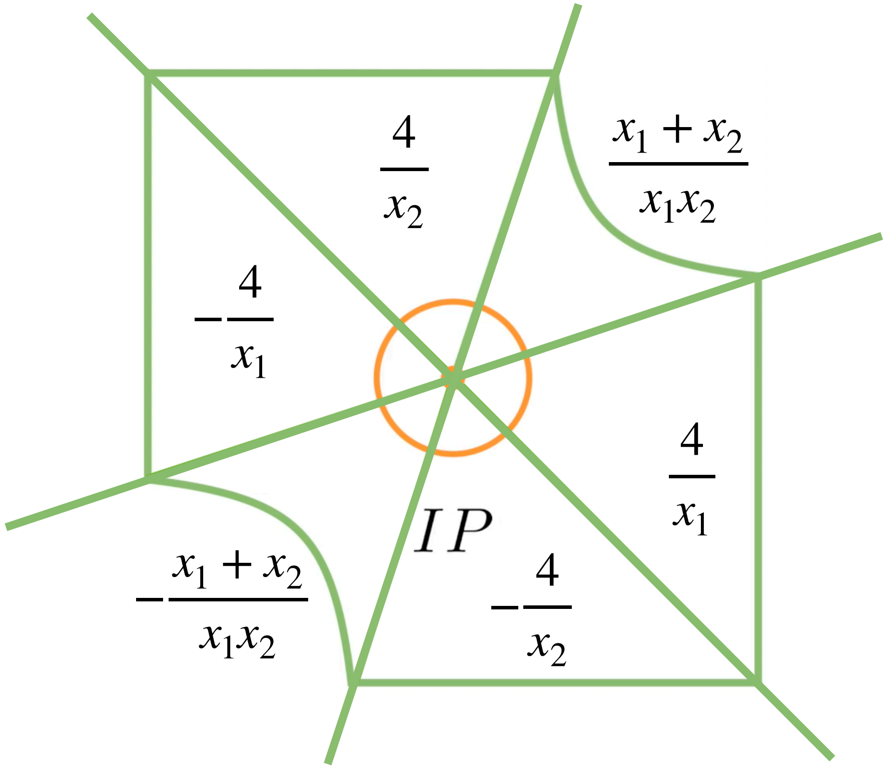

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Theorem (Berlow-B.-Meroni-Shankar):

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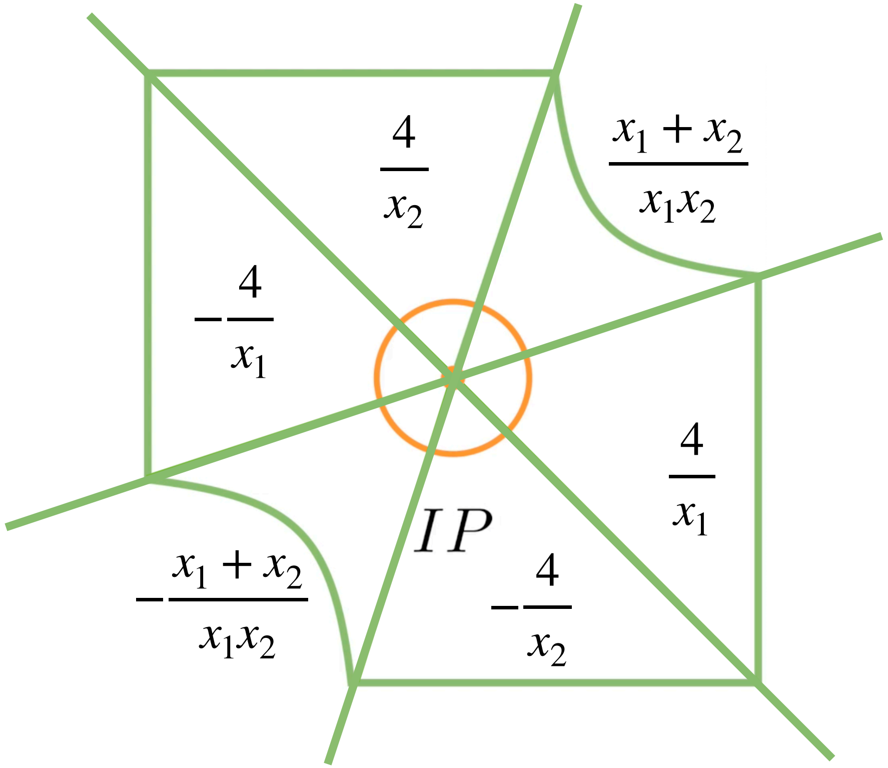
$$\rho(x) = \frac{1}{\|x\|}\mathrm{vol}(P \cap x^{\perp}) = \frac{p_C(x)}{q_C(x)} \quad \text{for } x \in C.$$

Corollary:

The intersection body of a polytope is a semialgebraic set,

i.e. a subset of $\mathbb{R}^d$ defined by finite unions and intersections of polynomial inequalities.



$$IP = \{x \in \mathbb{R}^d \mid \rho(x) \geq 1\}$$

Intersection Bodies of Translations

How does IP behave under translation of P by a vector $t \in \mathbb{R}^d$?

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- ρ is a piecewise rational function in variables $x_1, \ldots, x_d, t_1, \ldots, t_d$:

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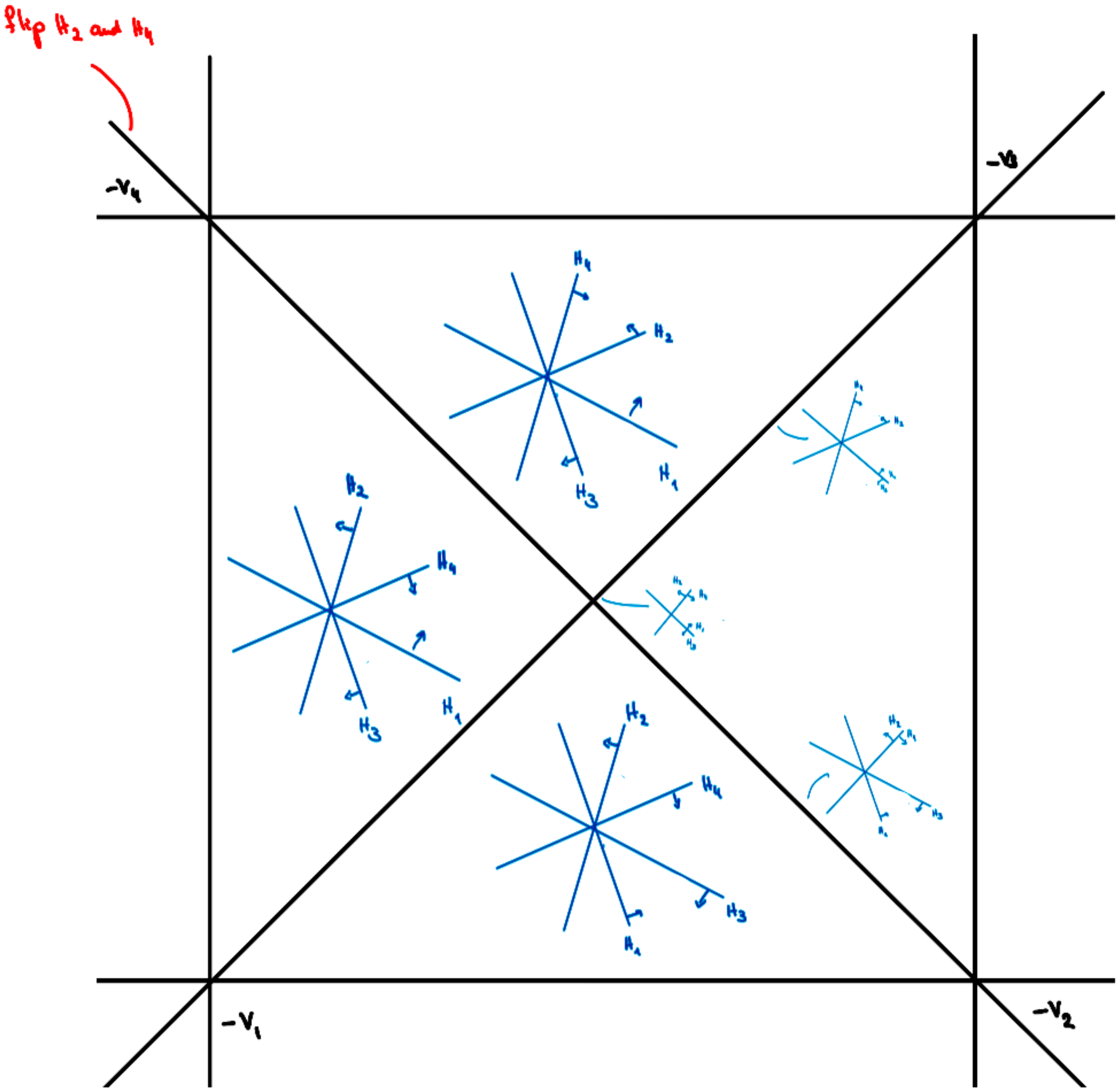
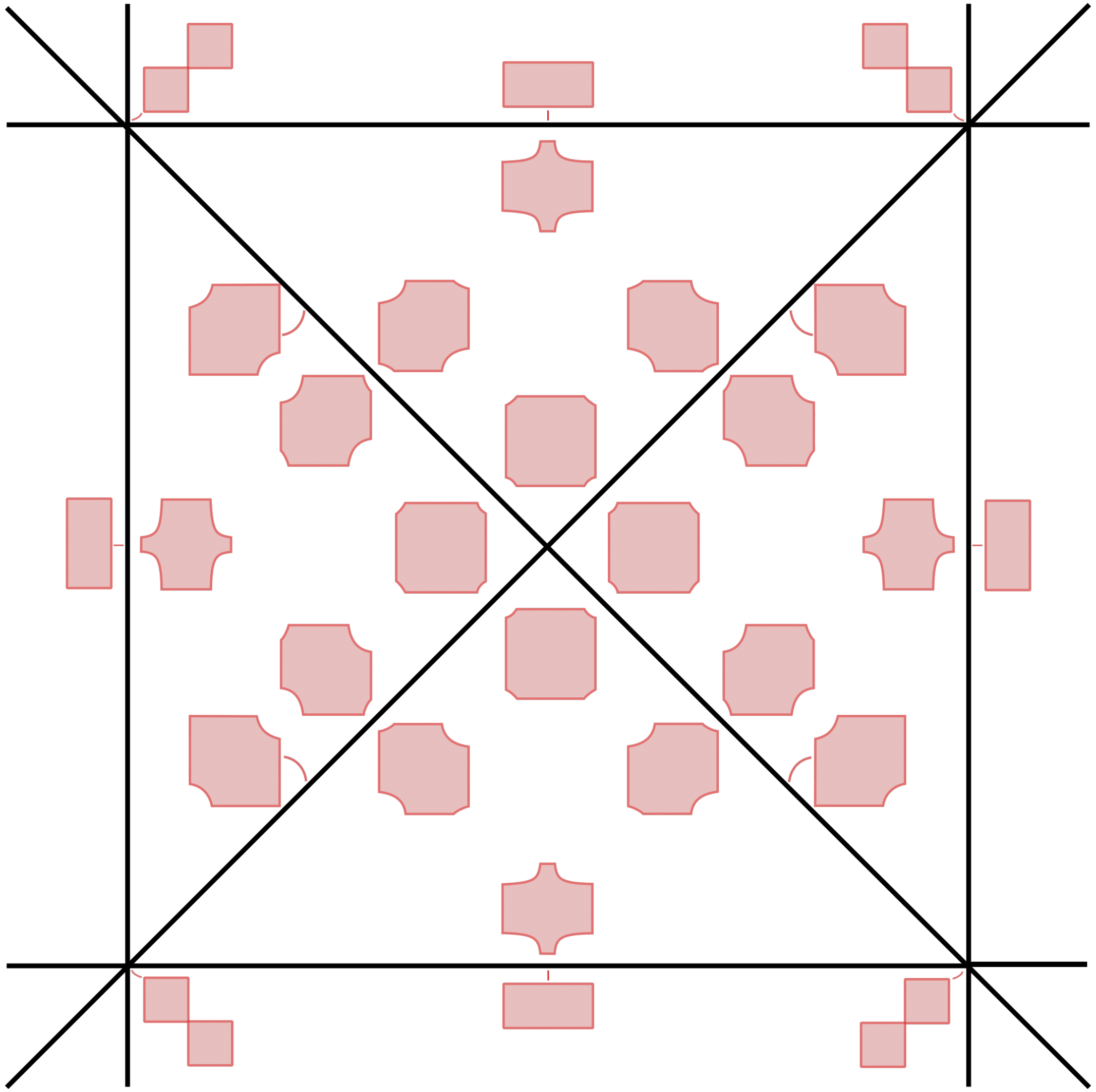
Intersection Bodies of Translations





Translations and Convexity

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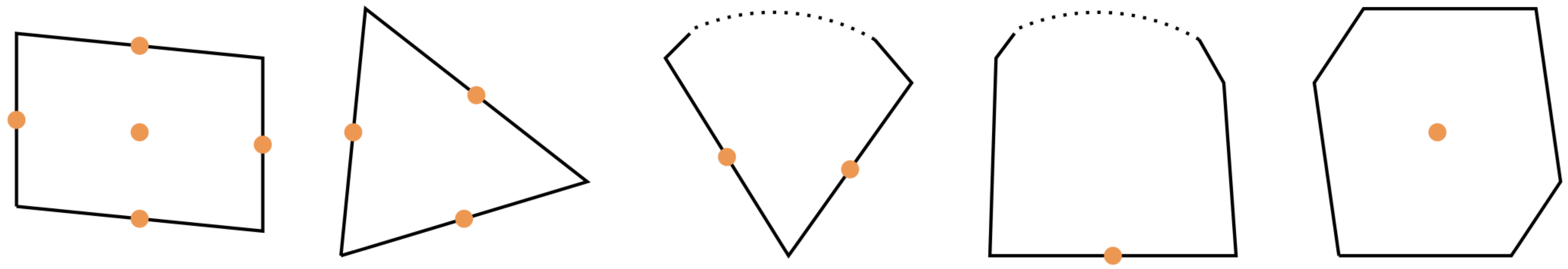
Corollary:

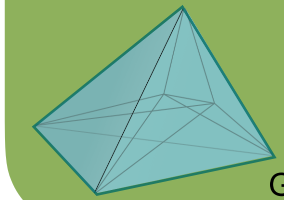
Let $k = |\{t \in \mathbb{R}^2 \mid I(P+t) \text{ is convex}\}|$. Then $k \leq 5$, and $k = 5 \iff P$ is a parallelogram.



Tropical Geometry

Semialgebraic Sets

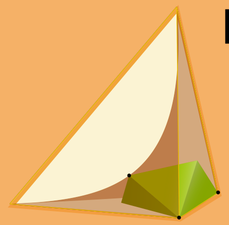
Tropical Positivity and Determinantal Varieties



joint work with
Georg Loho and Rainer Sinn

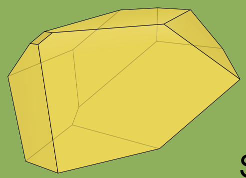
Polyhedral Geometry

Correlated Equilibrium Polytopes



joint work with
Benjamin Hollering and Irem Portakal[4]

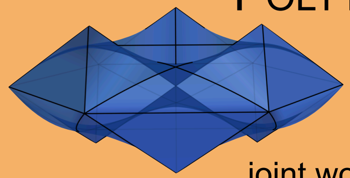
Volume Polynomials of Tropical Polytopes



joint work with
Sophia Elia and Leon Zhang

Volumes

Intersection Bodies of Polytopes



joint work with Katalin Berlow,
Chiara Meroni, and Isabelle Shankar

[4]M. Brandenburg, B. Hollering, and I. Portakal. *Combinatorics of Correlated Equilibria.* 2022. arXiv: 2209.13938

Game Theory

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GAME THEORY

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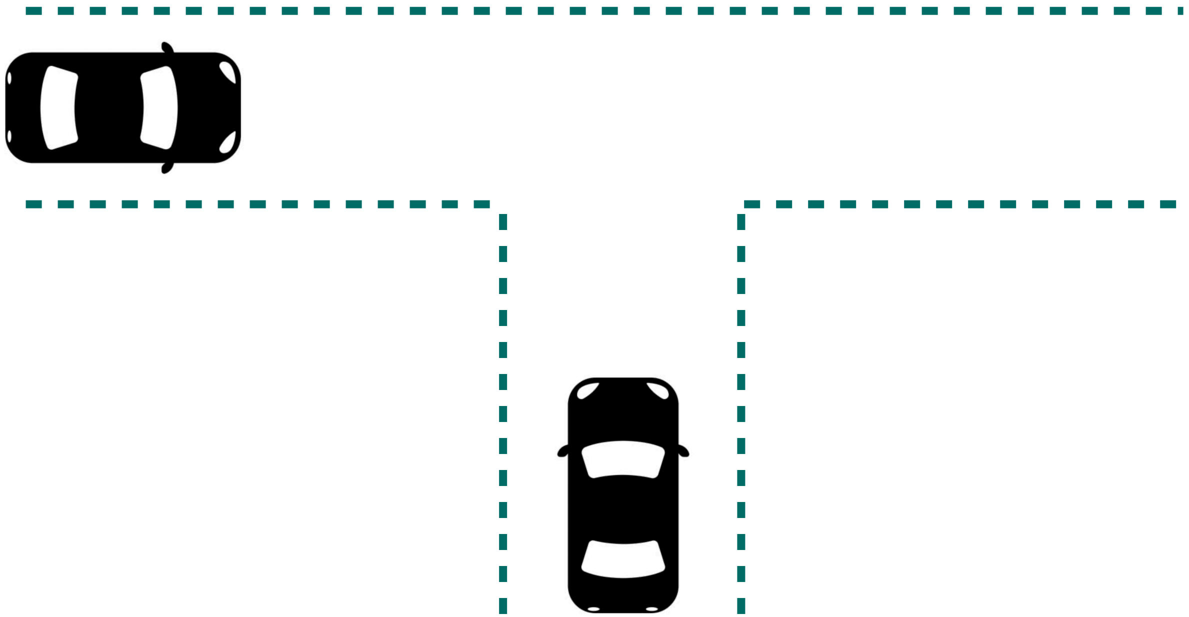

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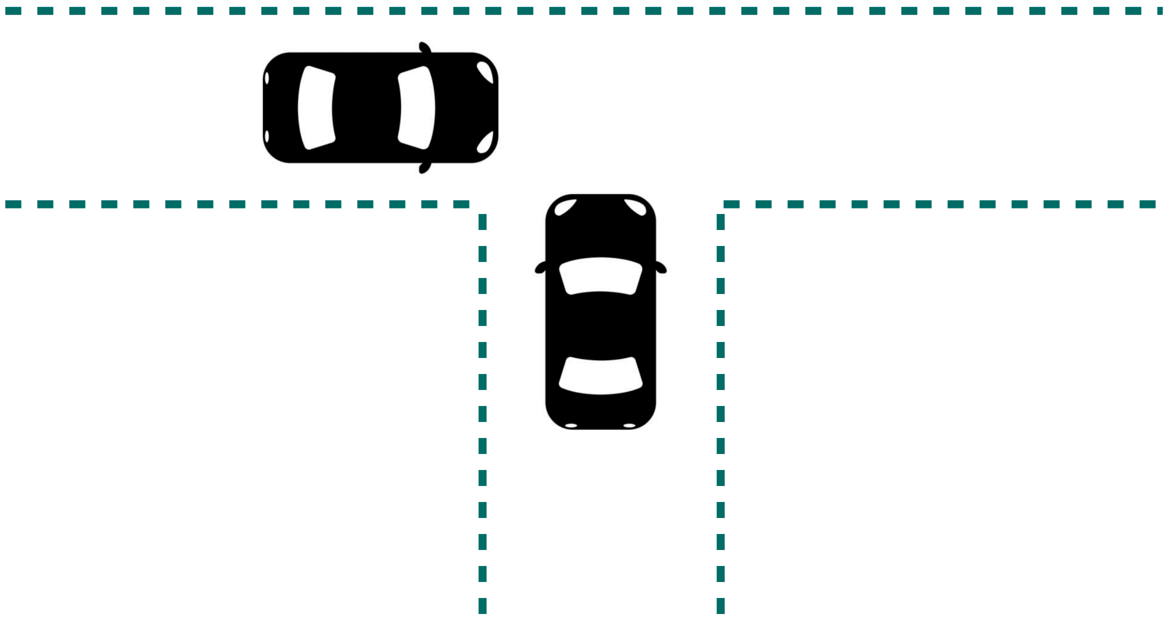

Player 1 / Player 2	stop	go
stop		
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Game Theory

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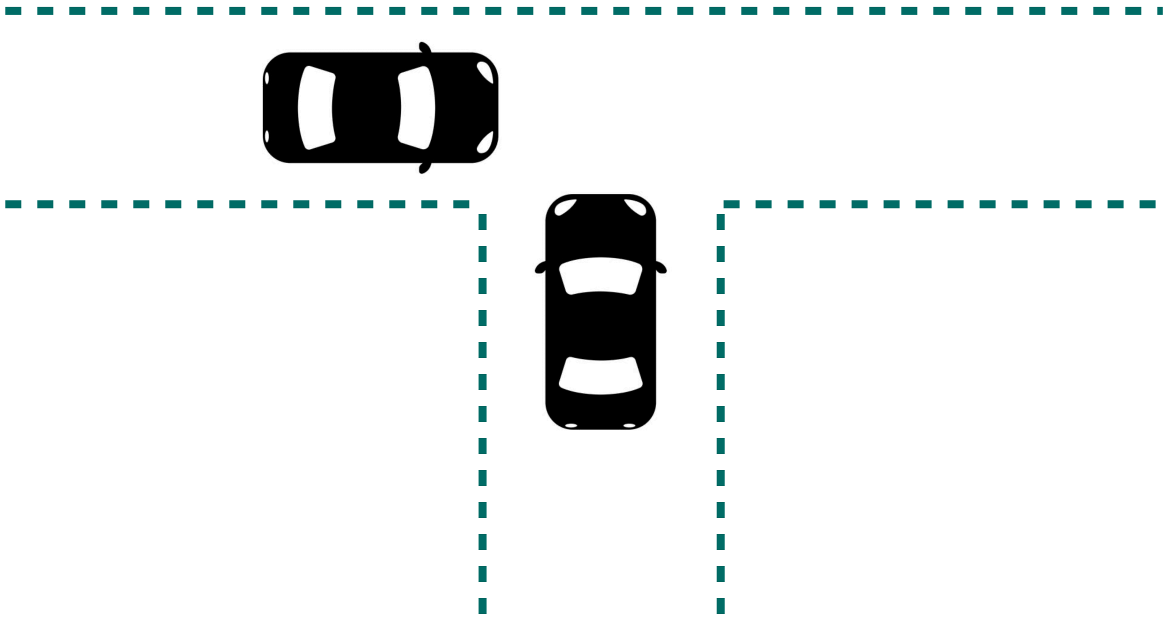

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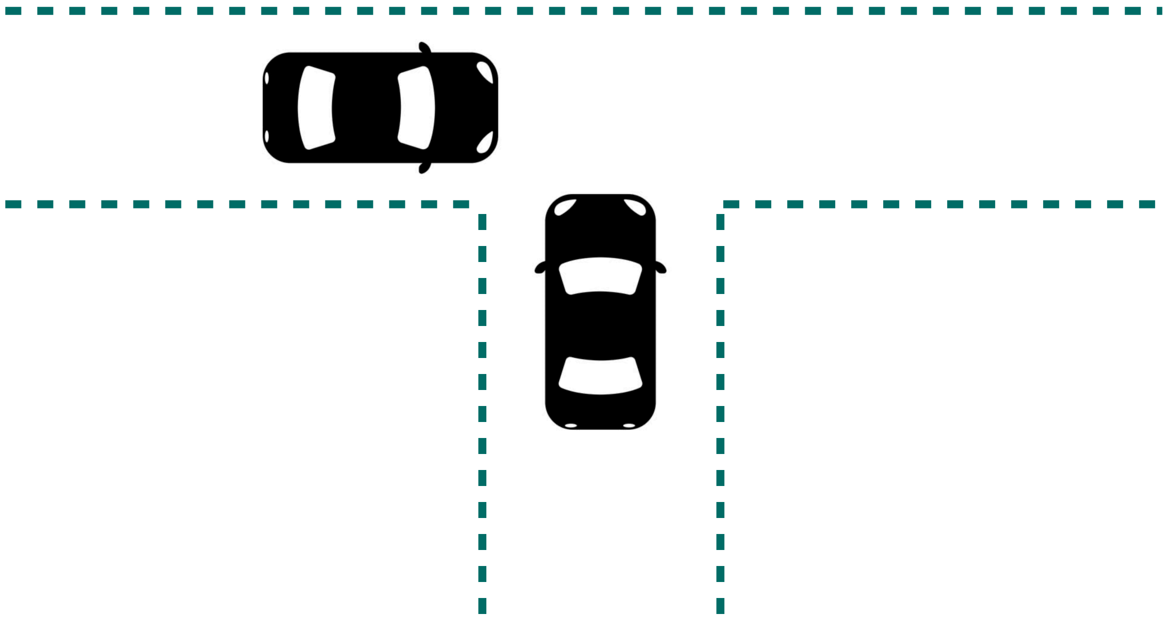

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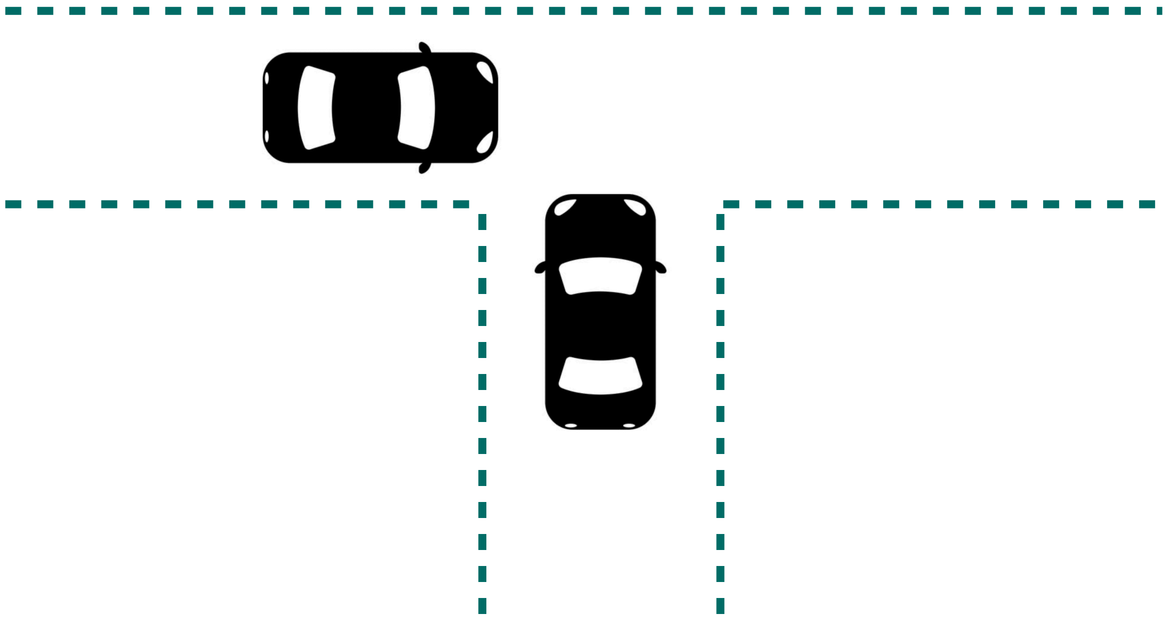

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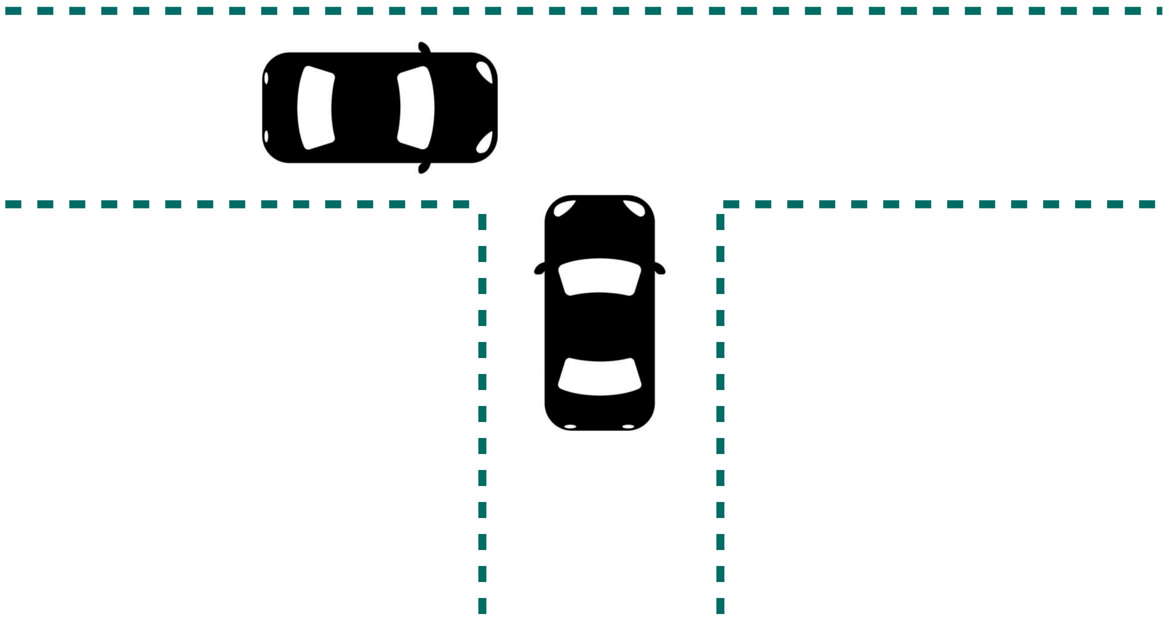
Idea: Third party draws recommendation with probability $p_{j_1 \dots j_n}$



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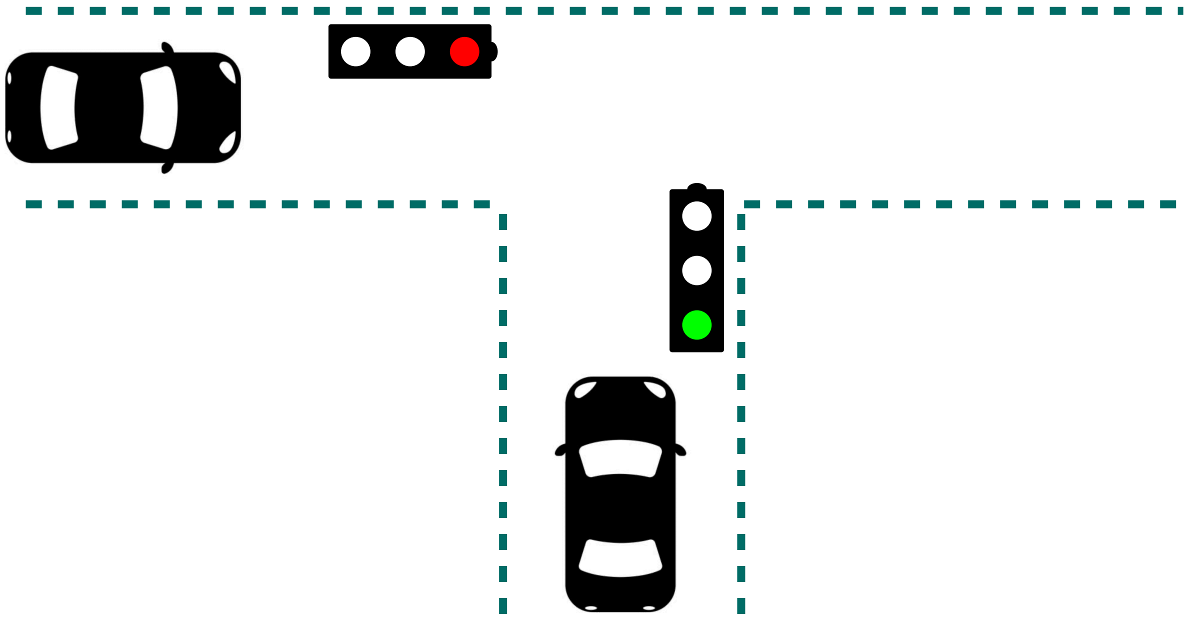

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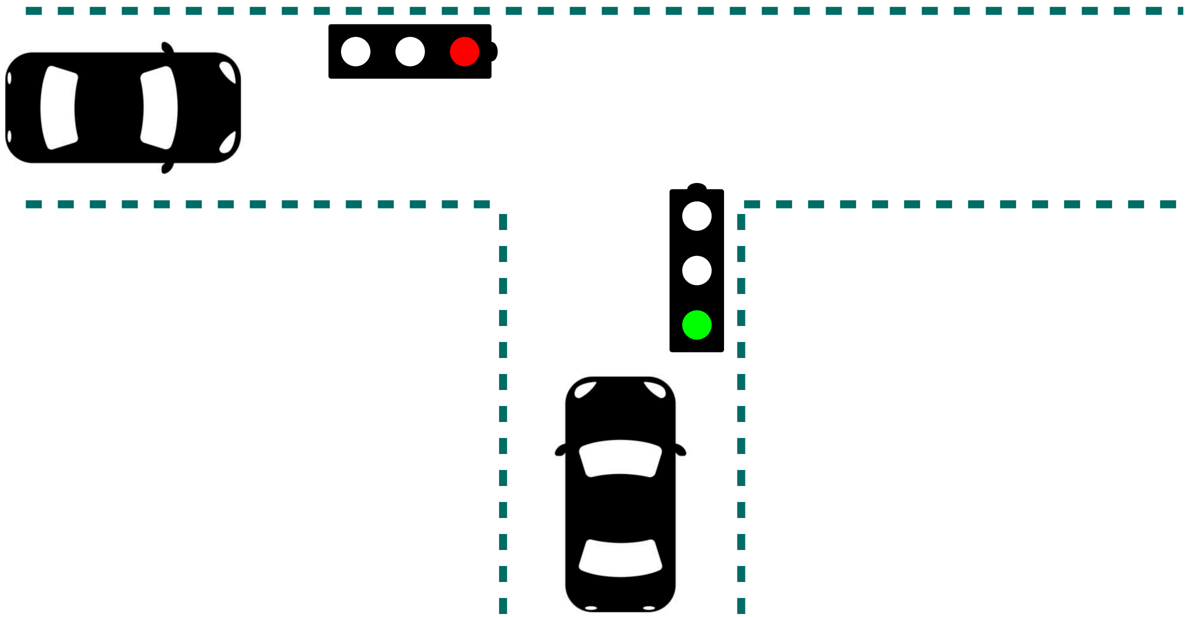

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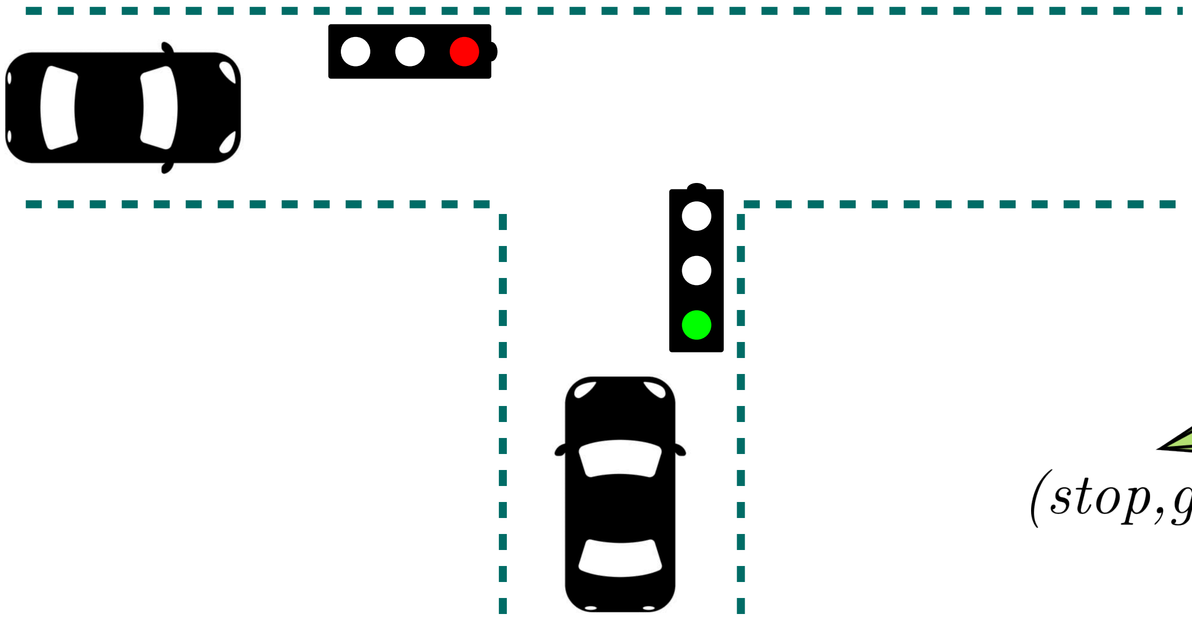
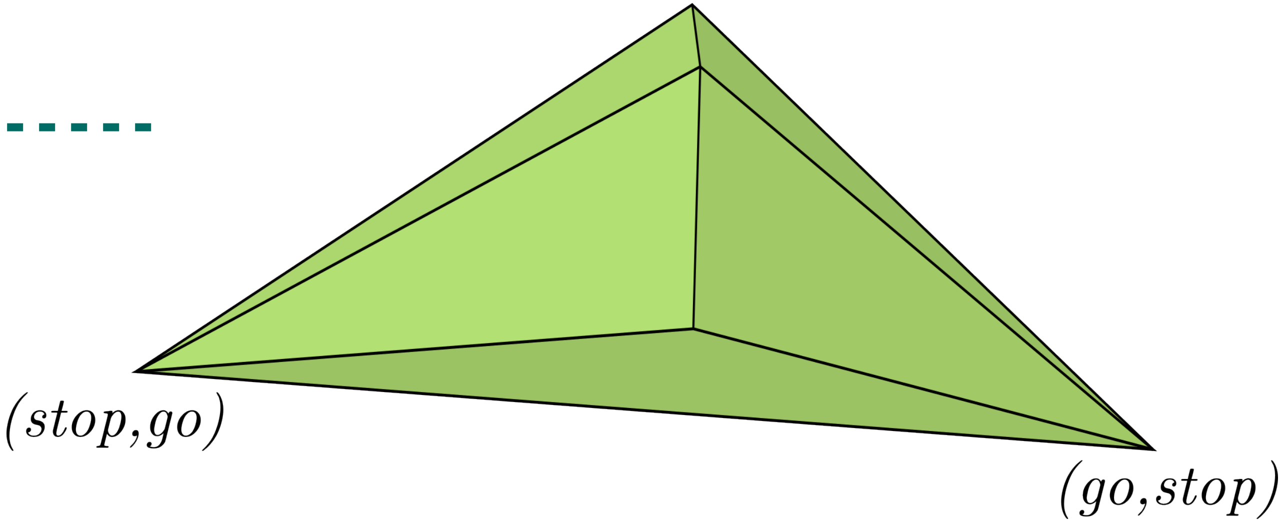
The set of correlated equilibria of a game is given by linear inequalities

$\rightarrow$ correlated equilibrium polytope P_G





Correlated equilibrium

The set of correlated equilibria is given by

$$p_{j_1\ldots j_n} \geq 0 \text{ for } j_i \in [d_i], i \in [n], \text{ and } \sum_{j_1=1}^{d_1} \cdots \sum_{j_n=1}^{d_n} p_{j_1\ldots j_n} = 1 \text{ and}$$

$$\sum_{j_1=1}^{d_1} \cdots \widehat{\sum_{j_i=1}^{d_i}} \cdots \sum_{j_n=1}^{d_n} \left(X^{(i)}_{j_1\cdots j_{i-1}kj_{i+1}\cdots j_n} - X^{(i)}_{j_1\cdots j_{i-1}lj_{i+1}\cdots,j_n} \right) p_{j_1\cdots j_{i-1}kj_{i+1}\cdots j_n} \geq 0 \text{ for all } k, l \in [d_i] \text{ and } i \in [n].$$

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Region of full-dimensionality: subset of payoff space

Theorem (B.-Hollering-Portakal)

The region of full-dimensionality forms a semialgebraic set (and can be explicitly described).

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THEOREM (B.-HOLLERING-PORTAKAL)

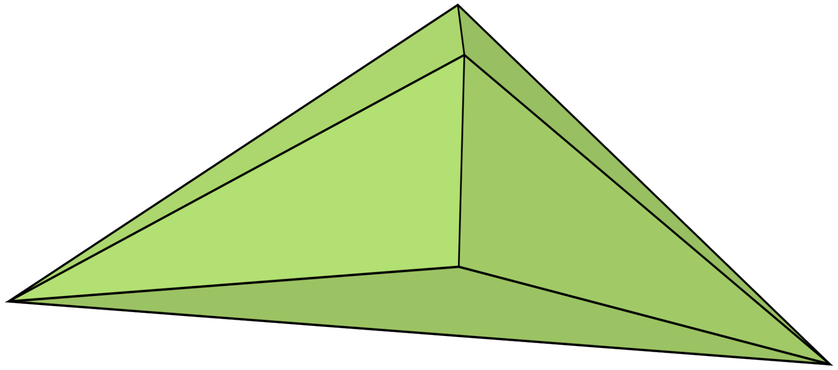
There exists a subdivision of the payoff space into semialgebraic sets (“oriented matroid strata”), such that within each cell the combinatorial type of P_G is fixed.

Correlated equilibrium

Theorem (CA03)

Let G be a $(2 \times 2)-$game. Then P_G is either

- a point, or
- $3-$dimensional (full-dimensional) bipyramid over a triangle.



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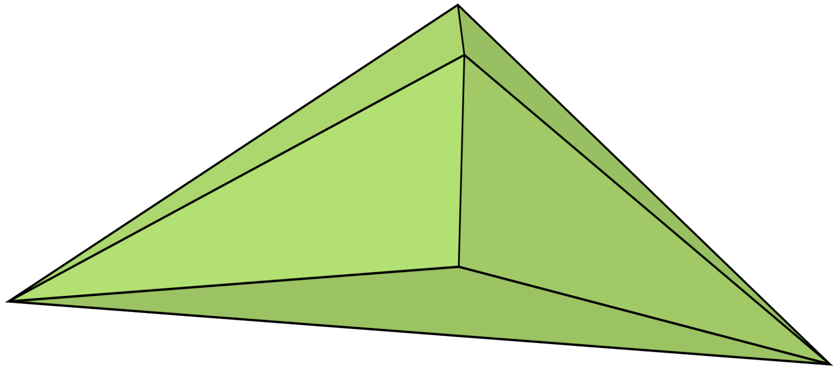
Proposition (B.-Hollering-Portakal):

For $(2 \times 2 \times 2)-$games, there are at least $14\ 949$ full-dimensional combinatorial types.



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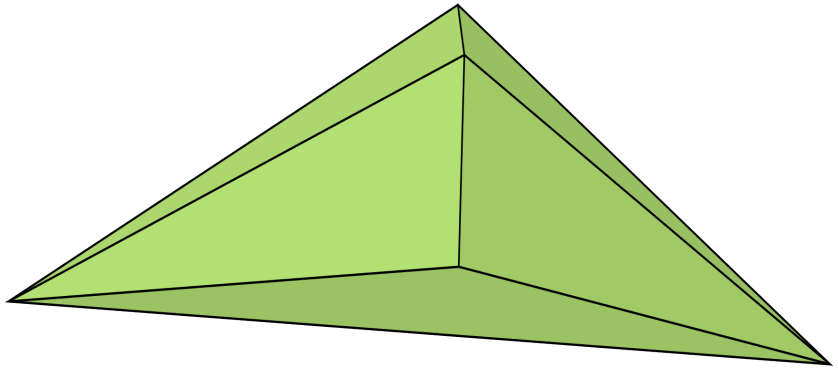
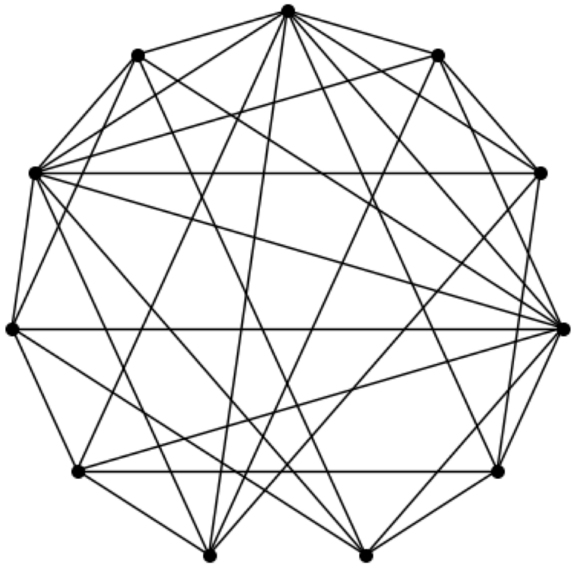


Theorem (B.-Hollering-Portakal)

Let G be a $(2\times 3)-$game. Then P_G is either

- a point, or
- a bipyramid over a triangle, or
- $5-$dimensional (full-dimensional) and of a unique combinatorial type.



Tropical Geometry

Semialgebraic Sets

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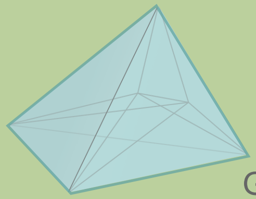
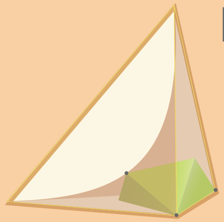


joint work with
Georg Loho and Rainer Sinn

Polyhedral Geometry

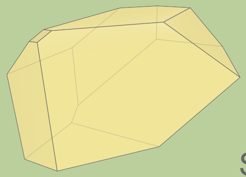
Correlated Equilibrium Polytopes



joint work with
Benjamin Hollering and Irem Portakal[4]

Thank You

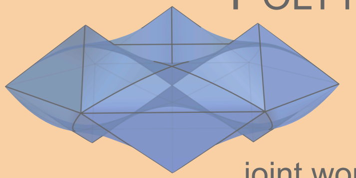
Volume Polynomials of Tropical Polytopes



joint work with
Sophia Elia and Leon Zhang

Volumes

Intersection Bodies of Polytopes

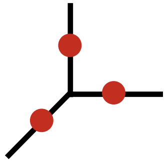


joint work with Katalin Berlow,
Chiara Meroni, and Isabelle Shankar

Starship Criterion

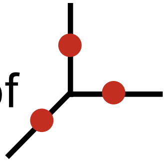
Starship Criterion (B.-Loho-Sinn):

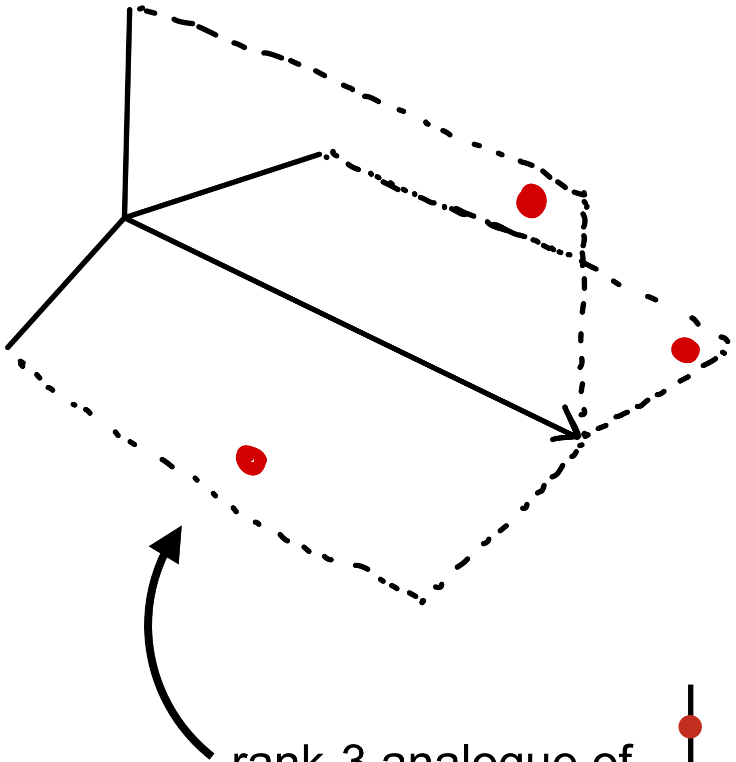
$A \in \text{trop}^+(V^3_{d\times n}) \implies$ the configuration of points on a tropical plane does not contain a starship.

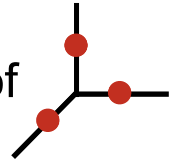
Rank 2 $\implies$ no 

Rank 3 $\implies$ no starship

Rank $k \geq 4$: There are examples of $A \in \text{trop}^+(V^k_{d\times n})$

such that the rank-k-analogue of  occurs.



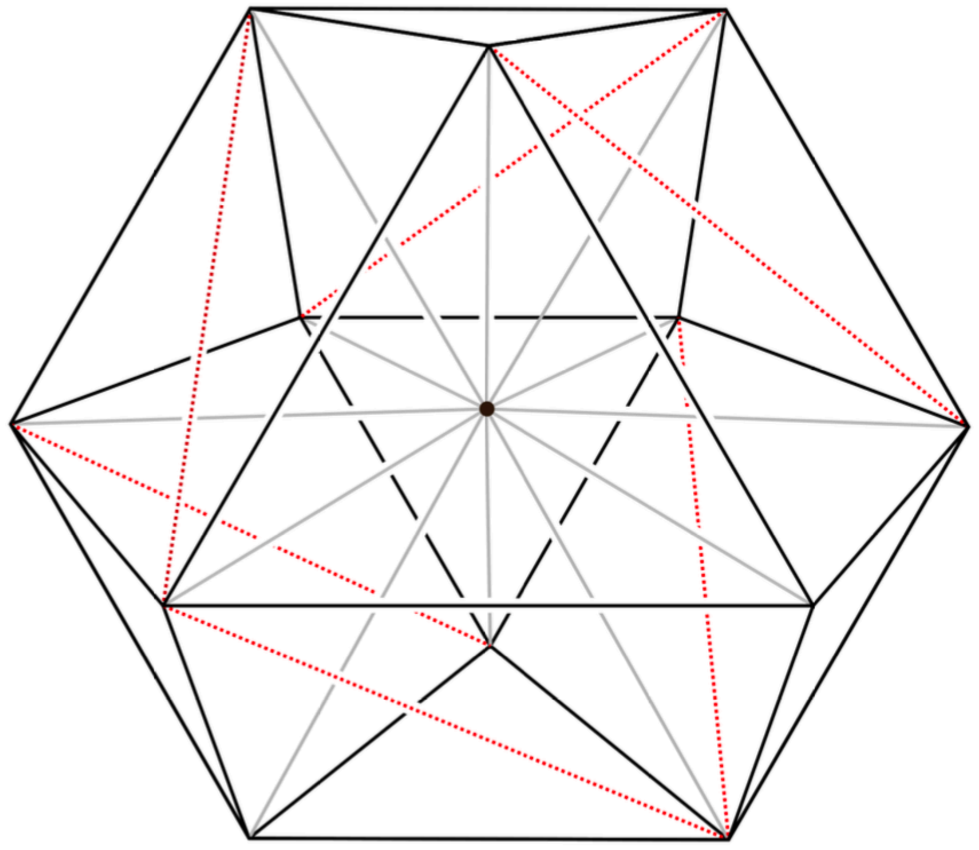
rank-3 analogue of 

Volume Polynomials of Polytropes

Theorem (B.-Elia-Zhang):

In the 8855-dimensional space of homogenous polynomials of degree 4, the $27\ 248$ volume polynomials span a 70-dimensional affine subspace.

Partition	Example monomial	Possible coefficients	Coefficient sum
4	a_{12}^4	$-6,-3,-2,-1,0,1,2,3$	-20
$3+1$	$a_{12}^3 a_{13}$	$-4,0,4,8$	320
$2+2$	$a_{12}^2 a_{13}^2$	$0,6$	300
$2+1+1$	$a_{12} a_{13} a_{14}^2$	$-12,0,12$	-2160
$1+1+1+1$	$a_{12} a_{13} a_{14} a_{15}$	$0,24$	1680

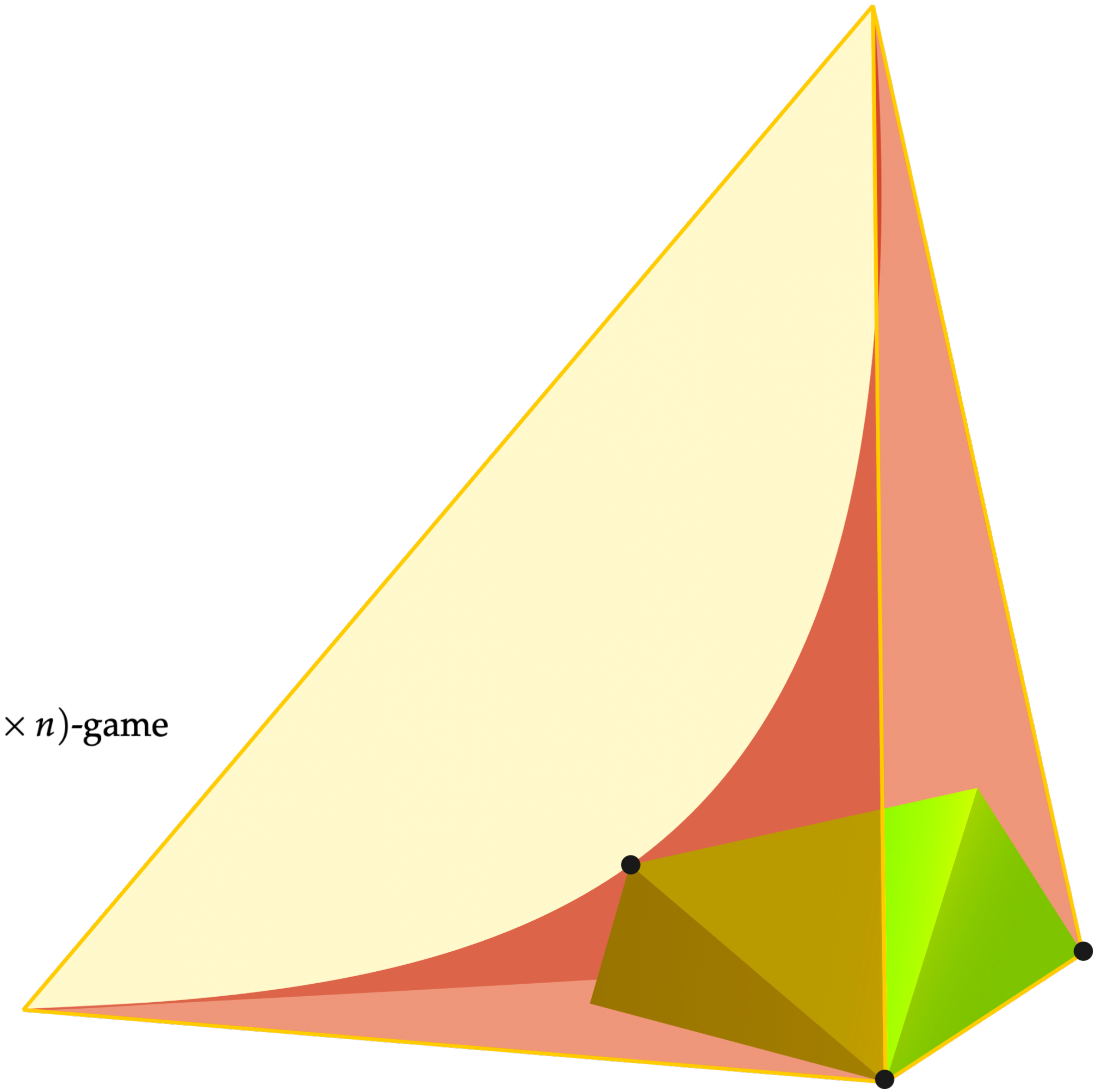


Regular triangulation of FP_3

Correlated Equilibrium

Unique Combinatorial Types by Dimension					
Dimension	0	3	5	7	9
(2×2)	1	1	0	0	0
(2×3)	1	1	1	0	0
(2×4)	1	1	1	3	0
(2×5)	1	1	1	3	4

The number of unique combinatorial types of P_G of each dimension for a $(2 \times n)$-game in a random sampling of size 100 000.



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