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Competitive Equilibrium and Lattice Polytopes

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Abstract

A competitive equilibrium is an envy-free allocation of goods. That is, given a price function that assigns a nonnegative value to each bundle of items, each agent prefers the allocated bundle to every other affordable bundle of goods, or is indifferent towards them. It is further required that the allocation clears the market, i.e. each item is allocated to exactly one agent. There is a particular interest in economics in conditions on the existence of a competitive equilibrium for indivisible goods. In a recent work, Condogan et al. study such conditions in the setting of graphical valuations, which is a class of valuations associated to a value graph, whose nodes correspond to items and whose edges encode (pairwise) complementarities or substitutabilities between items.

We explore a connection between these graphical valuations and regular subdivisions of a polytope that can be constructed from the value graph. We show that a competitive equilibrium exists for all sets of valuations and any bundle of indivisible goods, provided that, in the setting of anonymous graphical pricing, the value graphs of all agents correspond to complete graphs.

Contents

1 Introduction

Product-Mix Auctions were introduced by Paul Klemperer [5] for the Bank of England when the credit crisis emerged in 2007. Ever since, Klemperer's auction design has been actively used, e.g. by the Bank of England and the U.S. Treasury. As a single-round auction, it is resistant to many disadvantages that multi-round auctions have: The auction takes place at a single instant and the valuation of each agent is secret to all other participants of the auction. In fact, if a competitive equilibrium, i.e. an envy-free allocation of a set of goods, exists, then agents who want to maximize their profit benefit from bidding their true valuation [7]. In a recent work, Baldwin and Klemperer [1] explore a connection to tropical geometry and give a sufficient condition – the Unimodality Theorem – for the existence of a competitive equilibrium in Product-Mix Auctions of indivisible goods.

Tran and Yu [7] connect the Unimodality Theorem with an integer program and formulate the problem of finding a competitive equilibrium in Product-Mix Auctions for indivisible goods as a particular linear program, whose optimal integral solutions correspond to allocations that achieve a competitive equilibrium.

An extension of the above is the package assignment model, in which there is an underlying graph that models relationships between goods. We distinguish between *anonymous pricing*, in which there is a global pricing function that does not distinguish between agents and *agent-specific pricing*, in which there is one pricing function for each agent. Bikhchandan and Ostroy [2] show that in the single-seller setting, a competitive equilibrium with a nonnegative, agent-specific pricing rule exists.

In the setting of anonymous graphical pricing, Candogan, Ozdaglar and Parrilo [4] give an integer program for finding a competitive equilibrium for indivisible goods and describe the relationship between a Walrasian and a competitive equilibrium for series-parallel graphs, provided that the bundle of interest contains exactly one item per type. In this context, a Walrasian equilibrium is a competitive equilibrium in which the pricing rule of a bundle of goods is the sum of the prices of the single items. However, a Walrasian equilibrium does not exist for graphical valuations in general [3].

In this thesis, we consider polytopes that can be constructed from the underlying graph of the valuations. We give conditions on the polytope that guarantee the existence of a competitive equilibrium at a bundle of indivisible goods that is independent of the agents' valuations. Further, we show that if the underlying value graph is the complete graph, a competitive equilibrium always exists and so, in particular, the integer program [4, LP 2] always has an integral optimal solution.

Section 2.1 outlines the mathematical model of the auction and a competitive equilibrium. In 2.2, we describe the connection to tropical geometry and lay out the connection to lattice polytopes in 2.3. We explore the case in which the underlying graph is the complete graph in Section 3 and show that a competitive equilibrium always exists in 3.2. Sections 3.4 and 3.5 investigate generalizations of this question. Finally, we relate our results to [4].

2 Competitive Equilibrium

In the following, we lay out the mathematical model of graphical valuations and anonymous graphical pricing as introduced in [4]. Following [1] and [7], we describe the connection to tropical geometry, and, in 2.3, give conditions under which a competitive equilibrium is guaranteed to exist.

2.1 Product-Mix Auctions

A Product-Mix Auction is a single round auction. That is, given a bundle of goods, it is able to both set a price for each type of goods and determine the quantities of goods that are assigned to each bidder within a single instance. Opposed to the classical multi-round auction, in which the prices are raised in a series of bids, it resolves challenges that the classical setting faces. For example, in many scenarios agents participating in an auction are interested in buying only one out of two items, which are sold one after another in separate auctions. Thus, the agent has to give bids on the first item before the auction concerning the second item begins. At the same time, bidders are influenced by the bidding behavior of other agents, as well as by external factors. In particular, in contexts that are of public interest, the financial market can be influenced by the progression of the auction during multiple rounds, influencing the objectives of the bidding agents themselves in later rounds. It is thus hard to find a strategy that maximizes the profit of a single agent and to achieve an allocation that is desirable for the whole market in the classical setting.

Throughout this thesis, we assume that there are n types of indivisible goods and J agents who are interested in buying at most one good of each type. The auctioneer wants to sell a *bundle of goods* $a^* \in \mathbb{Z}^n_{\geq 0}$, where the value a^*_i is the quantity of goods of type i. At the beginning of the auction, each agent values all possible combinations of goods. The auctioneer collects these valuations and tries to find a distribution of the goods among the agents such that the bundle that is assigned to an agent maximizes the respective profit and is thus *demanded* by this agent. We further assume, that the valuation functions of all agents are *graphical valuations*, in which dependencies (complementarity or substitutability) between types of goods correspond to edges in a *value graph* on n vertices.

2.1.1 Graphical Valuations

Let G be a simple, undirected graph on n vertices with e edges, where each vertex represents one type of goods. Two vertices are connected by an edge if and only if there is a dependency between them, i.e. if agents or the auctioneer view the respective types of goods as pairwise complementary or substitutable. For an induced subgraph $G' \subseteq G$, we define the *characteristic vector* $\chi_{G'} \in \mathbb{R}^d,\ d = n + e$, by

$$(\chi_{G'})_i = \begin{cases} 1, & \text{if } i \in V(G') \\ 0, & \text{otherwise} \end{cases} \quad \text{for } i \in V(G), \qquad (\chi_{G'})_{ij} = \begin{cases} 1, & \text{if } ij \in E(G') \\ 0, & \text{otherwise} \end{cases} \quad \text{for } ij \in E(G).$$

We refer to indices that correspond to vertices by $i \in V(G) = [n]$ and to indices that correspond to edges by $ij \in E(G) \subseteq \binom{[n]}{2}$. We distinguish between vectors $a^* \in \mathbb{R}^n$ that are indexed by $V(G)$ and vectors $a \in \mathbb{R}^d$ that are indexed by $V(G) \cup E(G)$. Note that given an induced subgraph G', the vector $\chi_{G'}$ contains redundant information, since we can reconstruct the coordinates of $\chi_{G'}$ indexed by $E(G)$, given the coordinates indexed by $V(G)$ and the original graph G. However, this does not hold in general for convex combinations of such characteristic vectors.

We define the polytope of the graph G by

$$P(G) = \operatorname{conv}\left\{\chi_{G'} \,\middle|\, G' \text{ is an induced subgraph of } G\right\}.$$

By construction, $P(G)$ is a 0/1–polytope, and therefore

$$P(G) \cap \mathbb{Z}^d = \operatorname{vert}(P(G)) = \left\{\chi_{G'} \,\middle|\, G' \text{ is an induced subgraph of } G\right\}.$$

Each agent $j \in [J]$ is equipped with a *graphical valuation*

$$v^j : P(G) \cap \mathbb{Z}^d \to \mathbb{R}, \quad v^j(a) = \left\langle w^j, a \right\rangle,$$

where $w_i^j \geq 0$ and $w_{ik}^j \in \mathbb{R}$ for all $i \in V(G), ik \in E(G)$.

Such a valuation function can be interpreted as follows: If $w_i^j > 0$ for some vertex $i \in V(G)$, then agent j is interested in buying an item of type i and w_i^j is agent j's offer to purchase a single item of type i. That is, agent j is not willing to pay more than w_i^j for an item of type i. If $w_{ik}^j > 0$ for some edge $ik \in E(G)$, then agent j views items of types i and k as *complementary*, i.e. is rather interested in buying both items together than only a single one of them. The higher the value of w_{ik}^j, the higher is agent j's preference of buying both i and k at the same time. Similarly, if $w_{ik}^j < 0$, then items of types i and k are viewed as *substitutable*, i.e. agent j prefers to buy only one of the two items.

Let $J \cdot (P(G) \cap \mathbb{Z}^d) = \sum_{k=1}^{J} (P(G) \cap \mathbb{Z}^d)$ denote the Minkowski sum of $P(G) \cap \mathbb{Z}^d$ with itself. The *aggregate valuation* is given by

$$V(a) = \max \left\{ \sum_{j=1}^{J} v^j(a^j) \,\middle|\, a^j \in P(G) \cap \mathbb{Z}^d,\ \sum_{j=1}^{J} a^j = a \right\},$$

i.e. it is the maximum total bid for $a \in J \cdot (P(G) \cap \mathbb{Z}^d)$. If $a \notin J \cdot (P(G) \cap \mathbb{Z}^d)$, we define $V(a) = -\infty$.

Example 1. Let G be the graph consisting of 2 disjoint vertices. The corresponding polytope is

$$P(G) = \operatorname{conv}\left(\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \right).$$

This models an auction with two distinct types of items. Suppose there are two agents participating in the auction with the valuation functions

$$v^1(a) = \langle \left(\begin{smallmatrix} 3 \\ 5 \end{smallmatrix} \right), a \rangle, \quad v^2(a) = \langle \left(\begin{smallmatrix} 7 \\ 2 \end{smallmatrix} \right), a \rangle.$$

The respective aggregate valuation is defined on $2P(G)$ and given in Figure 1.

2.1.2 Demand Sets

Given the agents' valuations, the auctioneer needs to set a price $p \in \mathbb{R}^d_{\geq 0}$. The value p_i is the price of a single item of type $i \in V(G)$ offered by the auctioneer. The value p_{ik} for an edge $ik \in E(G)$ reflects constraints between items of types i and k that affect the total price of a bundle containing both items of types i and k (such as discounts or fees).

The *demand set* of agent j for a price p is

$$D_{v^j}(p) = \operatorname*{arg\,max}_{a \in P(G) \cap \mathbb{Z}^d} \left\{ v^j(a) - \langle p, a \rangle \right\}.$$

The *aggregate demand set* is given by

$$D_V(p) = \operatorname*{arg\,max}_{a \in J \cdot (P(G) \cap \mathbb{Z}^d)} \left\{ V(a) - \langle p, a \rangle \right\}.$$

It can be shown that the aggregate demand set is equal to the Minkowski sum of the demand sets of all agents, i.e.

$$D_V(p) = \sum_{j=1}^{J} D_{v^j}(p).$$

A proof of this is given in Lemma 3.

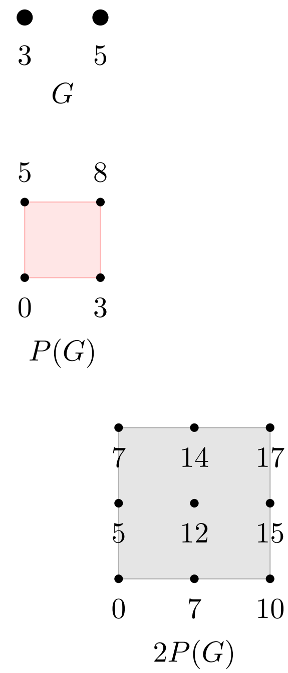


FIGURE 1. The top shows the graph G given in Example 1 with the respective weights w^1, w^2 on its vertices. Below is the polytope $P(G)$ with valuations v^1 (red) and v^2 (blue) given on all of its vertices. On the bottom is the dilated polytope $2P(G)$ with the aggregate valuation for each of its lattice points. The colors are consistent with the colors in Figure 2a and Figure 2b.

2.1.3 Anonymous Graphical Pricing

The auctioneer has a fixed bundle of goods $a^* \in \mathbb{Z}^n_{\geq 0}$ to sell (in economics application we often have $a^* \in \{0,1\}^n$). We consider the set of integer points in the J-th dilate of $P(G)$ which represent this bundle, i.e.

$$A^* = \left\{ a \in JP(G) \cap \mathbb{Z}^d \,\middle|\, a_i = a^*_i \ \forall i \in V(G) \right\}.$$

Given the agents' valuations, the auctioneer's goal is to find a price $p \in \mathbb{R}^d$ and a decomposition of the bundle a^* that induces a competitive equilibrium.

Definition 1. The set $\{v^j \mid j \in [J]\}$ of graphical valuations has a *competitive equilibrium* at $a \in \mathbb{Z}^d_{\geq 0}$ if there exists a price $p \in \mathbb{R}^d$ such that $a \in D_V(p)$. The set $\{v^j \mid j \in [J]\}$ has a *competitive equilibrium* at $a^* \in \mathbb{Z}^n_{\geq 0}$ if there exists a price $p \in \mathbb{R}^d$ and $a \in A^*$ such that $a \in D_V(p)$, i.e. where a competitive equilibrium is achieved at some $a \in A^*$. We say that a competitive equilibrium at a (resp. a^*) *always exists* if it exists for all sets of valuations.

Note that since $D_V(p) = \sum_{j=1}^J D_{v^j}(p)$, showing the existence of a competitive equilibrium is equivalent to finding a decomposition of a point $a \in A^*$ such that $a = \sum_{j=1}^J a^j$, $a^j \in D_{v^j}(p)$, i.e. agent j demands a^j at the given price $p \in \mathbb{R}^d$. Further, for a given $a^* \in \mathbb{R}^n$, if there is some $a \in A^*$ such that a competitive equilibrium always exists at a, then in particular it always exists at a^*.

Example 2. As pointed out in [7, Example 1], the question of the existence of a competitive equilibrium is different from the question of maximizing profit. As an example in the setup of graphical valuations, suppose there are only two items of the same type of goods to sell in an auction with two agents. That is, let $n = 1, J = 2, a^* = 2$, and the underlying graph G is the graph consisting of a single vertex. The respective polytope is $P(G) = \operatorname{conv}(\{0, 1\})$ and $A^* = \{2\}$. Suppose the valuation functions of the participating agents are

$$v^1(a) = a, \qquad v^2(a) = 3a.$$

For a fixed price $p \in \mathbb{R}$, the demand sets are given by

$$D_{v^1}(p) = \operatorname*{arg\,max}_{a \in \{0,1\}} \{(1-p)a\}, \quad D_{v^2}(p) = \operatorname*{arg\,max}_{a \in \{0,1\}} \{(3-p)a\}.$$

A competitive equilibrium can be achieved at $a^* = 2$ for the $p = 1$ with the decomposition $2 = 1 + 1$, i.e. one item is sold to each of the agents. The overall profit of the auctioneer in this case is 2. But if the auctioneer is willing to sell only one item to agent 2 and nothing to agent 1, then the profit can be raised to 3 by setting $p = 3$. Note that this distribution induces a competitive equilibrium at $a^* = 1$, since $D_{v^1}(3) = \{0\}$.

Example 3. A competitive equilibrium at a particular $a \in A^*$, $a^* \in \mathbb{Z}^n_{\geq 0}$ is not guaranteed to exist. An example of such an a^* and $a \in A^*$ in the case of graphical valuations is given in Section 3.3, where the underlying graph is the complete graph. However, by Theorem 6, there exists another $a' \in A^*$, where a competitive equilibrium can be attained. There is no known example for graphical valuations, where all valuations rely on the same value graph, all agents bid on at most one item per type and an $a^* \in \mathbb{Z}^n_{\geq 0}$ such that a competitive equilibrium does not exist at any $a \in A^*$.

2.2 Connections to Tropical Geometry

In this section, we connect the previously discussed content to tropical hypersurfaces and regular subdivisions of polytopes. This connection holds in a more general setting, so we relax the conditions on the valuations and include arbitrary valuations $v : P(G) \cap \mathbb{Z}^d \to \mathbb{R}$ in this section.

We consider the max-plus semiring $(\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$ with tropical addition and multiplication

$$r \oplus s = \max\{r, s\}, \qquad r \odot s = r + s.$$

Given a valuation $v : P(G) \cap \mathbb{Z}^d \to \mathbb{R}$, the *tropical Laurent Polynomial* with respect to v is

$$f_v(x) = \bigoplus_{a \in P(G) \cap \mathbb{Z}^d} v(a) \odot x^{\odot a} = \max_{a \in P(G) \cap \mathbb{Z}^d} \{v(a) + \langle a, x \rangle\}.$$

The *tropical hypersuface* $T(f_v)$ is the set of points $x \in \mathbb{R}^d$ such that the maximum of $\left\{ v(a) + \langle a, x \rangle \,\middle|\, a \in P(G) \cap \mathbb{Z}^d \right\}$ is attained at least twice. Let $p \in \mathbb{R}^d$ be a price vector. Then

$$f_v(-p) = \max_{a \in P(G) \cap \mathbb{Z}^d} \{v(a) - \langle a, p \rangle\}$$

and $p \in -T(f_v)$ if and only if $|D_v(p)| \geq 2$.

A subdivision of a polytope P is *regular*, if there is a height function $v : \operatorname{vert}(P) \to \mathbb{R}$ such that the subdivision is the projection of the upper hull of $\operatorname{conv}(\operatorname{lift}(P))$, where

$$\operatorname{lift}(P) = \left\{ \left(\begin{smallmatrix} a \\ v(a) \end{smallmatrix} \right) \,\middle|\, a \in \operatorname{vert}(P) \right\} \subseteq \mathbb{R}^{d+1}$$

and the *upper hull* is the polyhedral complex consisting of those facets of $\operatorname{conv}(\operatorname{lift}(P))$ whose normals have positive last coordinate.

We consider the regular subdivision of $P(G)$ that is induced by taking the valuation v as height function. Let $\tilde{v}$ be the smallest concave function such that $\tilde{v}(a) \geq v(a)$ for all $a \in P(G) \cap \mathbb{Z}^d$. We call a a *lifted point* if $\tilde{v}(a) = v(a)$, i.e. if $\left(\begin{smallmatrix} a \\ v(a) \end{smallmatrix} \right)$ lies in the upper hull of $\operatorname{conv}(\operatorname{lift}(P))$. Note that, in particular, a vertex of the regular subdivision is always a lifted point. Further, if the valuation is a linear function, then the induced regular subdivision is trivial. We define

$$\Delta_v = \left\{ \operatorname*{arg\,max}_{a \in P(G) \cap \mathbb{Z}^d} \{v(a) + \langle a, x \rangle\} \,\middle|\, x \in \mathbb{R}^d \right\} = \left\{ D_v(-p) \,\middle|\, p \in \mathbb{R}^d \right\}.$$

The tropical hypersurface $T(f_v)$ is a polyhedral complex of dimension $d - 1$ and induces a partition of $\mathbb{R}^d$, the space of all possible prices, into relatively open convex polyhedra. In this partition, two points $x, x' \in \mathbb{R}^d$ are contained in the same polyhedron if and only if

$$\operatorname*{arg\,max}_{a \in P(G) \cap \mathbb{Z}^d} \{v(a) + \langle a, x \rangle\} = \operatorname*{arg\,max}_{a \in P(G) \cap \mathbb{Z}^d} \left\{ v(a) + \left\langle a, x' \right\rangle \right\}.$$

This is equivalent to

$$D_v(-x) = D_v(-x'),$$

so each region in the partition can be written as

$$\left\{ x' \in \mathbb{R}^d \,\middle|\, D_v(-x') = D_v(-x) \right\}$$

for some $x \in \mathbb{R}^d$. Thus, the partition of $\mathbb{R}^d$ is *dual* to Δ_v via the bijection

$$\left\{ x' \in \mathbb{R}^d \,\middle|\, D_v(-x') = D_v(-x) \right\} \longmapsto D_v(-x) \in \Delta_v.$$

We show that there is a further bijection between Δ_v and the set of faces of the regular subdivision of $P(G)$ induced by v.

Lemma 1. *A point $a \in P(G) \cap \mathbb{Z}^d$ is a lifted point if and only if*

$$a \in \operatorname*{arg\,max}_{a \in P(G)\cap \mathbb{Z}^d} \{ v(a) + \langle\, a, x \,\rangle \}$$

for some $x \in \mathbb{R}^d$. The faces of the regular subdivision of $P(G)$ induced by v are in bijection with the demand sets. In particular, if the face F of the subdivision corresponds to $D_v(-x)$ for some $x \in \mathbb{R}^d$, then

$$D_v(-x) = \left\{ a \in F \cap \mathbb{Z}^d \,\middle|\, a \text{ is a lifted point} \right\}.$$

Proof. Suppose a maximizes $v(a) + \langle\, a, x \,\rangle$ for some $x \in \mathbb{R}^d$. That is, there exists some $\alpha \in \mathbb{R}$ such that $v(a) + \langle\, a, x \,\rangle = \alpha$ and $v(a') + \langle\, a', x \,\rangle \le \alpha$ for all $a' \in P(G) \cap \mathbb{Z}^d$. This is equivalent to

$$\left\langle \left(\begin{smallmatrix} a \\ v(a) \end{smallmatrix}\right), \left(\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right) \right\rangle = \alpha, \quad \left\langle \left(\begin{smallmatrix} a' \\ v(a') \end{smallmatrix}\right), \left(\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right) \right\rangle \le \alpha, \tag{1}$$

so $\left(\begin{smallmatrix} a \\ v(a) \end{smallmatrix}\right)$ lies in a face of the upper convex hull that is supported by the above hyperplane, and hence a is a lifted point. Conversely, if $\left(\begin{smallmatrix} a \\ v(a) \end{smallmatrix}\right)$ lies in the upper convex hull, then there is a supporting hyperplane with normal vector $\left(\begin{smallmatrix} x \\ x_{d+1} \end{smallmatrix}\right) \in \mathbb{R}^{d+1}, x_{d+1} > 0$ and $\alpha \in \mathbb{R}$ such that

$$\left\langle \left(\begin{smallmatrix} a \\ v(a) \end{smallmatrix}\right), \left(\begin{smallmatrix} \frac{x}{x_{d+1}} \\ 1 \end{smallmatrix}\right) \right\rangle = \frac{\alpha}{x_{d+1}}, \quad \left\langle \left(\begin{smallmatrix} a' \\ v(a') \end{smallmatrix}\right), \left(\begin{smallmatrix} \frac{x}{x_{d+1}} \\ 1 \end{smallmatrix}\right) \right\rangle \le \frac{\alpha}{x_{d+1}},$$

for all $a' \in P(G) \cap \mathbb{Z}^d$, i.e. a maximizes $v(a) + \left\langle a, \frac{x}{x_{d+1}} \right\rangle$. This proves the first statement. Let $F \in \Delta_v$, i.e. $F = \operatorname{arg\,max}_{a \in P(G)\cap\mathbb{Z}^d} \{ v(a) + \langle\, a, x \,\rangle \}$ for some $x \in \mathbb{R}^d$. Then all points $a \in F$ are lifted points, lying in the supporting hyperplane given in (1). Further, all lattice points that lie in the intersection of $\operatorname{lift}(P(G))$ and this hyperplane are contained in F. In particular, all vertices of the face that is supported by this hyperplane are contained in F and therefore $\operatorname{conv}(F)$ is a face of the regular subdivision. Conversely, is F is a face of the regular subdivision, then $\operatorname{conv}(\operatorname{lift}(F))$ is a face of the upper convex hull of $\operatorname{conv}(\operatorname{lift}(P))$. Hence, there exists a supporting hyperplane with normal vector $\left(\begin{smallmatrix} x \\ x_{d+1} \end{smallmatrix}\right) \in \mathbb{R}^{d+1}, x_{d+1} > 0$ and $\alpha \in \mathbb{R}$ such that for all lifted points $a \in F \cap \mathbb{Z}^d$ holds

$$\left\langle \left(\begin{smallmatrix} a \\ v(a) \end{smallmatrix}\right), \left(\begin{smallmatrix} \frac{x}{x_{d+1}} \\ 1 \end{smallmatrix}\right) \right\rangle = \frac{\alpha}{x_{d+1}}$$

and for all other points $P(G) \cap Z^d$ holds

$$\left\langle \begin{pmatrix} a' \\ v(a') \end{pmatrix}, \begin{pmatrix} \frac{x}{x_{d+1}} \\ 1 \end{pmatrix} \right\rangle < \frac{\alpha}{x_{d+1}}.$$

Given the supporting hyperplanes of F with normal vectors $\left(\frac{x}{x_{d+1}} \right), \left(\frac{x'}{x'_{d+1}} \right)$, we have $D_v(x) = D_v(x')$. Thus, for each face there is a unique corresponding set in Δ_v consisting of the lifted points of this face. $\square$

Recall the aggregate valuation

$$V(a) = \max \left\{ \sum_{j=1}^{J} v^j(a^j) \,\middle|\, a^j \in P(G) \cap \mathbb{Z}^d,\ \sum_{j=1}^{J} a^j = a \right\}.$$

Lemma 2. *Let $\left\{ v^j \,\middle|\, j \in [J] \right\}$ be valuations and V the respective aggregate valuation. Then the tropical Laurent polynomial $f_V(x)$ is equal to the tropical product*

$$f_V(x) = f_{v^1} \odot \cdots \odot f_{v^J} = \sum_{j=1}^{J} f_{v^j}(x).$$

Proof. Let $a \in J \cdot (P \cap \mathbb{Z}^d)$. We denote the sets of possible decompositions of a by $\operatorname{dec}(a) = \left\{ \left\{ a^1, \dots, a^J \right\} \,\middle|\, a^j \in P(G) \cap \mathbb{Z}^d,\ \sum_{j=1}^{J} a^j = a \right\}$. By definition,

$$\begin{aligned} f_V(x) &= \max_{a \in J \cdot (P \cap \mathbb{Z}^d)} \{ V(a) + \langle a, x \rangle \} \\ &= \max_{a \in J \cdot (P \cap \mathbb{Z}^d)} \left\{ \max_{\{a^j \,|\, j \in [J]\} \in \operatorname{dec}(a)} \left\{ \sum_{j=1}^{J} v^j(a^j) \right\} + \langle a, x \rangle \right\} \\ &= \max_{a \in J \cdot (P \cap \mathbb{Z}^d)} \left\{ \max_{\{a^j \,|\, j \in [J]\} \in \operatorname{dec}(a)} \left\{ \sum_{j=1}^{J} \left(v^j(a^j) + \langle a^j, x \rangle \right) \right\} \right\} \end{aligned}$$

and

$$\sum_{j=1}^{J} f_{v^j}(x) = \sum_{j=1}^{J} \max_{a^j \in P(G) \cap \mathbb{Z}^d} \left\{ v^j(a^j) + \langle a^j, x \rangle \right\}.$$

Let $\hat{a} \in J \cdot (P(G) \cap \mathbb{Z}^d)$ and $\left\{ \hat{a}^1, \dots, \hat{a}^J \right\} \in \operatorname{dec}(\hat{a})$ such that

$$\sum_{j=1}^{J} \left(v^j(\hat{a}^j) + \langle \hat{a}^j, x \rangle \right) = \max_{a \in J \cdot (P \cap \mathbb{Z}^d)} \left\{ \max_{\{a^j \,|\, j \in [J]\} \in \operatorname{dec}(a)} \left\{ \sum_{j=1}^{J} \left(v^j(a^j) + \langle a^j, x \rangle \right) \right\} \right\}.$$

Note that for each $j \in [J]$, we have

$$v^j(\hat{a}^j) + \langle \hat{a}^j, x \rangle \leq \max_{a^j \in P(G) \cap \mathbb{Z}^d} \left\{ v^j(a^j) + \langle a^j, x \rangle \right\}$$

and thus

$$f_V(x) = \sum_{j=1}^{J} \left(v^j(\hat{a}^j) + \langle \hat{a}^j, x \rangle \right) \leq \sum_{j=1}^{J} \max_{a^j \in P(G) \cap \mathbb{Z}^d} \left\{ v^j(a^j) + \langle a^j, x \rangle \right\} = \sum_{j=1}^{J} f_{v^j}(x).$$

Conversely, let $\hat{a}^j \in P(G) \cap \mathbb{Z}^d,\ j \in [J]$ such that

$$\sum_{j=1}^{J} \left(v^j(\hat{a}^j) + \langle \hat{a}^j, x \rangle \right) = \sum_{j=1}^{J} \max_{a^j \in P(G) \cap \mathbb{Z}^d} \left\{ v^j(a^j) + \langle a^j, x \rangle \right\}$$

and let $\hat{a} = \sum_{j=1}^{J} \hat{a}^j$. Then $\hat{a} \in J \cdot (P(G) \cap \mathbb{Z}^d)$, $\left\{ \hat{a}^1, \ldots, \hat{a}^J \right\} \in \operatorname{dec}(\hat{a})$, and therefore

$$\sum_{j=1}^{J} (v^j(\hat{a}^j) + \langle \hat{a}^j, x \rangle) \leq \max_{a \in J \cdot (P \cap \mathbb{Z}^d)} \left\{ \max_{\{a^j \mid j \in [J]\} \in \operatorname{dec}(a)} \left\{ \sum_{j=1}^{J} v^j(a^j) \right\} + \langle a, x \rangle \right\} = f_V(x).$$

□

Lemma 3. *The aggregate demand set is the Minkowski sum of the agents' demand sets, i.e.*

$$D_V(p) = \sum_{j=1}^{J} D_{v^j}(p).$$

Proof. Let $\hat{a} \in D_V(-x)$, i.e. $f_V(x) = V(\hat{a}) + \langle \hat{a}, x \rangle$ and let $\left\{ \hat{a}^1, \ldots, \hat{a}^J \right\} \in \operatorname{dec}(\hat{a})$ such that $V(\hat{a}) + \langle \hat{a}, x \rangle = \sum_{j=1}^{J} \left(v^j(\hat{a}^j) + \langle \hat{a}^j, x \rangle \right)$, where $\operatorname{dec}(\hat{a})$ is defined as in Lemma 2. By definition, we have

$$f_{v^j}(x) = \max_{a^j \in P(G)} v^j(a^j) + \langle a^j, x \rangle \geq v^j(\hat{a}^j) + \langle \hat{a}^j, x \rangle$$

for each $j \in [J]$. Lemma 2 implies

$$\sum_{j=1}^{J} f_{v^j}(x) = f_V(x) = \sum_{j=1}^{J} \left(v^j(\hat{a}^j) + \langle \hat{a}^j, x \rangle \right),$$

and therefore $f_{v^j}(x) = v^j(\hat{a}^j) + \langle \hat{a}^j, x \rangle$, i.e. $\hat{a}^j \in D_{v^j}(-x)$. Thus, $\hat{a} \in \sum_{j=1}^J D_{v^j}(-x)$. Conversely, let $\hat{a}^j \in D_{v^j}(-x)$ and $\hat{a} = \sum_{j=1}^J \hat{a}^j$. Then, by Lemma 2, we have

$$\begin{aligned}\sum_{j=1}^J v^j(\hat{a}^j) + \langle \hat{a}, x \rangle &= \sum_{j=1}^J f_{v^j}(x)\\ &= f_V(x)\\ &= \max_{a \in J\cdot(P\cap\mathbb{Z}^d)} \left\{ \max_{\{a^j \mid j\in[J]\}\in \mathrm{dec}(a)} \left\{ \sum_{j=1}^J v^j(a^j) \right\} + \langle a, x \rangle \right\},\end{aligned}$$

so $\hat{a}$, together with the decomposition $\{\hat{a}^1, \dots, \hat{a}^J\} \in \mathrm{dec}(\hat{a})$, maximize the above expression. Thus,

$$\sum_{j=1}^J v^j(\hat{a}^j) = \max_{\{a^j \mid j\in[J]\}\in \mathrm{dec}(a)} \sum_{j=1}^J v^j(a^j) = V(\hat{a})$$

and $f_V(x) = V(\hat{a}) + \langle \hat{a}, x \rangle$, i.e. $\hat{a} \in D_V(-x)$. □

Lemma 4. *The partition of $\mathbb{R}^d$ induced by the aggregate valuation V is the common refinement of the partitions induced by $v^1, \dots, v^J$.*

Proof. Note that if $|D_{v^j}(x)| \geq 2$ for some $x \in \mathbb{R}^d,\ j \in [J]$, then

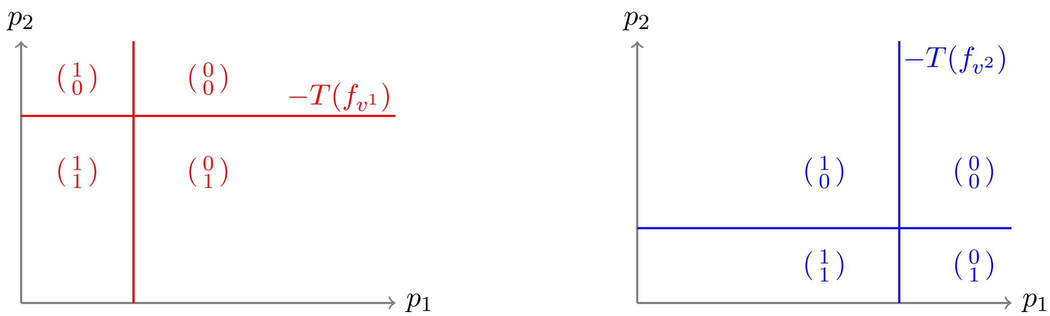
$$|D_V(x)| = \left| \sum_{j=1}^J D_{v^j}(x) \right| \geq 2.$$

Conversely, if $|D_{v^j}(x)| = 1$ for all $j \in [J]$, then $|D_V(x)| = 1$. Therefore, if $x \in T(f_{v^j})$, i.e. $|D_{v^j}(-x)| \geq 2$ for some $j \in [J]$, this is equivalent to $|D_V(-x)| \geq 2$, and hence $x \in T(f_V)$. Thus, we have $T(f_V) = \bigcup T(f_{v^j})$ and the partition of $\mathbb{R}^d$ which is induced by $T(f_V)$ agrees with the common refinement of the partitions induced by $T(f_{v^j}), j \in [J]$. □

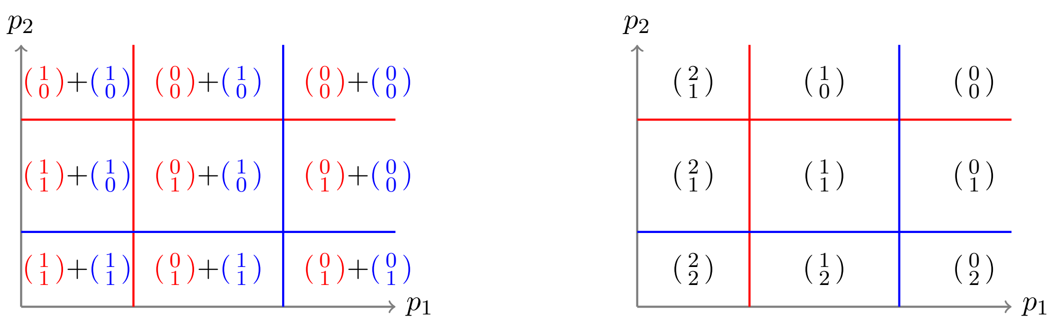
Example 4. Coming back to Example 1, the demand sets for a fixed price $p \in \mathbb{R}^2$ are

$$D_{v^1}(p) = \operatorname*{arg\,max}_{a\in P(G)\cap\mathbb{Z}^2} \left\{ \left\langle \begin{pmatrix} 3-p_1 \\ 5-p_2 \end{pmatrix}, \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \right\rangle \right\}, \quad D_{v^2}(p) = \operatorname*{arg\,max}_{a\in P(G)\cap\mathbb{Z}^2} \left\{ \left\langle \begin{pmatrix} 7-p_1 \\ 2-p_2 \end{pmatrix}, \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \right\rangle \right\}.$$

Figure 2a shows the price space $\mathbb{R}^2$ and the respective partitions into relatively open polyhedra, according to the agents' demand sets. For any two prices $p, p' \in R$ in such a region R holds $D_v(p) = D_v(p')$. The $d-1$-skeleton of the subdivision is given by the tropical hypersurface defined by the corresponding tropical polynomial. Each of the d-dimensional regions correspond to a demand set of size 1. Intersections of closures of regions correspond to unions of the respective demand sets. By Lemma 4, the partition of $\mathbb{R}^2$ induced by the



(A). The partitions of the price space $\mathbb{R}^2$ which are induced by v^1 (red) and v^2 (blue) respectively. The regions, i.e. the relatively open polyhedra, are in bijection with the demand sets. The maximal regions are labeled by the unique element that is contained in the respective demand set.



(B). Each of the regions of the common refinement is in bijection with a face of the regular subdivision of $2P(G)$ that is induced by the aggregate valuation V. The maximal regions are labeled by the unique element that is contained in the respective demand set.

FIGURE 2. The partitions of the price space $\mathbb{R}^2$ induced by the valuations given in Example 4.

aggregate demand set is the common refinement of the partitions given in Figure 2a. Let $p \in \mathbb{R}^2$ be a price in the relative interior of a 2-dimensional polyhedron of this common refinement. Then p is in the relative interiors of unique 2-dimensional cells of the partitions induced by v^1 and v^2 respectively. Thus, $D_{v^1}(p) = \left\{a^1\right\}, D_{v^2}(p) = \left\{a^2\right\}$ for some $a^1, a^2 \in P(G) \cap \mathbb{Z}^d$ and a competitive equilibrium, given the price p, can only be achieved at $a = a^1 + a^2$. On the other hand, a competitive equilibrium can be achieved at any $a \in \mathbb{Z}^2$ that appear as a label of a maximal region in Figure 2b, with the decomposition as indicated in the picture.

2.3 Conditions on the Polytope

Restricting again to graphical valuations, we can now summarize the above discussion to the following Lemma. The statement and proof is due to Ngoc Mai Tran (private communication).

Lemma 5. *A competitive equilibrium always exists at* $a \in \mathbb{R}^d$ *if and only if for any faces* $F_1, \ldots, F_J$ *of* $P(G)$ *holds: If* $a \in \sum_{j=1}^J F_j$ *then* $a \in \sum_{j=1}^J \mathrm{vert}(F_j)$.

Proof. First note, that by Lemma 1, each face F_j of the regular subdivision which is induced by a valuation v^j corresponds to a set in Δ_{v^j}, i.e. there exists a price $p^j \in \mathbb{R}^d$ such that $F_j = \mathrm{conv}(D_{v^j}(p^j))$. The valuations v^j are linear functions. Therefore, the regular subdivision induced by v^j on $P(G)$ is trivial, since all points satisfy $\left\langle \binom{w^j}{-1}, \binom{a}{v^j(a)} \right\rangle = 0$. Further, all lattice points of $P(G)$ are lifted, since all these points are vertices of $P(G)$ and vertices are always lifted points. Thus, $F_j \cap \mathbb{Z}^d = \mathrm{vert}(F_j) = D_{v^j}(p^j)$ and the set of lifted points of $P(G)$ is the set of vertices $\mathrm{vert}(P(G))$. Further, each face F of the regular subdivision of $JP(G)$ induced by the aggregate valuation V corresponds to a set $D_V(p)$ for some $p \in \mathbb{R}^d$, where $D_V(p)$ is the set of all lifted points in F. Thus, by Lemma 3 the set of lifted points of $JP(G)$ is

$$\bigcup_{p \in \mathbb{R}^d} D_V(p) = \bigcup_{p \in \mathbb{R}^d} \left(\sum_{j=1}^J D_{v^j}(p)\right) \subseteq \sum_{j=1}^J \mathrm{vert}(P(G)).$$

We now show the following equivalent statement:
A competitive equilibrium does not exist at $a \in \mathbb{R}^d$ *for some set of valuations if and only if there exist faces* $F_1, \ldots, F_J$ *such that* $a \in \sum_{j=1}^J F_j$ *and* $a \notin \sum_{j=1}^J \mathrm{vert}(F_j)$.

Suppose there is a set of valuations $\left\{v^j \,\middle|\, j \in [J]\right\}$ and $a \in JP(G)$ such that for all $p \in \mathbb{R}^d$ holds $a \notin D_V(p) = \sum_{j=1}^J D_{v^j}(p)$. Since a is contained in $JP(G)$, a lies in some face of the regular subdivision of $JP(G)$ induced by V. These faces are in bijection with the distinct sets $D_V(p)$, so there exists some $p \in \mathbb{R}^d$ such that $a \in \mathrm{conv}(D_V(p))$. The assumption $a \notin D_V(p)$ implies that a is not a lifted point. Note that, since Minkowski summation and the operator of forming convex hulls commute,

$$
\begin{aligned}
a \in \operatorname{conv}(D_V(p)) &= \operatorname{conv}(\sum_{j=1}^{J} D_{v^j}(p)) \\
&= \sum_{j=1}^{J} \operatorname{conv}(D_{v^j}(p)) \\
&= \sum_{j=1}^{J} F_j
\end{aligned}
$$

for some faces $F_1, \ldots, F_J$ of $P(G)$. Further, since $P(G)$ is a 0/1-polytope, for each $j \in [J]$ the vertices $\operatorname{vert}(F_j)$ are precisely the lifted points of F in the trivial subdivision induced by v^j, i.e. $\operatorname{vert}(F_j) = D_{v^j}(p)$. Therefore

$$
\sum_{j=1}^{J} \operatorname{vert}(F_j) = \sum_{j=1}^{J} D_{v^j}(p) = D_V(p),
$$

so $\sum_{j=1}^{J} \operatorname{vert}(F_j)$ is a set of lifted points in the subdivision induced by V. By assumption, a is not a lifted point, and thus $a \notin \sum_{j=1}^{J} \operatorname{vert}(F_j)$.

Conversely, suppose $a \in \sum_{j=1}^{J} F_j$ but $a \notin \sum_{j=1}^{J}(\operatorname{vert}(F_j))$. We show that there are graphical valuations $u^1, \ldots, u^J$ such that for any price $p \in \mathbb{R}^d$ we have $a \notin D_U(p)$. Let $\left\{ v^j(a) = \langle w^j, a \rangle \,\middle|\, j \in [J] \right\}$ be a fixed set of graphical valuations and $p \in \mathbb{R}^d$. For each $j \in [J]$ there exists some $p^j \in \mathbb{R}^d$ such that $F_j = \operatorname{conv}(D_{v^j}(p^j))$ and $\operatorname{vert} F_j = D_{v^j}(p^j)$. Thus, the assumptions are equivalent to $a \in \sum_{j=1}^{J} \operatorname{conv}(D_{v^j}(p^j))$ and $a \notin \sum_{j=1}^{J} D_{v^j}(p^j)$. We construct valuations

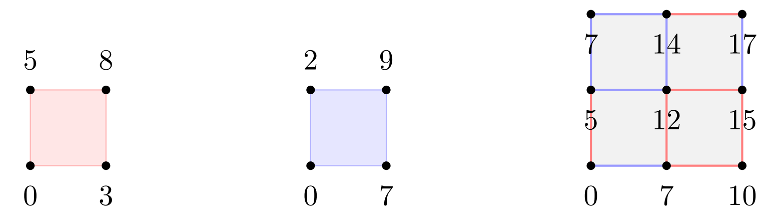
$$
u^j(a) = v^j(a) + \langle p - p^j, a \rangle = \langle w^j + p - p^j, a \rangle
$$

such that

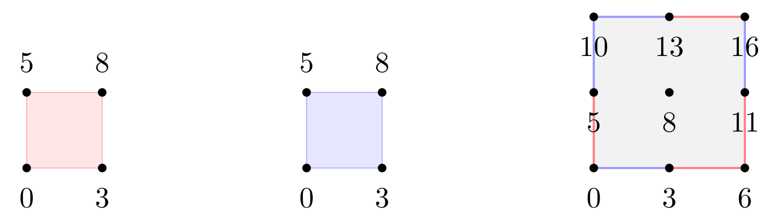
$$
D_{u^j}(p) = \operatorname*{arg\,max}_{a \in P(G)} \left\{ \langle w^j + p - p^j, a \rangle - \langle p, a \rangle \right\} = D_{v^j}(p^j).
$$

This is a set of valuations under which, for a particular p, $a \in \sum_{j=1}^{J} \operatorname{conv}(D_{v^j}(p))$ and $a \notin \sum_{j=1}^{J} D_{u^j}(p) = D_U(p)$. Therefore, there exists a face F corresponding to $D_U(p)$ in the regular subdivision of $JP(G)$ induced by the aggregate valuation U such that $a \in F$ but a is not a lifted point. In particular, since the set of lifted points in $JP(G)$ is the union of the demand sets $\bigcup_{p \in \mathbb{R}^d} D_U(p)$, it follows that $a \notin D_U(p)$ for any price $p \in \mathbb{R}^d$.

□



(A). The polytope $P(G)$ with valuations as given in Example 1 and the respective aggregate valuation on $2P(G)$. All lattice points of $2P(G)$ are vertices in the regular subdivision induced by the aggregate valuation.



(B). If the valuations of the two agents are equal, then the interior point of $2P(G)$ is not a vertex of the regular subdivision. However, it is a lifted point.

FIGURE 3

Example 5. Coming back to Examples 1 and 4, note that $P(G)$ is the Minkowski sum of the two unit vectors spanning $\mathbb{Z}^2$. Thus, for each lattice point in $2P(G)$ the condition of Lemma 5 holds and hence a competitive equilibrium exists. In fact, for the given valuations, all lattice points in $2P(G)$ are vertices of the regular subdivision (Figure 3a).
Note that it is not a necessary condition that a point is a vertex in the regular subdivision in order to guarantee a competitive equilibrium. This occurs if both valuations are equal (Figure 3b). In this example, the interior point $\binom{1}{1}$ is not a vertex of the regular subdivision, but it is a lifted point. Lemma 5 implies there is no set of valuations no point $a \in 2P(G)$ such that a is not a lifted point, since a competitive equilibrium always exists at all $a \in 2P(G)$.

3 Lattice Polytopes

Making use of Lemma 5, we can restrict to the analysis of lattice points in Minkowski sums of faces in order to show the existence of a competitive equilibrium. In the remaining of this thesis, we consider graphical valuations and anonymous graphical pricing having the complete graph K_n as the underlying value graph.

3.1 The Correlation Polytope

The polytope $P(K_n)$ is generally known as the *correlation polytope* or *boolean quadric polytope* and is isomorphic to the cut polytope [8]. It has been widely studied, yet, the hyperplane description of this polytope remains unknown. We thus make use of a linear relaxation given in [6].

Definition 2 ([8, Definition 20])**.** The n-th *correlation polytope* is the convex hull of all symmetric $n \times n$ 0/1-matrices of rank 1, i.e.

$$P(K_n) = \text{conv}\{vv^t | v \in \{0,1\}^n\}.$$

Let $x = vv^t$ for some $v \in \{0,1\}^n$. Note that $x_{ii} = v_i^2 = v_i$ and $x_{ij} = x_{ji} = v_i v_j$, so $x_{ij} = 1$ if and only if $v_i = v_j = 1$. Thus, we can indeed interpret x as the characteristic vector of a complete graph G for some vertex set $V(G) \subset [n]$, where $x_{ii} = 1$ if and only if $i \in V(G)$. As before, we write x_i for x_{ii} and interpret symmetric matrices as points in $\mathbb{R}^d$, $d = \binom{n}{2} + n$.

Consider the following sets of inequalities:

(i') $x_{ij} \geq 0$

(ii') $x_i - x_{ij} \geq 0$

(iii') $x_i + x_j - x_{ij} \leq 1$

(iv') $x_i + x_{jk} - x_{ij} - x_{ik} \geq 0$

(v') $x_i + x_j + x_k - x_{ij} - x_{ik} - x_{jk} \leq 1$

(i) $x_{ij} \geq 0$

(ii) $x_i - x_{ij} \geq 0$

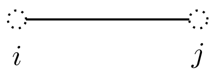
(iii) $x_i + x_j - x_{ij} \leq J$

(iv) $x_i + x_{jk} - x_{ij} - x_{ik} \geq 0$

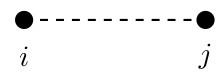
(v) $x_i + x_j + x_k - x_{ij} - x_{ik} - x_{jk} \leq J$

Since every vertex of the correlation polytope $P(K_n)$ is a characteristic vector of a complete graph, every point $x \in P(K_n)$ satisfies inequalities (i')-(v'). Conversely, a vector $x \in \{0,1\}^d$ satisfying inequalities (i') and (ii') can always be interpreted as the characteristic vector of a graph, for which the remaining inequalities define specific properties (Figure 4). By convexity, $JP(K_n)$ satisfies (i)-(v). For each $x \in \mathbb{R}^d$ satisfying the inequalities (i) and (ii) holds

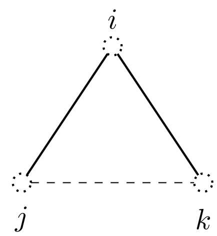
$$x_i \geq x_{ij} \geq 0.$$



(A). If the edge ij is present in the graph that is determined by x, i.e. $x_{ij} = 1$, then inequality (ii') or (ii) implies that $x_i = x_j = 1$ and hence i and j are vertices present in the graph.



(B). If i and j are vertices in the graph, then inequality (iii') implies that $x_{ij} = 1$, i.e. the edge ij is present.



(C). If the edges ij and ik are present in the graph, then inequality (iv') or (iv) implies that the vertex i and the edge jk are present. Inequality (ii') or (ii) then implies that j and k are also vertices of the graph. Therefore, the graph does not contain a path of length 2 as an induced subgraph.

FIGURE 4. We can interpret a $x \in \{0,1\}^d$ satisfying inequalities (i')-(v') as the characteristic vector of a graph with specific properties.

Inequality (iii) additionally implies

$$J \geq x_i + x_j - x_{ij} \geq x_j.$$

Thus, every $x \in \mathbb{R}^d$ satisfying inequalities (i)-(iii) is contained in the cube $[0, J]^d$, so the polyhedron defined by these inequalities is bounded. In fact, adding the constraint $x \in \mathbb{Z}^d$ to inequalities (i')-(iii') yields the vertices of correlation polytope, and for $n \leq 3$ the inequalities (i')-(v') suffice to describe the polytope completely ([6, Section 2]).

3.2 A Competitive Equilibrium Always Exists

In this section, we show the following theorem:

Theorem 6. *Let $a^* \in \mathbb{Z}^n \cap [0, J]^n$. If the underlying graph of all valuations is the complete graph, then $A^* \neq \emptyset$ and a competitive equilibrium at a^* always exists.*

That is, given $a^* \in \mathbb{Z}^n \cap [0, J]^n$, we show that for any set of valuations there exists a price $p \in \mathbb{R}^d$ and some $a \in A^*$ such that $a \in D_V(p)$. Note that if $a^* \in \mathbb{Z}^n_{\geq 00}$ and $a^* \notin [0, J]^n$, then there is a type i of goods of which the auctioneer wants to sell $a_i^* > J$ items. Since we assume that each of the J agents wants to buy at most one item of each type, the auctioneer can never sell all goods of the bundle a^* in a single auction, so a competitive equilibrium cannot exist.

The following proof consists of three steps. By Lemma 5, it suffices to show that there exists a point $a \in A^*$ such that for all faces $F_1, \ldots, F_J$ of $P(K_n)$ such that $a \in \sum_{j=1}^J F_j$ there exist vertices χ^j of F_j such that $x = \sum_{j=1}^J \chi^j$. First, in Proposition 7, we construct a particular $a \in A^*$ for which we show that it is the sum of J characteristic vectors of complete graphs, i.e. $a \in \sum_{j=1}^J \operatorname{vert}(P(K_n)) = J \cdot (P(K_n) \cap \mathbb{Z}^d)$. Second, assuming that a lies in the Minkowski sum $\sum_{j=1}^J F_j$, we show that whenever we write a as the sum of points $\sum_{p \in V(F_j)} \lambda_p p$ contained in the convex hulls of the faces $F_j, j \in [J]$, then the set of vertices involved in this representation equal the set of characteristic vectors that appear in the first step. This implies that a can be written as the sum of some vertices $\chi^j, j \in [J]$ of the faces $F_j, j \in [J]$. Finally, in Lemma 8 we show that we can find a labeling of the faces such that $\chi^j \in F_j$.

Proposition 7. *Let $a^* \in \mathbb{Z}^n \cap [0, J]^n$. Then there exists a point $a \in A^*$ such that the following holds: For all faces $F_1, \ldots, F_J$ of $P(K_n)$ containing $a \in \sum_{j=1}^J F_j$ in their Minkowski sum, there exist vertices χ^j of F_j such that $x = \sum_{j=1}^J \chi^j$.*

Proof. We define a as follows:

$$a_i = a_i^* \qquad a_{ij} = \min\{a_i^*, a_j^*\}.$$

We show that a is the sum of characteristic vectors of complete graphs $G_J \subseteq \cdots \subseteq G_1$. Let $t_0 = 0$ and $t_1 < t_2 < \cdots < t_s$ be the non-zero values of a^*, i.e. $\{a_i^* \mid i \in [n]\} \setminus \{0\} = \{t_l \mid l \in [s]\}$. Let χ^{t_l} be given by

$$\chi_i^{t_l} = \begin{cases} 1, & \text{if } a_i \geq t_l \\ 0, & \text{otherwise} \end{cases} \quad \text{for } i \in [n], \qquad \chi_{ij}^{t_l} = \chi_i^{t_l} \chi_j^{t_l} \quad \text{for } ij \in \binom{[n]}{2}.$$

Then χ^{t_l} is the characteristic vector of a complete graph G_{t_l} and for a non-zero $a_i = t_k$ we have

$$a_i = t_k = \sum_{l=1}^k t_l - t_{l-1} = \sum_{l=1}^k (t_l - t_{l-1}) \chi_i^{t_l} = \sum_{l=1}^s (t_l - t_{l-1}) \chi_i^{t_l}.$$

An analogous statement holds for $a_{ij} = \min\{a_i, a_j\}$ and thus

$$a = \sum_{l=1}^s (t_l - t_{l-1}) \chi^{t_l}$$

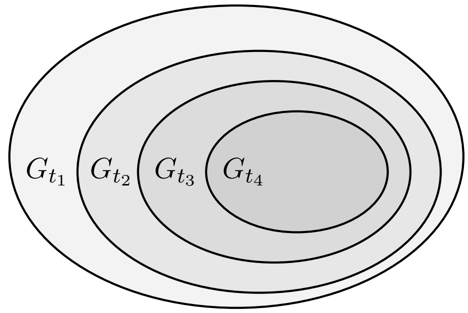


FIGURE 5. The vector a is the sum of characteristic vectors of nested complete graphs $G_{t_s} \subsetneq \cdots \subsetneq G_{t_1}$.

for some graphs $G_{t_s} \subsetneq \cdots \subsetneq G_{t_1}$. Taking $t_l - t_{l-1}$ copies of G_{t_l} and $J - t_s$ copies of the empty graph, we can write

$$a = \sum_{m=1}^{J} \chi^m$$

for graphs $G_J \subseteq \cdots \subseteq G_1$, where $G_m = G_{t_l}$ for any $m \in \mathbb{N}$ such that $t_l + 1 \leq m \leq t_l$ and χ^m denotes the characteristic vector of G_m. Thus, a is the sum of J vertices of $P(K_n)$ and is therefore contained in $A^* \subseteq JP(K_n)$.

Suppose $a \in (F_1 + \cdots + F_J)$, i.e.

$$a = \sum_{m=1}^{J} \sum_{p \in V(F_m)} \lambda_p p, \qquad \sum_{p \in V(F_m)} \lambda_p = 1,$$

where $V(F_m)$ is a subset of vertices of the face F_m and $\lambda_p > 0$ for all $p \in V(F_m)$. Note that all vertices of the polytope $P(K_n)$ satisfy inequalities (i')-(v'). We show that $\bigcup_{m=1}^{J} V(F_m) = \{\chi^m | m \in [J]\}$.

Let $\hat{p} \in V(F_m)$ be a non-zero vector and $i \in [n]$ such that $a_i = \min_{j \in [n]} \{a_j \,|\, \hat{p}_j = 1\}$. Note that since $\lambda_{\hat{p}} > 0$ and $\hat{p}_i = 1$, we have $a_i > 0$, so $a_i = t_k$ for some non-zero value. We show that $\hat{p} = \chi^{t_k}$. Let $j \in [n]$. If $a_j < t_k$, then $\hat{p}_j = 0$ by the choice of i and thus $\hat{p}_{ij} = 0$ by (iii'). If $a_j \geq t_k$, then $a_{ij} = a_i = t_k$ by the definition of a and so $\hat{p}_{ij} = \hat{p}_i = 1$, since otherwise (by (ii'))

$$t_k = a_{ij} = \sum_{m=1}^{J} \sum_{p \in V(F_m)} \lambda_p p_{ij} < \sum_{m=1}^{J} \sum_{p \in V(F_m)} \lambda_p p_i = a_i = t_k. \tag{2}$$

Hence, we have $\hat{p}_j = 1$ and so $\hat{p} = \chi^{t_k}$. Let $\hat{p} = \mathbf{0}$ be the zero vector and $i \in [n]$ such that $a_i = t_s$ attains the maximum value. Since $\hat{p}_i = 0$, $\lambda_{\hat{p}} > 0$ and $p_i \in \{0,1\}$ for all $p \in \bigcup_{m=1}^{J} V(F_m)$, we have

$$t_s = a_i = \sum_{m=1}^{J} \sum_{p \in V(F_m)} \lambda_p p_i < \sum_{m=1}^{J} \sum_{p \in V(F_m)} \lambda_p = \sum_{m=1}^{J} 1 = J \tag{3}$$

and therefore the number of characteristic vectors of the empty graph is $J - t_s > 0$, i.e. $\hat{p} = \mathbf{0} \in \{\chi^m \,|\, m \in [J]\}$. This implies $\bigcup_{m=1}^{J} V(F_m) \subseteq \{\chi^m \,|\, m \in [J]\}$.

Conversely, let t_k be a non-zero value of a. We show that there exists some $p \in \bigcup_{m=1}^{J} V(F_m)$ such that $p = \chi^{t_k}$. Let $i, j \in [n]$ such that $a_i = t_k, a_j = t_{k-1}$. Since $a_i > a_j$ and $\bigcup_{m=1}^{J} V(F_m) \subseteq \{\chi^m \,|\, m \in [J]\}$, this implies

$$\left\{\chi^{t_l} | t_l > t_{k-1}, i \in V(G_{t_l})\right\} \cap \bigcup_{m=1}^{J} V(F_m) \neq \emptyset.$$

Note that $i \in V(G_{t_l})$ implies that $a_i \geq t_l$. Thus, since otherwise $a_i > t_k$,

$$\left\{\chi^{t_l} | t_l = t_k, i \in V(G_{t_l})\right\} \cap \bigcup_{m=1}^{J} V(F_m) \neq \emptyset.$$

Suppose $\mathbf{0} \in \{\chi^m \,|\, m \in [J]\}$, i.e. there exists a characteristic vector of the empty graph and so $t_s < J$. Let $i \in [n]$ such that $a_i = t_s$. inequality (3) implies that there exists a $\hat{p} \in \bigcup_{m=1}^{J} V(F_m)$ such that $\hat{p}_i = 0$. By the above, we know that $\hat{p} \in \{\chi^m \,|\, m \in [J]\}$, which are characteristic vectors of nested graphs. In particular, all non-empty graphs contain the graph G_{t_s} and hence the vertex i. Since $\hat{p}$ is a characteristic vector of one of the nested graphs graphs, it must be the empty graph, i.e. $\hat{p} = \mathbf{0}$.

It remains to show that we can find a labeling of the faces such that $\chi^m \in F_m$ for all $m \in [J]$. This is done in the following Lemma 8.

□

Lemma 8. *Suppose $a = \sum_{m=0}^{J} \chi^m$, where χ^m is the characteristic vector of a complete graph G_m. Further, let a be contained in the Minkowski sum of faces $F_i, i \in [J]$, i.e.*

$$a = \sum_{m=1}^{J} \sum_{p \in V(F_m)} \lambda_p p, \quad \sum_{p \in V(F_m)} \lambda_p = 1$$

where $V(F_m)$ is a subset of vertices of the face F_m and $\lambda_p > 0$, such that $\bigcup_{m=1}^{J} V(F_m) = \{\chi^m \,|\, m \in [J]\}$. Then there is a labeling of the faces such that $\chi^m \in F_m$.

Proof. We seek to find a perfect matching in the bipartite graph B having vertices $V(B) = \{F_m | m \in [J]\} \sqcup \left\{(\chi^l, l) | l \in [J]\right\}$ where F_m and (χ^l, l) share an edge whenever $\chi^l \in V(F_m)$. In this graph, the neighborhood $N(X)$ of a subset $X \subseteq \left\{(\chi^l, l) | l \in [J]\right\}$ is the set of all faces that contain at least one of the respective vertices, i.e.

$$N(X) = \left\{F \in \mathcal{F} | \exists (\chi^l, l) \in X \text{ s.t } \chi^l \in F\right\},$$

where $\mathcal{F} = \{F_m \,|\, m \in [J]\}$. Let $l \in [m]$ and m_l denote the multiplicity of χ^l in the above representation, i.e.

$$m_l = \left|\left\{m \in [J] \,\middle|\, \chi^m = \chi^l\right\}\right|.$$

Let m_0 denote the number of empty graphs in this representation. By assumption,

$$\sum_{m=1}^{J} \chi^m = \sum_{m=1}^{J} \sum_{p \in V(F_m)} \lambda_p p \quad \text{and} \quad \bigcup_{m=1}^{J} V(F_m) = \{\chi^m \,|\, m \in [J]\}.$$

Therefore,

$$m_l \chi^l = \sum_{\substack{m \in [J] \\ \chi^m = \chi^l}} \chi^m = \sum_{m=1}^{J} \sum_{\substack{p \in V(F_m) \\ p = \chi^l}} \lambda_p p = \sum_{\substack{F \in \mathcal{F} \\ \chi^l \in V(F)}} \sum_{\substack{p \in V(F) \\ p = \chi^l}} \lambda_p p.$$

Let $\bar{X}$ denote the set of vectors of X with all multiplicities, i.e.

$$\bar{X} = \left\{(\chi^m, m) \,\middle|\, m \in [J], \exists (\chi^l, l) \in X \text{ s.t. } \chi^l = \chi^m\right\}.$$

Note that $N(X) = N(\bar{X})$. Let $k = \left|\left\{\chi^l | (\chi^l, l) \in \bar{X}\right\}\right|$ and $l_1, \ldots, l_k$ be the unique indices of the elements in $\left\{\chi^l | (\chi^l, l) \in \bar{X}\right\}$. For any $j \in [k]$ such that $\chi^{l_j} \neq \mathbf{0}$, we fix some vertex i_j present in the graph G_{l_j}. If X contains the zero vector, let $q_0 = 1$, otherwise $q_0 = 0$. Then

$$\begin{aligned}
|X| \leq \left|\bar{X}\right| &= \sum_{\substack{j \in [k] \\ \chi^{l_j} \neq \mathbf{0}}} \chi^{l_j}_{i_j} + m_0 q_0 \\
&= \sum_{\substack{j \in [k] \\ \chi^{l_j} \neq \mathbf{0}}} \sum_{\substack{F \in \mathcal{F} \\ \chi^{l_j} \in V(F)}} \sum_{\substack{p \in V(F) \\ p = \chi^{l_j}}} \lambda_p p_{i_j} + \sum_{\substack{F \in \mathcal{F} \\ \mathbf{0} \in V(F)}} \sum_{\substack{p \in V(F) \\ p = \mathbf{0}}} \lambda_p q_0 \\
&= \sum_{j \in [k]} \sum_{\substack{F \in \mathcal{F} \\ \chi^{l_j} \in V(F)}} \sum_{\substack{p \in V(F) \\ p = \chi^{l_j}}} \lambda_p
\end{aligned}$$

$$\leq \sum_{F\in N(X)} \sum_{p\in V(F)} \lambda_p$$
$$= \sum_{F\in N(X)} 1 = |N(X)|.$$

Thus, by Hall's Theorem, there exists a perfect matching in the bipartite graph. □

Note that if $a^* \in \{0,1\}^n$, then the procedure given in the proof of Proposition 7 picks the vector $a \in A^*$ that corresponds to a complete graph and gives a distribution of the goods in which one agent is assigned all items, while the others get nothing. However, there are more possibilities to split these goods and guarantee the existence of a competitive equilibrium at a^*, as given in Propositions 10 and 12.

3.3 The Integer Decomposition Property

Proposition 7 raises the question of the existence of more general results. The most general question is whether for all $J \in \mathbb{N}, a \in JP(K_n) \cap \mathbb{Z}^d$ and all faces $F_1, \ldots, F_J$ such that $a \in \sum_{j=1}^J F_j$ we have $a \in \sum_{j=1}^J \operatorname{vert}(F_j)$. Since we can choose $F_j = P$ for all $j \in [J]$, this is strictly stronger than the Integer Decomposition Property.

Definition 3. A d-dimensional lattice polytope $P \subset \mathbb{R}^d$ is said to posess the *Integer Decomposition Property (IDP)* if for every $J \in \mathbb{N}$ and for every lattice point $a \in JP \cap \mathbb{Z}^d$ there exist $v_1, \ldots, v_J \in P \cap \mathbb{Z}^d$ such that $a = \sum_{j=1}^J v_j$.

It is known that the correlation polytope $P(K_n)$ is not IDP. In fact, for $n = 4$, $J = 4$, the point

$$\begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}$$

is the sum of the midpoints of the four edges

$$\operatorname{conv}\left(\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right),$$
$$\operatorname{conv}\left(\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right),$$

$$\text{conv}\left(\begin{pmatrix}0&0&0&0\\0&1&1&1\\0&1&1&1\\0&1&1&1\end{pmatrix},\begin{pmatrix}1&0&1&1\\0&0&0&0\\1&0&1&1\\1&0&1&1\end{pmatrix}\right),$$

$$\text{conv}\left(\begin{pmatrix}1&1&0&0\\1&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}1&1&0&1\\1&1&0&1\\0&0&0&0\\1&1&0&1\end{pmatrix}\right),$$

but cannot be written as the sum of any 4 lattice points of $P(K_4)$. This can be verified by computing the Hilbert basis of the cone over $P(K_4)$. The given point is an element of the Hilbert basis at height 4. Thus, the strongest generalization of the conditions of Lemma 5 does not hold. Still, there are weaker generalizations, which are given in Propositions 10 and 12. The proofs of these statements are analogous to the proof of the previous Proposition 7.

3.4 Competitive Equilibrium at Constant-Size Bundles

Let $a^* \in \mathbb{Z}^n \cap [0, J]^n$ be a bundle of goods such that $a^* \in \{0, r\}^n$, $r \leq J$. This represents a set of items in which, for each type of goods, the auctioneer wants to either sell exactly r items or none. In the following, we show that at all $a \in A^*$ such that $a \in \{0, r\}^d$, a competitive equilibrium always exists. This gives a more general result than Proposition 7 in terms of possible allocations of these goods, since the previous procedure splits the items in a way such that r agents are assigned one of each of the items which are offered bz the auctioneer, while $J - r$ agents get nothing. The procedure given in the proof of Proposition 10 allows other distributions as well.

Again, the proof consists of three steps which are similar to the ones used in the proof of Proposition 7. First, in Lemma 9, we show that any $a \in A^* \cap \{0, r\}^d$ is the sum of characteristic vectors of complete graphs. In Proposition 10, we show that whenever we write a as the sum of points $\sum_{p \in V(F_j)} \lambda_p p$ contained in the convex hulls of faces F_j, $j \in [J]$, then the set of vertices involved in this representation equals the set of characteristic vectors that appear in the first step. Finally, we can apply Lemma 8 to show that we can find a labeling of the faces such that $\chi^j \in F_j$ for all $j \in [J]$.

Lemma 9. *Let $a \in \{0, r\}^d$ satisfying inequalities (i), (ii) and (iv) for some $J \geq r$. Then a is the sum of characteristic vectors of pairwise disjoint complete graphs $G_1 \ldots G_s$. More precisely,*

$$a = \sum_{t=1}^{s} r\chi^t,$$

where χ^t denotes the characteristic vector G_t and $s \leq n$.

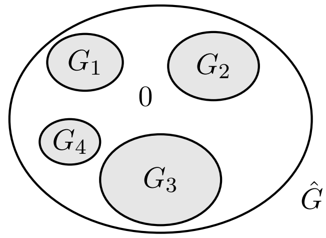


FIGURE 6. The vector a is the sum of r copies of characteristic vectors of pairwise disjoint complete graphs $G_1, \ldots, G_s$.

Proof. Since a satisfies inequalities (i) and (ii), $\frac{1}{r}a \in \{0,1\}^d$ satisfies inequalities (i') and (ii'). We can therefore interpret $\frac{1}{r}a$ as the characteristic vector of a (non-complete) graph $\hat{G}$, i.e $\frac{1}{r}a = \chi_{\hat{G}}$ and thus $a = r\chi_{\hat{G}}$ where $V(\hat{G}) \subseteq [n]$.

Inequality (iv) forbids an induced subgraph consisting of a path of length two (and not a triangle), since otherwise this would imply a set $\{i, j, k\}$ of indices such that

$$a_i = a_j = a_k = a_{ij} = a_{ik} = r, \quad a_{jk} = 0,$$

and thus

$$a_i + a_{jk} - a_{ij} - a_{ik} = -r.$$

Whenever two vertices are connected via a path of length two, they must be adjacent. Inductively follows, that whenever two vertices are connected via a path of arbitrary length, they must be adjacent as well. Therefore, $\hat{G}$ is the disjoint union of non-empty complete graphs $G_1, \ldots G_s$ and

$$a = \sum_{t=1}^{s} r\chi^t,$$

where χ^t denotes the characteristic vector of G_t. Further,

$$n \geq \left|V(\hat{G})\right| = \sum_{t=1}^{s} |V(G_t)| \geq \sum_{t=1}^{s} 1 = s.$$

□

Proposition 10. *Let $a \in \{0, r\}^d$. Then for all faces $F_1, \ldots, F_J$ of $P(K_n)$ containing $a \in \sum_{j=1}^{J} F_j$ in their Minkowski sum, there exist vertices χ^j of F_j such that $x = \sum_{j=1}^{J} \chi^j$.*

Proof. First, note that if there exist faces $F_1, \ldots, F_J$ of $P(K_n)$ such that $a \in \sum_{j=1}^{J} F_j$, then in particular $a \in JP(K_n)$. Thus, the inequalities (i)-(v) hold for a and by Lemma 9 we have

$$a = \sum_{t=1}^{s} r\chi^t,$$

where χ^t is the characteristic vector of a complete graph G_t for each $t \in [s]$ and $s \leq n$. Consider the following inequalities

$$\text{(vi')} \quad \sum_{i\in[s]} x_{k_i} - \sum_{ij\in\binom{[s]}{2}} x_{k_ik_j} \leq 1 \qquad \text{(vi)} \quad \sum_{i\in[s]} x_{k_i} - \sum_{ij\in\binom{[s]}{2}} x_{k_ik_j} \leq J.$$

Note that for any $s \leq n$ and $\{k_i \,|\, i \in [s]\} \in \binom{[n]}{s}$, each vertex of $P(K_n)$ satisfies the inequality (vi') and so $a \in JP(K_n)$ satisfies (vi). Choose $\{k_i \,|\, i \in [s]\}$ such that $k_i \in V(G_i)$ for each $i \in [s]$. Then $a_{k_i} = r$ and $a_{k_ik_j} = 0$ for all $i, j \in [s]$, and hence

$$\sum_{i\in[s]} a_{k_i} - \sum_{ij\in\binom{[s]}{2}} a_{k_ik_j} = s \cdot r - 0 \leq J.$$

We can thus write $a = \sum_{m=1}^{J} \hat{\chi}^m$, where in this representation we have r copies of χ^t for each $t \in [s]$, and $J - s \cdot r$ copies of the characteristic vector $\chi^0 = \mathbf{0}$ of the empty graph G_0. By assumption, $a \in (F_1 + \cdots + F_J)$, i.e.

$$a = \sum_{m=1}^{J} \sum_{p\in V(F_m)} \lambda_p p, \qquad \sum_{p\in V(F_m)} \lambda_p = 1,$$

where $V(F_m)$ is a subset of vertices of the face F_m and $\lambda_p > 0$ for all $p \in V(F_m)$. We now show that $\bigcup_{m=1}^{J} V(F_m) = \{\hat{\chi}^m | m \in [J]\}$.

Let $\hat{p} \in V(F_m)$ for some $m \in [J]$ such that $\hat{p} \neq \mathbf{0}$. Then there is some $i \in [n]$ such that $\hat{p}_i = 1$ and since $\lambda_{\hat{p}} > 0$ we have $a_i = r$. Hence, there is some $t \in [s]$ such that $i \in V(G_t)$. We now show that $\hat{p} = \chi^t$.
If $\hat{p}_j = 1$, then $\hat{p}_{ij} = 1$ and therefore $a_{ij} = r$. This implies that i and j are contained in the same connected component in $\hat{G}$, i.e. $j \in V(G_t)$. Note that if i is an isolated vertex in $\hat{G}$, then the assumption $p_j = 1$ immediately leads to a contradiction, so in this case we have indeed $\hat{p} = e_i$.
If $\hat{p}_j = 0$, then $0 = \hat{p}_{ij} < \hat{p}_i$. This implies $a_{ij} = 0$, since otherwise (by (ii'))

$$r = a_{ij} = \sum_{m=1}^{J} \sum_{p\in V(F_m)} \lambda_p p_{ij} < \sum_{m=1}^{J} \sum_{p\in V(F_m)} \lambda_p p_i = a_i = r.$$

Hence, $j \notin V(G_t)$ and so $\hat{p} = \chi^t$. We have shown that $\bigcup_{m=1}^{J} V(F_m) \subseteq \{\chi^0, \chi^1, \ldots, \chi^s\}$.
Conversely, let $t \in [s]$ and $i \in V(G_t)$. Then $a_i = r$, so there exists some $\hat{p} \in \bigcup_{m=1}^{J} F_m$ such that $\hat{p}_i = 1$. By the above, this implies $\hat{p} = \chi^t$, and hence $\{\chi^1, \ldots, \chi^s\} \subseteq \bigcup_{m=1}^{J} V(F_m)$.
For $t = 0, \ldots, s$, we define

$$\mu_t = \sum_{m=1}^{J} \sum_{\substack{p\in V(F_m) \\ p=\chi^t}} \lambda_p.$$

We show that $\mu_0 = J - s \cdot r$. This implies that $\mathbf{0} \in \bigcup_{m=1}^{J} V(F_m)$ if and only if $s \cdot r < J$, which is equivalent to $\mathbf{0} = \chi^0 \in \{\hat{\chi}^m | m \in [J]\}$. We can write

$$a = \sum_{m=1}^{J} \sum_{p \in V(F_m)} \lambda_p p \; = \sum_{t=0}^{s} \mu_t \chi^t.$$

For each $t \in [s]$, let $i \in V(G_t)$. Then

$$r = a_i = \sum_{t=0}^{s} \mu_t \chi_i^t = \mu_t$$

and so

$$s \cdot r + \mu_0 = \sum_{t=0}^{s} \mu_t = \sum_{t=0}^{s} \sum_{m=1}^{J} \sum_{\substack{p \in V(F_m) \\ p = \chi^t}} \lambda_p = \sum_{m=1}^{J} \sum_{p \in V(F_m)} \lambda_p = \sum_{m=1}^{J} 1 = J.$$

Thus, $\bigcup_{m=1}^{J} V(F_m) = \{\hat{\chi}^m | m \in [J]\}$ and Lemma 8 implies that there is a labeling of the faces such that $\hat{\chi}^m \in F_m$ for all $m \in [J]$. □

Note that if $a \in \{0,1\}^d$, then a corresponds to the disjoint union of complete graphs and the procedure in the proof of Proposition 10 assigns packages corresponding to connected components to some agents, while the others get nothing. If $a^* \in \{0,1\}^n$, then a choice of $a \in A^*$ corresponds to a choice of connected components.

3.5 Competitive Equilibrium at Arbitrary Bundles

In the previous sections, we explored the existence of a competitive equilibrium at specific $a \in \mathbb{Z}^d_{\geq 0}$. We now consider under which conditions a competitive equilibrium exists at arbitrary $a \in \mathbb{Z}^d$. If $a \notin JP(K_n)$, then a competitive equilibrium does not exists for any $J \in \mathbb{N}$. In the following, we show that if $J = 2$, then a competitive equilibrium always exists for all $a \in 2P(K_n) \cap \mathbb{Z}^d$.

3.5.1 Auctions with Two Agents

We show that in auctions with precisely two agents, a competitive equilibrium always exists at any $a \in 2P(K_n) \cap \mathbb{Z}^d$. Note that this condition implies that $a \in \{0,1,2\}^d$, i.e. the auctioneer offers either two, one or no item of each type of goods to the agents. Again, this gives a more general result than Proposition 7 in terms of possible allocations of the goods, since the previous procedure splits the items in a way such that one of the agents is assigned one item of each type that the auctioneer offers. The other agent is assigned an item if and only if the auctioneer has two goods of this type to sell.

The proof consists of four steps. First, we show in Lemma 11 that the values of $a \in 2P(K_n) \cap \mathbb{Z}^d$ have a particular structure. The following steps are again similar to the ones used in the proof of Proposition 7. In Proposition 12, we show that any a is the sum of characteristic vectors of complete graphs. As a third step, we show that whenever we write a as the sum of points $\sum_{p \in V(F_j)} \lambda_p p$ contained in the convex hulls of faces $F_j, j \in [J]$, the set of vertices involved in this representation equal the set of characteristic vectors that appear in the second step. We then apply Lemma 8 to show that we can find a labeling of the faces such that $\chi^j \in F_j$.

Lemma 11. *Let $a \in \mathbb{Z}^d$ satisfying (i)-(iii) for $J = 2$ such that $a_i = 2$ for some $i \in [n]$. Then $a_j = a_{ij}$ for all $j \in [n]$.*

Proof. Since a satisfies (i)-(iii) for $J = 2$, a is contained in $[0, 2]^d$, i.e. $a \in \{0, 1, 2\}^d$. Inequality (ii) implies

$$a_{ij} \leq a_j.$$

Since $a_i = 2$, inequality (iii) implies

$$a_i + a_j - a_{ij} = 2 + a_j - a_{ij} \leq 2$$

and hence

$$a_j \leq a_{ij}.$$

□

Proposition 12. *Let $a \in \mathbb{Z}^d$. Then for all faces F_1, F_2 of $P(K_n)$ containing $a \in F_1 + F_2$ in their Minkowski sum, there exist vertices χ^1 if F_1 and χ^2 of F_2 such that $x = \chi^1 + \chi^2$.*

Proof. If there exist faces F_1, F_2 containing $a \in F_1 + F_2$, then $a \in 2P(K_n)$ and a satisfies inequalities (i)-(v) for $J = 2$. We define a vector $v \in \{0, 1, 2\}^d$ by

$$v_i = \begin{cases} 2, & \text{if } a_i = 2 \\ 0, & \text{otherwise} \end{cases} \qquad v_{ij} = \begin{cases} a_{ij}, & \text{if } a_i = 2 \text{ or } a_j = 2 \\ 0, & \text{otherwise} \end{cases}$$

Then

$$(a - v)_i = \begin{cases} 0, & \text{if } a_i = 2 \\ a_i, & \text{otherwise.} \end{cases} \qquad (a - v)_{ij} = \begin{cases} 0, & \text{if } a_i = 2 \text{ or } a_j = 2 \\ a_{ij}, & \text{otherwise.} \end{cases}$$

Note that $a - v \in \{0, 1\}^d$ and $a - v$ satisfies (i)-(v) for $J = 2$. Thus, by Lemma 9, $a - v$ is the characteristic vector of the disjoint union of complete graphs. Inequality (v) implies that there are at most two connected components, so $a - v$ is the characteristic vector of

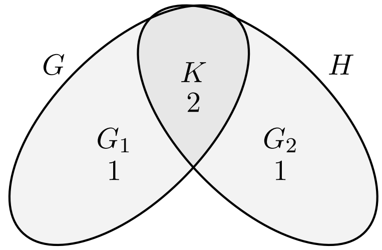


FIGURE 7. The vector a is the characteristic vector of a labeled complete graph K_n with label 2 on the vertices and edges of a complete graph K, label 1 on the vertices and edges of complete graphs G_1 and G_2, label 1 on the edges between K and G_1, K and G_2, and 0 otherwise.

the disjoint union of two complete graphs G_1 and G_2 (which can be equal to the empty graph). Further, if $v_i = 2$ for some $i \in [n]$, then $a_i = 2$ and

$$v_{ij} = a_{ij} = \min \{a_i, a_j\} = a_j.$$

Thus, v is a "characteristic vector" of a complete graph K that has label 2 on each of its vertices and edges, together with all edges between K and G_1, G_2, labeled by 1. Define $G = G_1 \cup K$ and $H = G_2 \cup K$. Then

$$a = \chi_G + \chi_H = \sum_{m=1}^{2} \sum_{p \in V(F_m)} \lambda_p p,$$

where $V(F_m)$ is a subset of vertices of the face $F_m, \lambda_p > 0$ for all $p \in V(F_m)$, and χ_G, χ_H denote the characteristic vectors of G and H. We show that $V(F_1) \cup V(F_2) = \{\chi_G, \chi_H\}$.

Let $L \in \{G,\ H,\ G_1,\ G_2,\ K,\ O\}$, where O denotes the empty graph with characteristic vector $\chi_O = \mathbf{0}$. For $i, j \in V(L)$ we have $a_i = a_j = a_{ij}$. Thus, for any $p \in V(F_1) \cup V(F_2)$, inequality (2) implies $p_i = p_j = p_{ij}$, i.e. p is constant on L. If i is not contained in $G \cup H$, then $a_i = 0$ and thus $p_i = 0$. Therefore, $V(F_1) \cup V(F_2) \subseteq \{\chi_L |\ L \in \{G,\ H,\ G_1,\ G_2,\ K,\ O\}\}$. Let

$$\mu_L = \sum_{m=1}^{2} \sum_{\substack{p \in V(F_m) \\ p = \chi_L}} \lambda_p.$$

Then

$$a = \mu_G \chi_G + \mu_H \chi_H + \mu_{G_1} \chi_{G_1} + \mu_{G_2} \chi_{G_2} + \mu_K \chi_K + \mu_O \chi_O.$$

If $i \in V(K), j \in V(G_1)$, then

$$\begin{aligned} 1 = a_j &= \mu_G + \mu_{G_1} \\ 1 = a_{ij} &= \mu_G \end{aligned}$$

and thus $\mu_{G_1} = 0, \mu_G = 1$. Similarly $\mu_{G_2} = 0, \mu_H = 1$, and since $\chi_O = \mathbf{0}$, we have

$$a = \chi_G + \chi_H + \mu_K \chi_K,$$

which implies $\mu \chi_K = 0$. Hence, we have $V(F_1) \cup V(F_2) \subseteq \{\chi_G, \chi_H, \chi_O\}$ and it remains to show that $\mu_O = 0$. This is given by

$$\mu_O + 2 = \sum_{L \in \{G,H,O\}} \mu_L = \sum_{L \in \{G,H,O\}} \sum_{m=1}^{2} \sum_{\substack{p \in V(F_m) \\ p = \chi_L}} \lambda_p = \sum_{m=1}^{2} \sum_{p \in V(F_m)} \lambda_p = 2,$$

so $V(F_1) \cup V(F_2) = \{\chi_G, \chi_H\}$ and Lemma 8 yields the desired result. □

3.5.2 Auctions with More Than Two Agents

For an auction with 4 agents and 4 types of items, the example in Section 3.3 is a point $a \in 4P(K_4)$ which cannot be written as the sum of any 4 vertices of $P(K_4)$, i.e. $a \notin 4 \cdot (P(K_4) \cap \mathbb{Z}^{10})$, and so a competitive equilibrium at a does not exist. This example can be extended to an example for any $n \geq 4$ by adding rows and columns with values $x_{ij} = 0$ for $4 < i, j \leq n$ to each of the points x defining the edges given in the example, as well as the point a itself. Since $P(K_n)$ is 3-connected, the convex hulls of pairs of these points are edges in $P(K_n)$ for all $n \geq 4$. For $J \geq 4$, we can add $J - 4$ copies of the face consisting of the vertex $\mathbf{0}$ in order to extend the above to an example for any $n \geq 4, J \geq 4$, i.e. an auction with at least 4 agents and at least 4 items, in which there exists no set of valuations such that a competitive equilibrium at a can be achieved. This implies that an analogue of Proposition 12 does not hold in general for $J \geq 4, n \geq 4$. Hoewever, for $n \leq 3$, all elements of the Hilbert basis of the cone over $P(K_n)$ indeed lie at height 1.
It is unknown whether we can make a statement similar to Proposition 12 for $J = 3$ and arbitrary $n \in \mathbb{N}$ or for $n \leq 3$ and $J \geq 3$.

3.6 The Efficient Allocation Problem

Candogan, Asuman and Parillo give a linear program [4, LP2] to find a competitive equilibrium at a given point $a \in \mathbb{R}^{n+e}$. They generally assume $a^* = (1, \dots, 1)^T$ and that the graphical valuations that have non-zero weights on all edges of the underlying graph. Let M be the collection of sets containing all *efficient allocations* of a. In the setting of complete graphs, M is the collection of sets $\mu = \left\{ s^1, \dots, s^J \,\middle|\, s^j \in \text{vert}(P(K_n)),\ \sum_{j=1}^{J} s^j = a \right\}$.

We introduce variables $x^j(s)$ for all $s \in \text{vert}(P(K_n)), j \in [J]$ and $\delta(\mu)$ for all $\mu \in M$. Rewritten in our notation, the linear program is

$$\max \sum_{j=1}^{J} \sum_{s \in \text{vert}(P(K_n))} x^j(s) v^j(s) \tag{4}$$

$$\text{s.t.} \sum_{\mu \in M} \delta(\mu) \leq 1 \tag{5}$$

$$x^j(s) \leq 1 \qquad \forall j \in [J], s \in \text{vert}(P(K_n)) \tag{6}$$

$$\sum_{j=1}^{J} \sum_{\substack{s \in \text{vert}(P(K_n)) \\ s_k = 1}} x^j(s) = \sum_{j=1}^{J} \sum_{\substack{\mu \in M \\ \mu = \{s^j \mid j \in [J]\} \\ s^j_k = 1}} \delta(\mu) \qquad \forall k \in [d] \tag{7}$$

$$\delta(\mu) \geq 0, \quad x^j(s) \geq 0 \qquad \forall \mu \in M, j \in [J], s \in \text{vert}(P(K_n)) \tag{8}$$

A solution for this linear program has variables $x^j(s), \delta(\mu)$ for all $\mu \in M, j \in [J]$, and $s \in \text{vert}(P(K_n))$. Given an integral solution in which $\delta(\hat{\mu}) = 1$ for some $\hat{\mu} \in M$, this can be interpreted as the auctioneer choosing the allocation $\hat{\mu}$. (5) then implies that $\delta(\mu) = 0$ for all other allocations $\mu \in M, \mu \neq \hat{\mu}$, so the auctioneer does not choose any other allocation. If $x^j(\hat{s}) = 1$, then this can be interpreted as bundle $\hat{s}$ being assigned to agent j. Then (6) implies that $x^j(s) = 0$ for all other bundles $s \in \text{vert}(P(K_n)), s \neq \hat{s}$, so this is the only bundle that is assigned to agent j. (7) satisfies that $x^j(\hat{s}) = 1$ if and only if $\hat{\mu}$ allocates $\hat{s}$ to agent j. The feasible solutions of the dual of this linear program can be interpreted as prices $p \in \mathbb{R}^d$.

Theorem 13 ([4, Theorem 3.2])**.** *The linear program has an optimal solution that is integral if and only if a pricing equilibrium with anonymous graphical pricing (competitive equilibrium) exists. Moreover, if a pricing equilibrium with anonymous graphical pricing exists, then the prices at an optimal solution of the dual linear program, and the allocation suggested by an integral optimal solution of the linear program constitute a pricing equilibrium.*

Given $a \in JP(K_n) \cap \{0,1\}^d$, Proposition 10 implies that for all faces $F_1, \ldots, F_J$ containing $a \in \sum_{j=1}^J F_j$ in their Minkowski sum, we also have $a \in \sum_{j=1}^J \text{vert}(F_j)$. Hence, by Lemma 5, there exists a price $p \in \mathbb{R}^d$ and a decomposition $a = \sum_{j=1}^J a^j$ such that $a^j \in D_{v^j}(p)$, so there exists a competitive equilibrium with respect to the allocation $\mu = \{a^1, \ldots, a^J\}$. By Theorem 13, this gives an optimal integral solution for the linear program having dual optimal solution $p \in \mathbb{R}^d$.

3.7 Competitive and Walrasian Equilibrium

Definition 4 ([4, Section 2.1])**.** Let $\{v^j \mid j \in [J]\}$ be a set of graphical valuations with underlying value graph G. There exists a *Walrasian equilibrium* at $a \in \mathbb{Z}^d_{\geq 0}$ if there is a price $p \in \mathbb{R}^d$ such that $p_{ij} = 0$ for all $ij \in E(G)$ and $a \in D_V(p)$.

Thus, the existence of a Walrasian equilibrium implies a competitive equilibrium. If G is the graph on n vertices having $E(G) = \emptyset$, then, by definition, these concepts coincide.

Theorem 14 ([4, Thereom 3.7(ii)])**.** *Assume that there are at least three agents, an anonymous graphical pricing equilibrium (competitive equilibrium) exists, and the underlying pricing graph is series-parallel. Then a pricing equilibrium with anonymous item pricing (Walrasian equilibrium) also exists.*

Note that in the above setting, it is assumed that for each graphical valuation $v(a) = \langle w, a \rangle$ for each $ij \in E(G)$ there exists positive or negative edge weights, i.e. $w_{ij} \neq 0$ ([4, Definition 2.1]). We consider an example of a competitive equilibrium for a series-parallel graph.

Example 6 ([3, Example 3.2],[4, Example 3.13])**.** Let $G = K_3$ be the complete graph on three vertices $V(K_3) = \{A, B, C\}$, $J = 3$ and $a^* = (1, 1, 1)$. The agent's valuations are given by the weight vectors

$$w^1 = (0,0,0,1,0,0)^T, \quad w^2 = (0,0,0,0,1,0)^T, \quad w^3 = (0,0,0,0,0,1)^T,$$

i.e. the first agent has weight one for edge AB, the second agent has weight one for edge AC, the third agent has weight one for edge BC and all remaining weights are zero. If we pick the graphical price vector

$$p = (0,0,0,1,1,1)^T,$$

then for each $j \in \{1, 2, 3\}$ we have

$$D_{v^j}(p) \supseteq \left\{ \begin{pmatrix} 0\\0\\0\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 1\\0\\0\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\1\\0\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\1\\0\\0\\0 \end{pmatrix} \right\}.$$

Thus, we can decompose $a = (1, 1, 1, 0, 0, 0)^T$ by assigning one item to each agent (where each agent is charged the price 0) in order to achieve a competitive equilibrium at a. In fact, this particular p induces a competitive equilibrium at all $a \in A^* \cap 3P(K_n)$.

It was established in [3, Example 3.2] that a Walrasian equilibrium for this example does not exist. However, we have found a competitive equilibrium and the graph is series-parallel. This indicates that in Theorem 14 the assumption on the weights w_{ij} to be non-zero for all edges and all agents is indeed needed.

4 Conclusion

We have shown, that in the setting of anonymous graphical pricing and graphical valuations for which the underlying value graph is the complete graph, a competitive equilibrium can always be achieved by constructing a particular distribution of items given a (reasonable) bundle of goods. Further, we have shown generalizations, which give the auctioneer several options to allocate the items in order to achieve competitive equilibrium, while restricting either on the structure of the given bundle or the number of agents participating in the auction. The proofs of all of these results have a similar structure, following the same steps and yield the same result. Therefore, it is likely that these three cases are connected by a unifying property that is used in all three proofs. One of these properties is

$$a_{ij} \in \{\min(a_i, a_j), 0\}$$

for a particular $a \in JP(K_n) \cap \mathbb{Z}^d, i, j \in [n]$. This implies that a can be interpreted as labeled graph. Further, all of these vectors satisfy

$$\text{If } \ a_j \geq a_i > 0, \ a_k, a_{ij}, a_{ik} > 0 \ \text{ then } \ a_{jk} > 0.$$

This is a property that is similar to inequality (iv'), since these two properties together imply that whenever a subgraph G of constant label t and a subgraph H of constant label t' share an edge, then any vertex of G is connected to all vertices of H by an edge with label $\min(a_i, a_j)$. These or similar properties could unify the three cases.

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